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ON THE NON-FRAGILE ROBUST CONTROLLER DESIGN

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Abstract: *In the last years the problem of fragile controllers (high sensitivity of closed-loop stability and/or performance against small changes in controller coefficients) designed by using robust and optimal control methods has generated some controversial discussions. In this paper, we analyse the parametric stability margins for some optimal controllers by using different measures. So, we show that the normalized* $\boldsymbol{l}^2$ *parametric stability margin is not suitable when the controller coefficients lie in very different ranges and, in this case, we propose a new measure to appreciate the parametric stability margin. Also, we propose some reformulations of optimization problem such that to increase the parametric stability margins around the coefficients of the designed optimal controller. An algorithm to determine an extended range of robust stability is presented. Also, an iterative algorithm that allows computing controllers with increased parametric stability margins is proposed. We illustrate these design techniques by numerical examples.*

Keywords: *Robust stability, optimality, fragility, stability margins.*

1. INTRODUCTION

There are several techniques for designing linear time-invariant control systems that are optimum and robust. These are based on the Youla parameterization of all stabilizing controllers for a nominal (fixed) linear time-invariant plant, which provides a free parameter. This parameter is determined by minimizing the $\mathbf{H}_2$ or $\mathbf{H}_\infty$ norm of an appropriate closed-loop transfer function to obtain robust stability and/or robust performance with respect to plant perturbations. In this type of methodology, an implicit assumption is that the controller that is designed will be implemented exactly. This assumption is usually reasonable, since clearly, the plant uncertainty is the most significant source of uncertainty in the control system, while controllers are implemented with high precision hardware. However, there will inevitably be some amount of uncertainty in the controller: if the controller is implemented by analogue means, there are some tolerances in the analogue components; if the controller will be implemented digitally, there will be uncertainty involved with the quantization in the analogue-digital conversion and rounding in the parameter representation and in the numerical computations. In practice, due to the imprecision of controller implementation or to the requirements for readjustment of its coefficients, it is necessary that any controller be able to tolerate some uncertainty in its coefficients. This translates to the requirement that an adequate stability and performance margin be available around the transfer function coefficients of the designed nominal controller.

Recently it has been claimed in the literature [1] that several optimal and robust control synthesis techniques ($\mathbf{H}_2$, $\mathbf{H}_\infty$, $\mathbf{l}^1$ and $\boldsymbol{\mu}$) tend produce fragile controllers (that is, controllers resulting in high sensitivity of closed-loop stability and/or performance against small changes in controller coefficients). Unfortunately, the standard optimal and robust control design methods involve optimization criteria that do not include any explicit penalty terms against such fragility. This paper has generated some controversial discussions [2] and some techniques to increase the controller robustness against uncertainties in the controller coefficients were proposed [3-5].

Incited by these results, in this paper, we make a detailed analysis of one of the examples given in [1] and some conclusions are outlined. So, we show that the normalized $\mathbf{l}^2$ parametric stability margin is not suitable when the controller coefficients lie in very different ranges and, in this case, we propose a new measure to appreciate the parametric stability margin. Also, we propose some reformulations of optimization problem such that to increase the parametric stability margin around the coefficients of the designed optimal controller. Then, an algorithm to determine an extended range (of box type) of parametric robust stability is presented. Also, an iterative algorithm that allows computing controllers with increased parametric stability margins is proposed. Finally, some simulation results that illustrate the ideas of this paper are presented.

2. STABILITY MARGINS

The robustness of a closed-loop system can be defined both with respect to plant uncertainty and with respect to perturbations of the controller coefficients.

2.1. *The maximum stability margin with respect to plant uncertainty*

In this paragraf the stability margin for plant (unstructured) uncertainties is considered. Common approaches are to model the plant uncertainty in a multiplicative or additive way with respect to the nominal plant or as stable additive perturbations on the factors in a coprime factorization of the nominal plant. For each of these three classes of uncertainties the model of the perturbed system can be represented as an Upper Linear Fractional Transformation (see Fig. 1):

$$\Im_U\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}, \Delta\right) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}$$

where P is the standard plant (that depends on the model of uncertainty) and Δ ($\|\Delta\|_\infty < \varepsilon$) is the corresponding model of the (admissible) uncertainty.

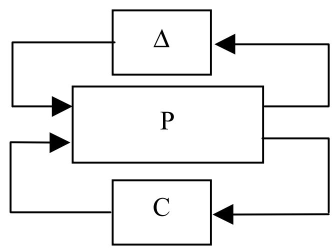

Fig.1. The generalized model of the uncertainty.

The largest stability margin for unstructured uncertainties is given by [6]:

$$\varepsilon_{\max} = \left(\inf_C \|\Im_L(P, C)\|_\infty\right)^{-1} \tag{1}$$

where C is chosen from all controllers which stabilize $\Im_U(\mathrm{P}, 0)$ and $\Im_L$ is the Lower Linear Fractional Transformation:

$$\Im_L\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}, C\right) = P_{11} + P_{12}C(I - P_{22}C)^{-1}P_{21}$$

For a given controller, the stability margin for unstructured uncertainties of the plant is given by

$$\varepsilon = \left(\|\Im_L(\mathrm{P}, \mathrm{C})\|_\infty\right)^{-1} \tag{2}$$

2.2. *The normalized* $\mathbf{l}^2$ *parametric stability margin*

Now, we consider the transfer function coefficients of the controller to be a parameter vector $\mathbf{q} = [q_1, q_2, \cdots q_l]$, with its nominal value being $\mathbf{q}^0$, and let $\Delta\,\mathbf{q}$ be the vector representing perturbations in $\mathbf{q}$. Then, the characteristic polynomial of the closed-loop system is

$$n(s, \mathbf{q}) = n_{\mathrm{P}}(s)n_{\mathrm{C}}(s, \mathbf{q}) + m_{\mathrm{P}}(s)m_{\mathrm{C}}(s, \mathbf{q}) \tag{3}$$

where (m_P, n_P) and (m_C, n_C) are the nominator and denominator polynomials of the plant and, respectively, controller. The $\mathbf{l}^2$ parametric stability margin around the nominal point is defined as being the radius of the largest stable hypersphere in the space of parameters $\mathbf{q}$ for the characteristic polynomial of the closed-loop system. Since the coefficients of this polynomial have affine dependency on $\mathbf{q}$ and based on the Boundary Crossing Theorem, the stability radius ρ_s is found as [7]

$$\rho_s = \min\left\{r_0,\ r_\infty,\ r_C(\omega^*)\right\} \tag{4}$$

where r_0, r_∞ and $r_C(\omega^*)$ are the distances of the origin from the hyperplanes that define the stability boundaries corresponding to the following three possibilities for the polynomial to become unstable: a) a real zero goes through the origin; b) a real zero goes through infinity; c) a pair of conjugate zeros crosses the imaginary axis. The normalized stability radius is defined as [1]:

$$\rho_{sn} = \rho_s \Big/ \left\|\mathbf{q}^0\right\|_2 \tag{5}$$

2.3. *The largest stable box in the parameter space*

We consider the closed-loop characteristic polynomial family $N(s, Q)$ defined by

$$N(s,Q) = \{n(s,\mathbf{q}) \mid \mathbf{q} \in Q\} \tag{6}$$

with

$$n(s,\mathbf{q}) = a_0(\mathbf{q}) + a_1(\mathbf{q})s + \cdots + a_n(\mathbf{q})s^n,$$
$$q_i \in \left[q_i^-, q_i^+\right],\ i = 1, 2, \cdots, l \tag{7}$$

Since the polynomial coefficients have an affine dependency on the controller coefficients, the polynomial family $N(s,Q)$ can be rewritten in the following form

$$n(s,\mathbf{q}) = n_0(s) + \sum_{i=1}^{l} q_i n_i(s) \tag{8}$$

with

$$n_0(s) = a_0^0 + a_1^0 s + \ldots a_n^0 s^n,\quad a_n^0 > 0 \tag{9}$$

$$n_i(s) = a_0^i + a_1^i s + \ldots a_n^i s^n,\quad i = 1, 2, \ldots l \tag{10}$$

Remark: Shifting the origin and scaling the q_i-axes in the parameter space, the box Q is transformed to an l-dimensional cube with sidelength two and center at $\mathbf{q} = [0 \ldots 0]$.

Let ω_s be a common real zero of the rational functions

$$\mathrm{Im}\left(n_k(j\omega)/n_0(j\omega)\right) = 0,\quad k = 1, 2, \ldots l \tag{11}$$

Now a real-valued function $\tau(\omega)$ is defined by

$$\tau(\omega) = \max_{1\le k\le l} \frac{\left|\mathrm{Im}\left(n_0(j\omega)/n_k(j\omega)\right)\right|}{\sum_{i=1}^{l}\left|\mathrm{Im}\left(n_i(j\omega)/n_k(j\omega)\right)\right|},$$
$$0 < \omega < \infty,\quad \omega \neq \omega_s \tag{12}$$

$$\tau(\omega) = \frac{\left|n_0(j\omega)\right|}{\sum_{i=1}^{l}\left|n_i(j\omega)\right|},\quad \omega = \omega_s \tag{13}$$

$$\tau(0) = \frac{\left|a_0^0\right|}{\sum_{i=1}^{l}\left|a_0^i\right|},\ \tau(\infty) = \frac{\left|a_n^0\right|}{\sum_{i=1}^{l}\left|a_n^i\right|} \tag{14}$$

For $\omega = 0$, $\omega = \infty$ and $\omega = \omega_s$ the function τ is, in general, discontinuous. The first two cases correspond to roots at $s = 0$ and $s = \infty$. Now, the following theorem [7] can be presented.

Theorem 1 (Tsypkin and Polyak). The polynomial family $N(s,Q)$ is stable, if and only if

1. $n_0(s)$ is stable;
2. $\tau(\omega) > 1$, $0 \le \omega \le \infty$.

Assume that $n(s, \mathbf{q})$ is stable for the nominal point $\mathbf{q}^0$ of the Q box. Then, the box may be blown up by a factor ρ, i.e. the polynomial family

$$N(s,\rho Q) = \{n(s,\mathbf{q}) \mid \mathbf{q} \in \rho Q\} \tag{15}$$

is considered. By increasing ρ a value ρ_{max} must be reached, where a member of the polynomial family becomes unstable, i. e. the box ρQ hits a stability boundary; it is a measure for the smallest destabilizing perturbation.

In this paper we choose the Q box having the edges proportional to the nominal values of $\mathbf{q}^0$; so, we consider

$$q_i \in \left[q_i^0 - \frac{\rho}{100}q_i^0,\ q_i^0 + \frac{\rho}{100}q_i^0\right],\ i = 1, 2, \cdots, l \tag{16}$$

Based on the Tsypkin and Polyak Theorem, the largest stable box of this type (defined by $\rho = \rho_{max}$) can be determined.

3. EXTENDED PARAMETRIC STABILITY MARGINS

Here, we propose the following algorithm to extend the stability box given by (16):

Algorithm 1.

1. *The largest stable box of type* (16) *is determined*;
2. *This stable box is extended by successively extending of each edge* q_i *in each direction by a very small factor. If the box is destabilized then the corresponding limit of* q_i *is fixed (precisely* ρ_i^- *and/or* ρ_i^+*)*;
3. *The steps* 1 *and* 2 *are repeated for the edges (and directions) unfixed while all these limits are fixed*.

Finally, the resulting stable box is characterized by

$$q_i \in \left[q_i^0 - \frac{\rho_i^-}{100} q_i^0,\ q_i^0 + \frac{\rho_i^+}{100} q_i^0 \right],\ \ i = 1, 2, \cdots, l \tag{17}$$

Obviously, ρ_{max} for the largest stable box of type (16) is given by

$$\rho_{max} = \min_i \left\{ \rho_i^-, \rho_i^+ \right\} \tag{18}$$

Sometimes, in order to obtain a controller with increased parametric stability margins (that is, to increase ρ_{max}), the following iterative method can be tried (the improvement of stability margin is not always possible by this method):

Algorithm 2.

1. *Apply the Algorithm* 1 *and determine the stable box of type* (17);
2. *Compute the centred controller in the box* (17) *characterized by*

$$q_i^c = q_i^0 + \frac{\rho_i^+ - \rho_i^-}{2} \times \frac{q_i^0}{100},\quad i = 1, 2, \cdots, l \tag{19}$$

3. *Repeat, if it is necessary, the steps* 1 *and* 2 *for the new centred controller obtained in the step* 2.

Remark: Generally, the increasing of the controller parametric stability margin lead to decreasing of the plant stability margin against unstructured uncertainty.

4. A WORKING EXAMPLE

4.1. *Upper gain margin optimization*

This example, originally proposed in [8] is treated in [1, 9, 10]. The plant to be controlled is

$$P(s) = \frac{s-1}{s^2 - s - 2} \tag{20}$$

and the controller, designed to give an upper gain margin of 3.5, is obtained by optimizing the $\mathbf{H}_\infty$ norm of the complementary sensitivity function as follows

$$C(s) = \frac{q_6^0 s^6 + q_5^0 s^5 + q_4^0 s^4 + q_3^0 s^3 + q_2^0 s^2 + q_1^0 s + q_0^0}{p_6^0 s^6 + p_5^0 s^5 + p_4^0 s^4 + p_3^0 s^3 + p_2^0 s^2 + p_1^0 s + p_0^0} \tag{21}$$

where:

$q_6^0 = 379$; $q_5^0 = 39383$; $q_4^0 = 192306$;
$q_3^0 = 382993$; $q_2^0 = 383284$; $q_1^0 = 192175$;
$q_0^0 = 38582$; $p_6^0 = 3$; $p_5^0 = -328$; $p_4^0 = -38048$;
$p_3^0 = -179760$; $p_2^0 = -314330$; $p_1^0 = -239911$;
$p_0^0 = -67626$.

The $\mathbf{l}^2$ parametric stability margin around the nominal point $\mathbf{q}^0 = [q_6^0 \dots q_0^0;\ p_6^0 \dots p_0^0]$ is found as $\rho_s \approx 0.158$ and the normalized stability radius is $\rho_{sn} \approx 2.103 \times 10^{-7}$. To illustrate the fragility of this controller, in [1] is constructed a destabilizing controller whose parameters are very closed to the nominal ones, except p_6 that is $p_6 \approx 3.158$. The relative change in p_6 is $\Delta p_6 / p_6^0 \approx 5.27\%$ and, in this case, the normalized stability radius (that is very small) is not adequate to measure the parametric stability margin for the controller. This is specifically for the cases where the controller coefficients lie in very different ranges.

The fragility of this controller is determined by an inadequate formulation of optimization problem, where only the upper gain margin is considered. So, the resulting lower gain margin and phase margin are very small (see, Fig. 2).

Based on *Theorem* 1, we have determined the largest stable box of type (16) to be given by

$\rho = \rho_{\max} \approx 0.038$, value that illustrates more adequately the possibilities for readjustment of nominal controller coefficients. The function τ given by (12)-(14), for $\rho = \rho_{\max}$, is represented in Fig. 3.

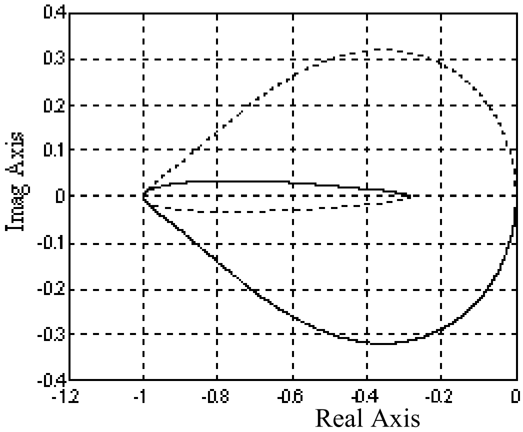

Fig.2. Nyquist plot of P(s)C(s).

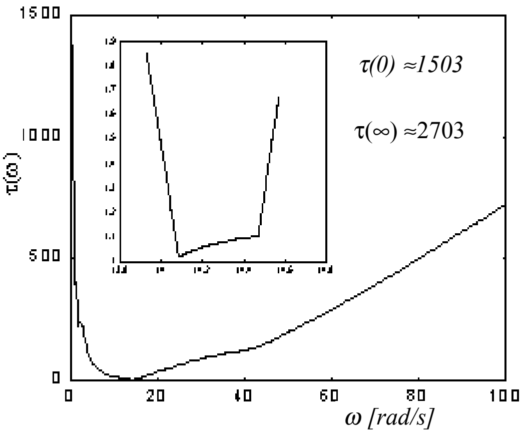

Fig.3. The distance function $\tau(\omega)$.

4.2. *Gain margin optimization*

Here, we propose to extend the robust stability problem formulated in Section 4.1 for the family

$$\wp = \{ kP : 1 \le k \le k_1 \} \tag{22}$$

to the following family

$$\wp = \{ kP : k_0 \le k \le k_1 \},\ 0 < k_0 < 1 < k_1 \tag{23}$$

where the interval $[k_0, k_1]$ must be centred in 1, in the sense that $k_0 k_1 = 1$. This family can be reduced to (22) by scaling, where the scaling factor is $k_0 = 1/\sqrt{k_{\inf} k_{\sup}}$ and $k_{\inf}$, $k_{\sup}$ are the lower and, respectively, upper gain margins obtained in Section 4.1. Now, the controller is obtained as $C(s)/k_0$, where $C(s)$ is given by (21). The resulting gain margins are now $k_{\inf} \approx 0.534$ and $k_{\sup} \approx 1.872$ (see, Fig. 4). The $\mathbf{l}^2$ parametric stability margin is now $\rho_s = 3$ and the normalized stability radius is about 2.455×10^{-6}.

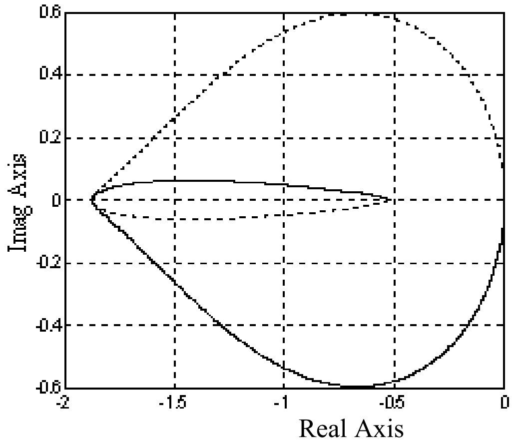

Fig.4. Nyquist plot of P(s)C(s).

Remark: In this case, the $\mathbf{l}^2$ parametric stability margin is determined by a real zero that goes through infinity corresponding to $p_6 = 0$. So, this destabilizing perturbation of controller coefficients is very strong ($\left|\Delta p_6 / p_6^0\right| = 100\%$) and the normalized stability radius is not an adequately measure for the controller robustness.

We have determined the largest stable box of type (16) to be given by $\rho = \rho_{\max} \approx 0.85$, value that illustrates more adequately the possibilities for readjustment of nominal controller coefficients. The function τ given by (12)-(14), for $\rho = \rho_{\max}$, is represented in Fig. 5.

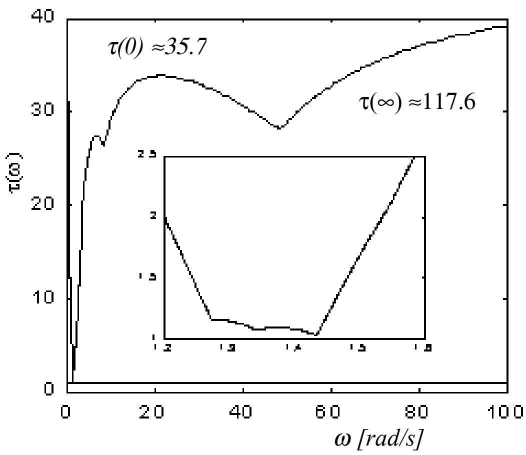

Fig.5. The distance function $\tau(\omega)$.

Though we have obtained an improvement in stability margins, the robustness in respect with the coefficients of the nominal controller is yet very small. This result confirms us that good gain and phase margins are not necessarily reliable indicators of robustness, because the uncertainties can affect simultaneously the gain and phase of the system. However, poor gain and/or phase margins are accurate indicators of fragileness!

4.3. *Maximum stability margin*

Since the coefficient perturbations of the transfer function of the nominal controller modify both the gain and the phase of the system, these perturbations can be treated as unstructured uncertainties and can be included in the initial uncertainty of the plant. So, in this section, we design a controller to obtain a maximum stability margin with respect to unstructured uncertainty of the plant. The maximum stability margin depends of the type of uncertainty modelling. Here, we use the normalized coprime factor plant description [6]. Then, for the same nominal plant (20), we obtain the maximum stability margin (defined by (1)) $\varepsilon_{max} = 1/\gamma_{min} = 1/13.429 \approx 0.0744$, which show that the plant, which is unstable and non-minimum phase, is relatively difficult to control.

By choosing $\gamma = 13.5 > \gamma_{min}$, the following controller is obtained:

$$C(s) = \frac{q_1^0 s + q_0^0}{p_2^0 s^2 + p_1^0 s + p_0^0} \qquad (24)$$

where, $q_1^0 = q_0^0 = 5137.2$; $p_2^0 = 1$; $p_1^0 = 387.06$; $p_0^0 = -3064$. The $\mathbf{l}^2$ parametric stability margin is now $\rho_s = 1$ and the normalized stability radius is $\rho_{sn} \approx 1.2667 \times 10^{-4}$, that is approximately 600 times bigger than in Section 4.1. The resulting lower and upper gain margins are $k_{inf} \approx 0.732$ and $k_{sup} = 1.193$ (see Fig. 6).

Remark: In this case, the $\mathbf{l}^2$ parametric stability margin is determined by a real zero that goes through infinity corresponding to $p_2 = 0$. So, this destabilizing perturbation of controller coefficients is very strong and the normalized stability radius is not an adequately measure for the controller robustness. The largest stable box of type (16) it is given by $\rho_{max} \approx 8.79$, value that illustrates more adequately the possibilities for readjustment of nominal controller coefficients. The function τ given by (12)-(14), for $\rho = \rho_{max}$, is represented in Fig. 7.

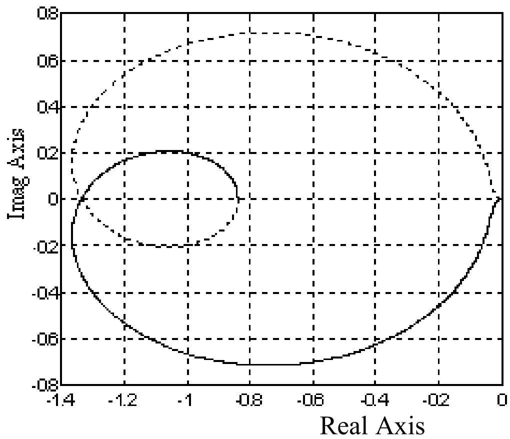

Fig.6. Nyquist plot of P(s)C(s).

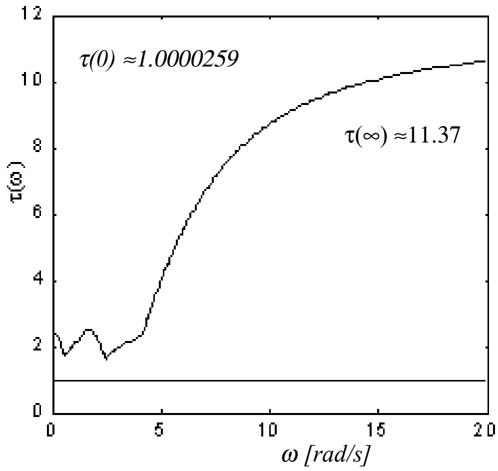

Fig.7. The distance function $\tau(\omega)$.

4.4. *Extended stability margins*

By using the *Algorithm* 1 proposed in Section 3 we have determined an extended stable box characterized by the following lower and upper limits (ρ_i^- and ρ_i^+):

$$\boldsymbol{\rho}^- \approx [14.41 \quad 14.41; \quad 99.99 \quad 98.20 \quad 8.79],$$

$$\boldsymbol{\rho}^+ \approx [19.18 \quad 8.79; \quad 98.47 \quad 14.41 \quad 14.41]$$

Then, by applying the *Algorithm* 2 we obtained several centred controllers with increased stability margins (see, Table 1).

The Nyquist plots for the centred controller obtained after iteration 4 together with that for the initial controller – given by (24) are presented in Fig. 8.

Table 1.

Iteration	1	2	3	4
ρ_{max}	11.57	14.83	16.92	18.22
k_{inf}	0.682	0.674	0.672	0.672
k_{sup}	1.262	1.348	1.408	1.446

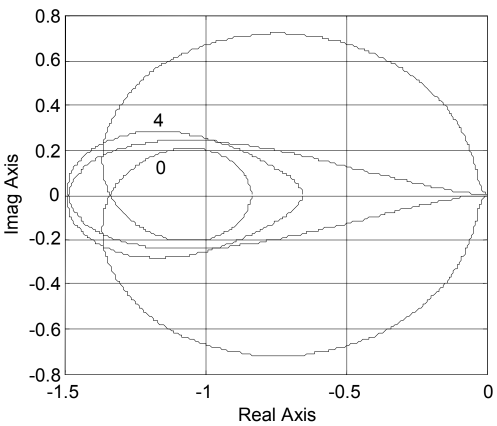

Fig.8. Nyquist plot of P(s)C(s).

5. CONCLUSIONS

In this paper we have proposed some reformulations of optimization problems such that to increase the parametric stability margin around the coefficients of the designed optimal controller. Also, we have presented some remarks about the parametric stability margin measures. So, we have showed that the normalized parametric stability radius is not suitable when the coefficients lie in very different ranges; in this case, more adequately is the maximum relative change accepted for controller coefficients to preserve stability. Also, we presented an algorithm to determine an extended range of robust stability. Finally, we proposed an algorithm to determine a centred controller with increased parametric stability margins. A possibility to maintain the stability/ performance robustness against the plant uncertainties is to use the free parameter from $\mathbf{H}_\infty$ optimization method in the controller design process.

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