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PERIODIC MODES ANALYSIS IN A TWO-RELAY FEEDBACK SYSTEM

Alina BESANÇON-VODĂ

Laboratoire d'Automatique de Grenoble

Alina.Besancon@inpg.fr

Abstract: *The limit cycles behavior for a class of two-relay systems is investigated in this paper. The analysis, which is performed both in frequency domain and in time domain, is using exact methods, which are essentially extensions of those presented for one-relay feedback systems. Simulation results are given to substantiate the proposed methods.*

Keywords: *Relay systems, limit cycles, periodic modes, Tsypkin locus, state space representation.*

1. INTRODUCTION

Relay feedback systems have both theoretical and practical great interest. They have been widely used in control applications, owing to:

a) their simplicity and low cost,

b) proportional amplification is realized with vibrating relays,

c) optimization methods lead in many cases to control inputs of rectangular form (which thus *a posteriori* justifies the interest of relays in control systems), and

d) recent use of relay for auto-tuning techniques, almost since 1984 [8].

Moreover, relays are important elements in variable-structure systems [10] and they represent a special class of switched systems with simple characteristic. However, even if they represent a classical topic in control theory, open theoretical interesting problems still exist (see [6]).

In many practical situations, more than one relay type element may be found in a closed-loop system, and some results available for one-relay systems can be extended for theoretical analysis of systems containing several relays [9], [1].

A particular class of such systems is shown in fig. 1, named two-relay feedback system. It contains an inner relay with amplitude f and an outer relay with amplitude d, with hysteresis ε or with time lag τ (not shown on the figure). This con¯guration with two relays can represent, for example, a system containing Coulomb friction (inner relay), controlled by another relay (outer relay).

In some previous studies dealing with this configuration, it has been shown how to simply determine the amplitude of the inner relay, if unknown (Coulomb friction force is in general unknown in a mechanical servo-system). Time domain methods [3] or frequency domain methods, like the describing function [2], [5], [4] have been developed. However, the exact analysis of the periodic solution has not been yet realized in frequency domain, in the sense that until present, there is no mean to calculate a priori the frequency of the limit cycle for such a two-relay system (describing function method is an approximate one).

It has been shown in [9] that the hodographs of a relay control system (known since then as *Tsypkin locus*) constitute a very convenient tool in quantitative and qualitative analysis of the influence of the structure and parameters of the linear part and of the relay element on the frequency and the form of self-oscillations.

In this paper, Tsypkin locus is constructed, in order to give the limit cycle conditions for this class of two-relay systems using an exact analysis method in frequency domain. The aim is to find the frequency of the periodic solution for the two-relay system, using similar ideas as in the existing results for one-relay systems [9]. The advantage of this method over the describing function is that the limit cycle frequency is found more accurately.

Moreover, a time-domain version of these frequency-domain methods is given, using some previous existing results for one relay [7], and for two relays [3].

Both methods (frequency and time-domain) are also developed in the case of a system containing a time-delay in the outer relay, instead of the hysteresis.

Illustrative examples are given to show the accuracy of these methods.

2. TWO-RELAY FEEDBACK SYSTEM AND PERIODIC MODES

2.1. *System structure and relay commutations*

In this paper we are interested in a class of two-relay feedback systems which consist of a linear part with a transfer function $G(s) = \dfrac{N(s)}{sD(s)}$ containing an inner loop with a first relay (inner relay) and in closed loop with a second relay (outer relay) as described by the block diagram in fig. 1.

The inner loop is made up of :

- a transfer function $\dfrac{N(s)}{D(s)}$, assumed to be asymptotically stable (*D(s)* is a Hurwitz polynomial) and to have a positive steady-state gain, and
- an inner ideal relay feedback of amplitude *f*.

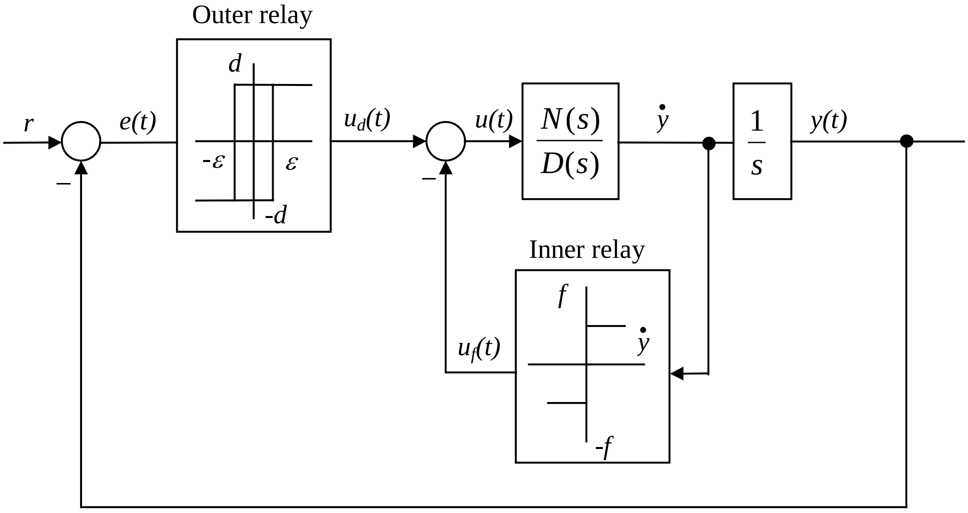


Fig. 1. Feedback system structure.

The input signal to the outer relay is the error *e(t)* (where $e(t) = r(t) - y(t)$), the output is $u_d(t)$, while the input signal to the outer relay is $\dot{y}(t)$ and the output is $u_f(t)$.

The input to the linear part of the feedback system, denoted by *u(t)*, is the difference between the outer and inner relay outputs:

$$u(t) = u_d(t) - u_f(t)$$

For sake of simplicity, in the sequel it is assumed that the reference signal *r(t)* is zero

($r(t) \equiv 0$). This implies that $e(t) \equiv -y(t)$.

Relay systems behavior is described by the relay commutations (or switchings). In the case of this two-relay system, the conditions for the kth commutation (k = 0; 1; …) are described as functions of *e(t)* and its two derivatives, and can be expressed mathematically as follows:

1. for the outer relay, at the time instants $t_k^{'}$, the switching conditions are:

$$\begin{aligned} & e(t_k^{'}) = (-1)^k \varepsilon \\ & \dot{e}(t_k^{'})(-1)^k > 0 \end{aligned}, \tag{1}$$

   where it has been assumed that $\dot{e}(t_1^{'}) < 0$ or $\dot{y}(t_1^{'}) > 0$;

2. for the inner relay, at the time instants t_k the switching conditions are:

$$\begin{aligned} & \dot{e}(t_k) = 0 \\ & \ddot{e}(t_k)(-1)^k > 0 \end{aligned}. \tag{2}$$

Equalities in conditions (1) and (2) are referred to as "*commutation conditions*" and the inequalities are called "*complementary conditions*". The commutation instants are denoted by t_k (inner relay switching instants) and $t_k^{'}$ (outer relay switching instants).

In the case of periodic oscillations, the commutations alternate regularly, i.e. the time interval between two consecutive commutations of each relay is constant ($t_{k+1}^{'} - t_k = h_1 = const.$ and $t_k - t_k^{'} = h_2 = const.$, for k=0, 1, …).

A particularity of the relay feedback systems (as well as of many other non-linear systems) consists of the periodic modes (periodic oscillations), which are not due to external periodic action *r(t)*, but to forces that depend on the state of the system. These oscillations have been called self-oscillations by Andronov. The present study consists of establishing their existence and in determining their frequency. The investigation of their form and stability are beyond of the scope of this paper.

2.2. *Periodic modes analysed in frequency domain*

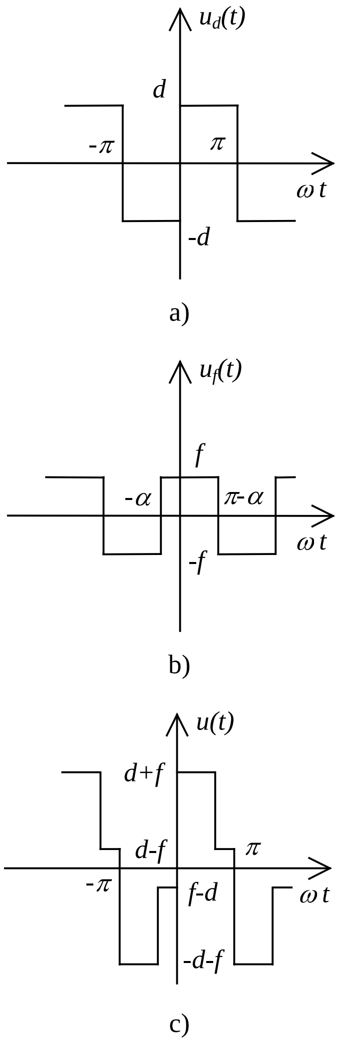


Fig. 2. Input to the linear part of the system: a) outer relay, b) inner relay and c) the result of signals composition.

In order to construct Tsypkin loci for our system, it is useful to firstly present the output signals from both relays, that are generated during the limit cycles.

Suppose that the limit cycles in the system are symmetrical and simple (there is, minimum number of commutations per period). Two parameters characterize the limit cycle: the frequency ω, and the mutual shift α of two (synchronized) rectangular signals u_d and u_f corresponding to the outer and inner relay outputs, respectively (see figure 2). The second parameter α is also equal to the phase shift of the first harmonics of both signals.

In the case of a symmetric periodic mode, like the one studied in this paper, the conditions of proper switching time and direction are derived from the general ones for transient processes (1, 2), by putting the following values for the switching instants:

$$t_k = \frac{k\pi}{\omega},\ k = 0,1,... \qquad t_k^{'} = \frac{(k-1)\pi + \alpha}{\omega},\ k = 0,1,... \tag{3}$$

The switching conditions (1, 2) become :

- for the outer relay:

$$e\left(\frac{(k-1)\pi + \alpha}{\omega}\right) = (-1)^k \varepsilon, \qquad \dot{e}\left(\frac{(k-1)\pi + \alpha}{\omega}\right)(-1)^k > 0 \tag{4}$$

- for the inner relay:

$$\dot{e}\left(\frac{k\pi}{\omega}\right) = 0, \qquad \ddot{e}\left(\frac{k\pi}{\omega}\right)(-1)^k > 0 \tag{5}$$

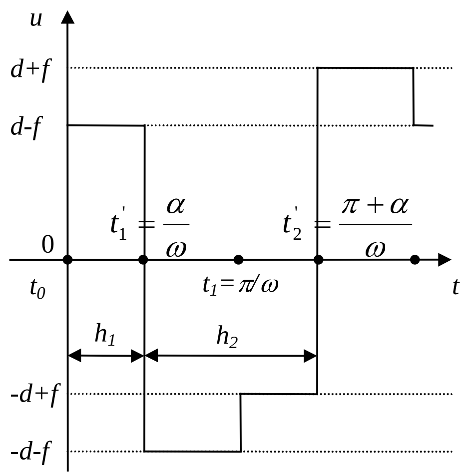


Fig. 3. First commutation instants (k=0, 1, 2), input $u(t)$ to the linear part.

The time intervals between two consecutive commutations of each relay become $h_1 = \frac{\alpha}{\omega}$ and $h_2 = \frac{\pi - \alpha}{\omega}$ (see fig. 3).

Moreover, in the periodic modes considered here, as in [9], the periodicity conditions for $e(t)$, $\dot{e}(t)$ and $\ddot{e}(t)$ can be simplified naturally by taking a fixed value for k, in particular $k = 1$.

Substituting $k = 1$ in (4, 5), it is obtained:

- for the outer relay:

$$e\left(\frac{\alpha}{\omega}\right) = -\varepsilon, \qquad \dot{e}\left(\frac{\alpha}{\omega}\right) < 0 \tag{6}$$

- for the inner relay

$$\dot{e}\left(\frac{\pi}{\omega}\right) = 0, \qquad \ddot{e}\left(\frac{\pi}{\omega}\right) < 0 \tag{7}$$

2.2.1. Definition of Tsypkyn Locus of a relay system

As it has been already mentioned, Tsypkin loci [9] allow to analyze self-oscillations of frequency ω in a relay controlled system, providing also a graphical interpretation. This approach is referred to in this paper as a frequency domain method. Moreover, in this paper, Tsypkin loci is constructed using the frequency characteristics of the linear dynamics $G(s)$. However, Tsypkin loci can also be derived from time-domain expressions of the signals generated during the periodic modes.

The definition of Tsypkin locus, or hodograph, for a system containing only one relay is (see also [9], chapter 6):

$$J(\omega) = -\frac{1}{\omega}\dot{y} - \left(\frac{\pi}{\omega}\right) - j \cdot y\left(\frac{\pi}{\omega}\right) \tag{8}$$

The imaginary part of $J(\omega)$, denoted here by $\Im(J(\omega))$, is equal to the value of the output $y(t)$ of the linear part of the system at $t=\pi/\omega$ (taken with opposite sign). The real part of $J(\omega)$, which is denoted by $\Re(J(\omega))$, is equal to the value of the derivative of y(t) with respect to ωt at $t=\pi/\omega$-0 (also with opposite sign) divided by the frequency ω, that is, to the value of the derivative of $y(t)$ with respect to ωt at $t=\pi/\omega$-0:

$$\begin{aligned} &\Im(J(\omega)) = -y\left(\frac{\pi}{\omega}\right) \\ &\Re(J(\omega)) = -\frac{1}{\omega}\dot{y}^{-}\left(\frac{\pi}{\omega}\right) \end{aligned} \tag{9}$$

The hodograph of a relay without dead-zone is fully determined by the response of the corresponding open-loop system to a periodic action and has the dimension of the output.

For a fixed value of ω the hodograph $J(\omega)$ is a vector. When ω varies, the end point of this vector describes a curve on which every point corresponds to a definite value of ω.

2.2.2. Tsypkin Loci of two-relay system, case without time delay in the outer relay

In the following, Tsypkin loci is derived for a system structure shown in figure 1.

In the case of the two-relay system from this paper, a useful analogy with the relays containing a dead-zone (see [9], chapter 6.3) will be used. In both two-relay system and in a relay system with a dead-zone, two types of commutations must be accounted for. In a relay containing a dead-zone, two different types of commutations occur. In our case also two different types of commutations occur: outer and inner relay commutations, respectively. Then two system characteristics (or hodographs) can be constructed.

For the outer relay, the Tsypkin locus is denoted J_1 and for the inner relay, it is denoted J_α. They are defined by the following expressions (Tsypkin loci), which are complex functions of or two real variables (ω and α):

$$J_1(\omega,\alpha) = -\frac{1}{\omega}\dot{y}^{-}\left(\frac{\pi}{\omega}\right) - j\cdot y\left(\frac{\pi}{\omega}\right) \tag{10}$$

$$\begin{aligned} J\alpha(\omega,\alpha) = &-\frac{1}{\omega^2}\ddot{y}^{-}\left(\frac{\pi-\alpha}{\omega}\right) - \\ &- j\frac{1}{\omega}\dot{y}\left(\frac{\pi-\alpha}{\omega}\right) \end{aligned} \tag{11}$$

Proposition 2.1 *A limit cycle with frequency ω_0 and phase shift α_0 exists if the Tsypkin loci satisfy respectively*:

$$\begin{aligned} &\Im(J_1(\omega_0,\alpha_0)) = -\varepsilon \\ &\Re(J_1(\omega_0,\alpha_0)) < 0 \end{aligned} \tag{12}$$

$$\begin{aligned} &\Im(J_\alpha(\omega_0,\alpha_0)) = 0 \\ &\Re(J_\alpha(\omega_0,\alpha_0)) < 0 \end{aligned} \tag{13}$$

Proof

The proposition is very easy to prove.

The conditions (4) and (5) for limit cycle existence with frequency ω_0 and phase shift α_0 can be rewritten as follows:

1) for the outer relay, at the time instants $t_k^{'} = \frac{(k-1)\pi+\alpha_0}{\omega_0}$, (k=0, 1, …), the switching conditions are:

$$\begin{aligned} &e\left(\frac{(k-1)\pi+\alpha_0}{\omega_0}\right) = (-1)^k\varepsilon \\ &\dot{e}\left(\frac{(k-1)\pi+\alpha_0}{\omega_0}\right)(-1)^k > 0 \end{aligned} \tag{14}$$

2) for the outer relay, at the time instants $t_k = \frac{k\pi}{\omega_0}$, (k=0, 1, …), the switching conditions are:

$$\begin{aligned} &\dot{e}\left(\frac{k\pi}{\omega_0}\right) = 0 \\ &\ddot{e}\left(\frac{k\pi}{\omega_0}\right)(-1)^k > 0 \end{aligned} \tag{15}$$

or, for k=1, conditions (6, 7) can be rewritten as:

- for the outer relay:

$$\begin{aligned} &e\left(\frac{\alpha_0}{\omega_0}\right) = -\varepsilon \\ &\dot{e}\left(\frac{\alpha_0}{\omega_0}\right) < 0 \end{aligned} \tag{16}$$

- for the inner relay:

$$\begin{aligned} &\dot{e}\left(\frac{\pi}{\omega_0}\right) = 0 \\ &\ddot{e}\left(\frac{\pi}{\omega_0}\right) < 0 \end{aligned} \tag{17}$$

Now keeping into account that $y(t) = -e(t)$, these conditions can be re-written as:

- for the outer relay

$$-y\left(\frac{\alpha_0}{\omega_0}\right) = -\varepsilon$$
$$-\dot{y}\left(\frac{\alpha_0}{\omega_0}\right) < 0 \quad (18)$$

- for the inner relay

$$-\dot{y}\left(\frac{\pi}{\omega_0}\right) = 0$$
$$-\ddot{y}\left(\frac{\pi}{\omega_0}\right) < 0 \quad (19)$$

Looking to the expressions of $J_1(\omega, \alpha)$ and of $J_\alpha(\omega, \alpha)$, one can see that conditions (18), (19) are equivalent with (12), (13), because ω is positive.

This shows that verification of conditions (12) and (13) ensures the existence of limit cycles with frequency ω_0 and phase shift α_0.

End of proof.

2.2.3 Graphical interpretation

The search for the limit cycles with frequency ω_0 and phase shift α_0.by means of the Tsypkin loci has a graphical interpretation that is similar to that of the describing function approach.

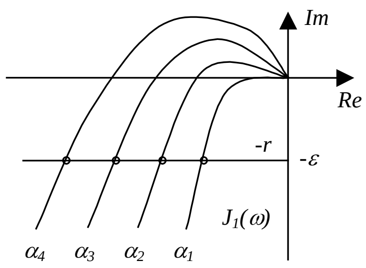
For the outer relay, the frequency of the self-oscillations is determined as follows. In the same plane as $J_1(\omega, \alpha)$, one has to draw a line parallel to the real axis at a distance $-\varepsilon$ from it, called the $-\varepsilon$ line, for short. The points in the left half-plane where this line intersects $J_1(\omega)$ determine the frequency of the self-oscillations that may occur in the system.

For the inner relay, the frequency of the self-oscillations can be determined in a similar way as for the outer relay. In the same plane as $J_\alpha(\omega, \alpha)$, the negative real axis has to be drawn (the inner relay has no hysteresis), and the intersection between $J_\alpha(\omega, \alpha)$ and this axis determine the frequency of the self-oscillations.

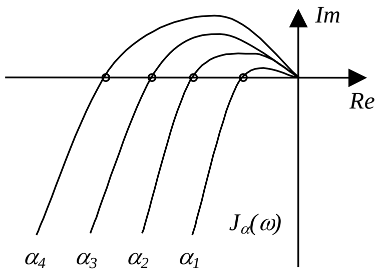
The only difficulty with the two-relay system is that both Tsypkin loci $J_1(\omega, \alpha)$ and $J_\alpha(\omega, \alpha)$ depend not only on the frequency ω, but they depend also on the phase shift α. In order to determine graphically both limit cycle characteristics, i.e. ω_0 and α_0, two parametric curves of parameter α, one from the J_1 loci and the second from J_α loci have to be constructed.

The first curve, $A_1(\alpha)$ represent a function $\omega(\alpha)$ obtained from the intersection points between the Tsypkin loci and the horizontal line -(ε). Indeed, for any value α_i of the parameter α, the corresponding Tsypkin loci $J_1(\omega; \alpha_i)$ is drawn. The frequency corresponding to the intersection between Tsypkin loci and the horizontal line -(ε) is denoted by ω_i (see fig. 4, left side). The function A_1 is then given by all the pairs $[\omega_i; \alpha_i]$, representing the pairs for which conditions (12) are fulfilled.

The same idea is applied for the second Tsypkin loci, $J_\alpha(\omega; \alpha_i)$, and a function A_α is found, given by all the pairs $[\omega_j; \alpha_j]$, for which conditions (13) are fulfilled (see also fig. 4, right side).



a)



b)

Fig. 4. Graphic interpretation of the system characteristics: a) J_1 and b) J_α.

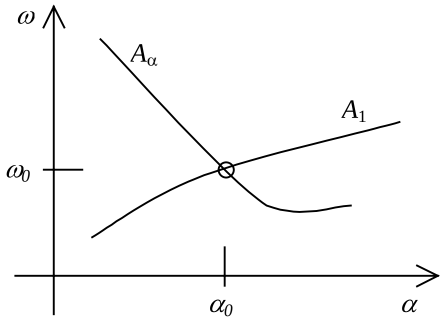


Fig. 5. Global intersection of two sets of intersections corresponding to limit cycles $[\omega_0; \alpha_0]$.

Finally, to find the solution [ω_0; α_0] that determines the limit cycles in the system, one must than look for the global intersection of both sets A_1 and A_α (see figure 5), representing the values for which both conditions (12) and (13) are fulfilled .

2.2.4 Analytical expressions of Tsypkin loci

In order to derive an expression for both Tsypkin loci, we first look for the expression of the system output *y(t)* in the case of a limit cycle with frequency ω. This expression is found as a response of a linear system to an input *u(t)*, which is a periodic sequence of pulses. This sequence of pulses can be expressed by means of two Fourier series as two sums of harmonic components with multiple frequencies $m\omega$. Using similar calculations to [9] (see chapter 5 of this book), *u(t)* can be expressed as in (20).

The response of the linear part of the system to this excitation (20) is given in (21).

From equation (21) one can easily express the first and second derivatives of the system output, too ($\dot{y}(t)$, $\ddot{y}(t)$).

Using equation (21), the characteristics J_1 and J_α can be expressed as infinite series depending on the linear part $G(j\omega) = U(\omega) + j\cdot V(\omega)$ of the system (relations 22 to 25).

$$\begin{aligned} u(t) &= u_d(t) - u_f(t) = \\ &= \sum_{m=1}^{\infty} \frac{4}{\pi(2m-1)} \left\{ d \cdot \sin[(2m-1)\omega t] + f \cdot \sin[(2m-1)(\omega t + \alpha)] \right\} \end{aligned} \tag{20}$$

$$\begin{aligned} y(t) = \sum_{m=1}^{\infty} \frac{4G_0((2m-1)\omega)}{\pi(2m-1)} \{ d \cdot \sin[(2m-1)\omega t +]\} + \theta((2m-1)\omega)] + \\ + \sum_{m=1}^{\infty} \frac{4G_0((2m-1)\omega)}{\pi(2m-1)} \{ f \cdot \sin[(2m-1)(\omega t + \alpha)] + \theta((2m-1)\omega)\} \end{aligned} \tag{21}$$

$$\begin{aligned} \Re(J_1(\omega,\alpha)) = \frac{4}{\pi} \sum_{m=1}^{\infty} \{ U((2m-1)\omega \cdot [d + f \cdot \cos((2m-1)\alpha)]\} - \\ - \frac{4}{\pi} \sum_{m=1}^{\infty} \{ V((2m-1)\omega \cdot f \cdot \sin((2m-1)\alpha)\} \end{aligned} \tag{22}$$

$$\begin{aligned} \Im(J_1(\omega,\alpha)) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \{ U((2m-1)\omega \cdot f \cdot \sin((2m-1)\alpha)]\} + \\ + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \{ V((2m-1)\omega \cdot [d + f \cdot \cos((2m-1)\alpha)]\} \end{aligned} \tag{23}$$

$$\begin{aligned} \Re(J_\alpha(\omega,\alpha)) = \frac{4}{\pi} \sum_{m=1}^{\infty} (2m-1) \{ U((2m-1)\omega) \cdot d \cdot \sin((2m-1)\alpha)]\} - \\ - \frac{4}{\pi} \sum_{m=1}^{\infty} (2m-1) \{ V((2m-1)\omega) \cdot [d \cdot \cos((2m-1)\alpha) + f]\} \end{aligned} \tag{24}$$

$$\begin{aligned} \Im(J_\alpha(\omega,\alpha)) = \frac{4}{\pi} \sum_{m=1}^{\infty} \{ U((2m-1)\omega \cdot [d \cdot \cos((2m-1)\alpha) + f]\} + \\ + \frac{4}{\pi} \sum_{m=1}^{\infty} \{ V((2m-1)\omega \cdot d \cdot \sin((2m-1)\alpha)\} \end{aligned} \tag{25}$$

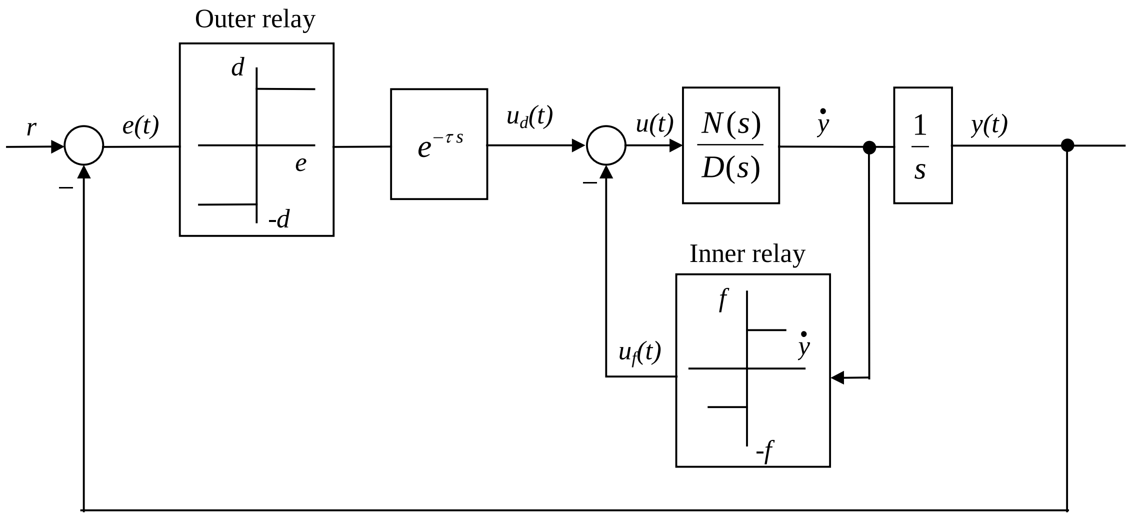


Fig. 6. System with time delay τ in the outer relay.

2.2.5 Tsypkin Loci, case with time delay in the outer relay

Unlike the preceding paragraph, suppose now a non-zero time delay $\tau \neq 0$ included in the outer relay (e.g. due to the discrete implementation of the outer relay), like shown in figure 6.

To be able to cope with this modification, an additional hypothesis on the limit cycles is necessary. As above, the self-oscillations are supposed to be symmetrical and simple.

Moreover, the system should oscillate "slowly enough", so that the time delay never includes more than one commutation. The assumption of slow oscillations can be expressed as follows: An outer relay commutation can be succeeded by an inner relay commutation not sooner than in τ seconds, where τ is the pure time lag (in seconds) included in the outer relay.

Keeping the notation of switching instants introduced in section 2 (t_k for an inner relay switch and $t_k^{'}$ for an outer relay switch), the above hypothesis yields (26).

Then, although the order of the system is now infinite, it can be can analyzed with very few modifications.

The same commutation and complementary conditions (equations (12) and (13)) are used and the same system characteristics J_1 and J_α (equations (10) and (11)). The time lag τ will appear in the mathematical expressions of the input *u(t)* and the output *y(t)* in the terms corresponding to the outer relay (equations (27) and (28)).

$$\begin{aligned} \left|t_k - t_k^{'}\right| \le \tau \\ \left|t_{k+1}^{'} - t_k\right| \le \tau \end{aligned} \qquad (26)$$

$$u(t) = \sum_{m=1}^{\infty} \frac{4}{\pi(2m-1)} \left\{ d \cdot \sin[(2m-1)\omega(t-\tau)] + f \cdot \sin[(2m-1)(\omega t + \alpha)] \right\} \qquad (27)$$

$$\begin{aligned} y(t) = \sum_{m=1}^{\infty} \frac{4G_0((2m-1)\omega)}{\pi(2m-1)} \{ d \cdot \sin[(2m-1)\omega(t-\tau)]\} + \theta((2m-1)\omega)] + \\ + \sum_{m=1}^{\infty} \frac{4G_0((2m-1)\omega)}{\pi(2m-1)} \{ f \cdot \sin[(2m-1)(\omega t + \alpha)] + \theta((2m-1)\omega)]\} \end{aligned} \qquad (28)$$

Using (28) along with definitions (10) and (11), the system characteristics is then as represented by equations (29) to (32).

The graphic interpretation and the use of the Tsypkin loci is the same as in section 2.2.3.

3. TIME-DOMAIN ANALYSIS

3.1. *Analysis of the system without time lag*

Another possibility to find exact parameters of limit cycles in a relay system is related with a state representation of the system. Consider the linear part of the system being expressed by the equations (33) where *A*, *B* and *C* satisfy equation (34) with *D(s)* a Hurwitz polynomial in *s*.

The limit cycle is supposed to begin with the inner relay commutation (i.e. $\dot{y} = 0$). Two limit cycle parameters are looked for: h_1, which is the time between an inner relay commutation and the following outer relay commutation, and h_2, which is the rest of one half-period (i.e. the limit cycle period is $T = 2 \cdot (h_1 + h_2)$). These parameters have been already defined in section 2.2 (see also fig. 3). Parameters h_1 and h_2 must verify the conditions (35) (issued from switching conditions for both relays), where *x(t)* is the state vector.

The third condition (35) ensures the odd symmetry of the limit cycles.

$$\begin{aligned}\Re(J_1(\omega,\alpha)) &= \frac{4}{\pi}\sum_{m=1}^{\infty}\{U((2m-1)\omega\cdot[d\cdot\cos((2m-1)\omega\tau)+f\cdot\cos((2m-1)\alpha)]\}-\\ &-\frac{4}{\pi}\sum_{m=1}^{\infty}\{V((2m-1)\omega\cdot[d\cdot\sin((2m-1)\omega\tau)-f\cdot\sin((2m-1)\alpha)]\}\end{aligned} \tag{29}$$

$$\begin{aligned}\Im(J_1(\omega,\alpha)) &= \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\{U((2m-1)\omega\cdot[-d\cdot\sin((2m-1)\omega\tau+f\cdot\sin((2m-1)\alpha)]\}+\\ &+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\{V((2m-1)\omega\cdot[d+f\cdot\cos((2m-1)\alpha)]\}\end{aligned} \tag{30}$$

$$\begin{aligned}\Re(J_\alpha(\omega,\alpha)) &= \frac{4}{\pi}\sum_{m=1}^{\infty}(2m-1)\{U((2m-1)\omega)\cdot d\cdot\sin((2m-1)(\alpha+\omega\tau))]\}-\\ &-\frac{4}{\pi}\sum_{m=1}^{\infty}(2m-1)\{V((2m-1)\omega)\cdot[d\cdot\cos((2m-1)(\alpha+\omega\tau))+f]\}\end{aligned} \tag{31}$$

$$\begin{aligned}\Im(J_\alpha(\omega,\alpha)) &= \frac{4}{\pi}\sum_{m=1}^{\infty}\{U((2m-1)\omega\cdot[d\cdot\cos((2m-1)(\alpha+\omega\tau))+f]\}+\\ &+\frac{4}{\pi}\sum_{m=1}^{\infty}\{V((2m-1)\omega\cdot d\cdot\sin((2m-1)(\alpha+\omega\tau))\}\end{aligned} \tag{32}$$

$$\begin{cases}\dot{x}(t) = Ax(t) + Bu(t)\\ y(t) = Cx(t)\\ u(t) = (d - f\cdot sign(\dot{y}(t)))\cdot sign(r(t) - y(t))\end{cases} \tag{33}$$

$$C(sI - A)^{-1}B = \frac{N(s)}{s\cdot D(s)} \tag{34}$$

$$\begin{aligned}&\dot{y}(0) = \dot{y}(h_1 + h_2) = 0\\ &y(h_1) = \varepsilon\\ &x(h_1 + h_2) = -x(0)\end{aligned} \tag{35}$$

From the solution of (33) with conditions (35) (see [3]), the expressions (36) and (37) define the time parameters h_1 and h_2, where $u_1 = d - f$ and $u_2 = - d - f$ are the values of the input $u(t)$ during the first part (h_1) and the second part (h_2) of one half-period, respectively (see also fig. 3). The total time of one half-period is denoted by h^*, where $h^* = h_1 + h_2$.

Note that the Tsypkin method (which is a frequency domain method) and the state space method (a time domain method) are equivalent, in the sense that the limit cycle conditions are the same. Indeed, conditions (36), (37) are the time version of equality conditions (13), (12), which correspond to the equality conditions in (18), (19).

This time-domain analysis can be used to determine stability of limit cycles in two-relay systems [3].

3.2. *Analysis of the system with time delay*

The same analytical approach can be applied for the case of an outer relay with pure time delay τ. The assumption made in section 2.2.5 on the limit cycles frequency must be again taken into account - the self-oscillations must be slow enough for the time between an outer relay switch and the following inner relay switch to be at least τ (see equation (26)). These conditions can be rewritten in terms of parameters h_1 and h_2:

$$h_1 \leq \tau \qquad\qquad h_2 \leq \tau$$

The solution of state equations (34) is given by the equations (38) to (40).

The input value (which is equal to $u_1 = d - f$ during the first part h_1 of the half-period) changes only after $h_1 + \tau$ where the instant h_1 corresponds to the outer relay commutation.

The second value $u_2 = - d - f$ appears on the system input during $h_2 - \tau$.

Substituting (40) in (39) and using the third switching condition (35), the state vector is described by the equations (41) and (42).

Considering (41) and (42) together with the first two switching conditions (35) gives relations (43) and (44).

The preceding expressions determine limit cycles parameters h_1 and h_2 which must be found numerically as a solution of the system composed by the two nonlinear equations (43) and (44).

$$E_1(h_1,h_2) = -CA\left(e^{Ah^*}+I\right)^{-1}\left(e^{Ah_2}\int_o^{h_1} e^{At}Bdt \cdot u_1 + \int_o^{h_2} e^{At}Bdtu_2\right) = 0 \tag{36}$$

$$E_2(h_1,h_2) = C\left(e^{Ah^*}+I\right)^{-1}\left(e^{-Ah_1}\int_o^{h_2} e^{At}Bdt \cdot u_2 + \int_o^{h_1} e^{At}Bdt \cdot u_1\right) = \varepsilon \tag{37}$$

$$x(h_1) = e^{Ah_1}\cdot x(0) + \int_0^{h_1} e^{At}Bu_1dt \tag{38}$$

$$x(h_1+h_2) = e^{A(h_2-\tau)}\cdot x(h_1+\tau) + \int_0^{h_2-\tau} e^{At}Bu_2dt \tag{39}$$

$$x(h_1+\tau) = e^{A(h_1+\tau)}\cdot x(0) + \int_0^{h_1+\tau} e^{At}Bu_1dt \tag{40}$$

$$x(h_1) = C\left(e^{Ah^*}+I\right)^{-1}\left(-e^{A(h_1+h_2-\tau)}\int_0^{\tau} e^{At}Bu_1dt - e^{Ah_1}\int_0^{h_2-\tau} e^{At}Bu_2dt + \int_0^{h_1} e^{At}Bu_1dt\right) \tag{41}$$

$$x(0) = \left(e^{Ah^*}+I\right)^{-1}\left(e^{A(h_2-\tau)}\int_0^{h_1+\tau} e^{At}Bu_1dt + \int_0^{h_2-\tau} e^{At}Bu_2dt\right) \tag{42}$$

$$E_1(h_1,h_2) = -CA\left(e^{Ah^*}+I\right)^{-1}\left(e^{A(h_2-\tau)}\int_o^{h_1+\tau} e^{At}Bdt \cdot u_1 + \int_o^{h_2-\tau} e^{At}Bdt \cdot u_2\right) = 0 \tag{43}$$

$$E_2(h_1,h_2) = C\left(e^{Ah^*}+I\right)^{-1}\left(-e^{A(h_1+h_2-\tau)}\int_o^{\tau} e^{At}Bdt \cdot u_1 - e^{Ah_1}\int_o^{h_2-\tau} e^{At}Bdt \cdot u_2 + \int_0^{h_1} e^{At}Bdt \cdot u_1\right) = \varepsilon \tag{44}$$

4. EXAMPLES

Simulations are aimed to verify the results that are obtained from the equations of the preceding sections. The value of the limit cycle frequency ω_0 is calculated for three examples of linear dynamics G(s) and outer relay with hysteresis or with time lag. Several values of the outer relay amplitude d have been used.

Table 1 presents the limit cycle frequency ω_0 for the two analytical methods presented above (Tsypkin loci and state-space), compared with the describing function approach (DF) applied to this class of systems ([5]. All these results are also compared with the true values, given by measuring the limit cycle characteristics in simulation .

Tsypkin loci have been obtained numerically in Matlab, based on equations (22, 23) and (24, 25) in the case of systems without time delay and based on equations (29, 30) and (31, 32) in the case of systems with time delay τ .

The limit cycle frequency ω_0 and phase shift α_0 have been determined from the intersections of A_1 with A_α as it has been presented in section 2.2.3.

Table 1: Comparison of different analytical methods with simulation results. The transfer function of the linear part is denoted *G(s)*, the outer relay has an hysteresis ε and a pure time lag τ, limit cycle frecquency ω_0 (rad/d).

G(s)	d	DF	Tsypkin loci	State space	Simul
		ω_0	ω_0	ω_0	ω_0
$\frac{1}{s^2}$, $\varepsilon = 0.02$	0.2 0.3 0.4 0.5	2.348 2.451 2.482 2.498	2.153 2.342 2.405 2.434	2.152 2.341 2.407 2.434	2.123 2.310 2.327 2.398
$\frac{1}{s(s+1)}$, $\varepsilon = 0.02$	0.2 0.3 0.4 0.5	2.508 3.027 3.339 3.574	2.315 2.905 3.257 3.517	2.307 2.909 3.255 3.530	2.310 2.910 3.210 3.490
$\frac{1}{s^2}$, $\tau = 0.2\ s$	0.2 0.3 0.4 0.5	2.618 1.699 1.263 1.006	2.168 1.404 1.037 0.821	2.162 1.403 1.036 0.821	2.166 1.408 1.072 0.842
$\frac{1}{s(s+1)}$, $\tau = 0.2\ s$	0.2 0.3 0.4 0.5	3.807 3.152 2.868 2.710	3.292 2.828 2.615 2.494	3.286 2.827 2.615 2.493	3.378 2.830 2.618 2.533
$\frac{1}{s(s+1)^2}$	0.2 0.3 0.4 0.5	1.618 1.387 1.280 1.219	1.605 1.365 1.256 1.195	1.603 1.366 1.257 1.195	1.605 1.365 1.256 1.195
$\frac{1}{s(s+1)^2}$, $\varepsilon = 0.02$	0.2 0.3 0.4 0.5	1.273 1.199 1.155 1.127	1.245 1.174 1.131 1.103	1.245 1.174 1.131 1.103	1.237 1.168 1.126 1.078

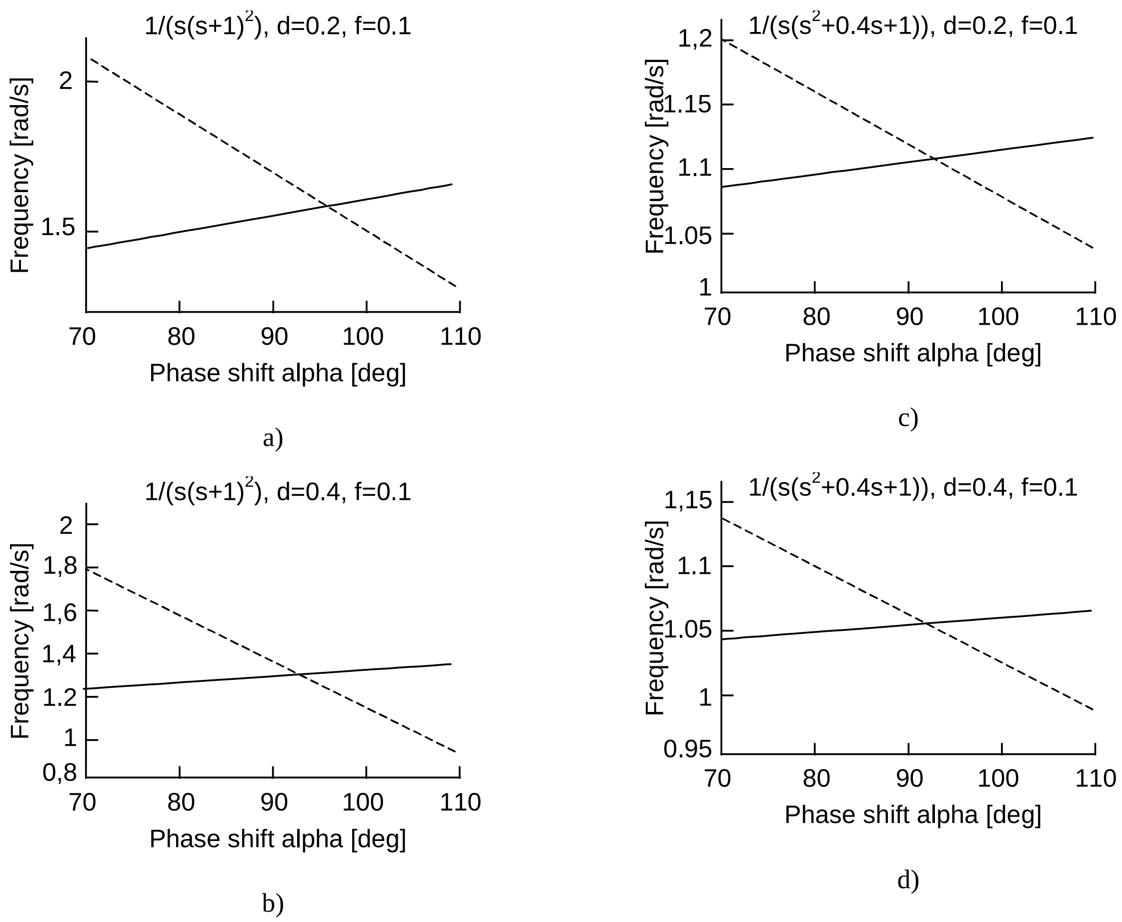


Fig. 7. Auto oscilations analysis by Tsipkin loci.

Figure 7 presents auto oscilations analysis by Tsipkin loci for systems with G(s) = $\frac{1}{s(s+1)^2}$ (*a)* and *b)* cases) and G(s) = $\frac{1}{s(s^2+0.4s+1)}$ (*c)* and *d)* cases); d = 0.2 (*a)* and *c)* cases) and d = 0.4 (*b)* and *d)* cases); f = 0.1. Solid line: intersection of J_1 loci with -ε line (function A_1). Discontinuous line: intersection of J_α loci with the real negative axis (function A_α).

For the state-space method, the limit cycle frequency is calculated as $\omega_0 = \frac{\pi}{h_1 + h_2}$, where h_1 and h_2 are calculated as shown in section 3.

The numerical values obtained show that there is a very good accordance between the exact analytical techniques (Tsypkin loci and state space approach) and the simulation. The describing function method show the same tendency of increasing or decreasing ω_0 with increasing d as the analytical methods, but the numerical values have a lower accuracy.

For more details on the tendency of the limit cycle frequency ω_0 with increasing *d* see [4].

5. CONCLUSIONS

Limit cycles in a two-relay system are analyzed by means of two exact approaches - the Tsypkin loci and the state-space analysis. Both techniques are inspired from existing approaches of one-relay system analysis. In this paper these methods have been extended for two-relay systems, eventually with time lag τ in the outer relay (the time lag is either due to implementation or introduced artificially for estimation of the inner relay amplitude *f*, like in [5]). The analytical results have been verified on some numerical examples that show the improved accuracy of the proposed methods compared with approximation methods, like describing function (harmonic balance).

Using the proposed methods, stability analysis of the two-relay system can further be developed, like for example it has been done in time-domain, for the case without time delay in the outer relay (see [3]).

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