Right Bounds for Eigenvalue Ranges of Interval Matrices – Estimation Principles vs Global Optimization

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Abstract: The paper develops a study on the evaluation of right bounds for the eigenvalue ranges of interval matrices. Given an arbitrary interval matrix $\mathcal{A}$, a right bound approximates the right end point of the eigenvalue range - defined as an exact value, denoted by $I(\mathcal{A})$, which, generally speaking, is not directly calculable. We consider two classes of methods providing right bounds: (*i*) $I(\mathcal{A})$ is approximated by a value $I_e(\mathcal{A})$ (with $I(\mathcal{A}) \le I_e(\mathcal{A})$), which is calculable from a mathematical expression, especially constructed as an estimation of $I(\mathcal{A})$ by majorization; (ii) $I(\mathcal{A})$ is approximated by a value $I_c(\mathcal{A})$ which is computable as the solution of a global optimization problem with constraints given by the interval coefficients of $\mathcal{A}$. For our study on right bounds, we use three estimation principles, based on different majorization approaches – corresponding to the class of methods (i), and a genetic-algorithm-based optimizer that masters non-smooth cost functions – corresponding to the class of methods (ii). The tests performed on a relevant collection of interval matrices (most of them selected from literature) yield a thorough comparative analysis revealing drawbacks and advantages equally unexpected at a first glance.

Keywords**:** interval matrices, (bounds of) eigenvalues, eigenvalue estimation by majorization, global optimization, genetic-algorithm-based optimization.

1. INTRODUCTION

The current paper considers interval matrices and interval systems defined as follows.

A family (set) of real square matrices

$$\mathcal{A} = [\boldsymbol{A}^-, \boldsymbol{A}^+] = \boldsymbol{A}^0 + [-\boldsymbol{R}, \boldsymbol{R}],\ \boldsymbol{A}^-, \boldsymbol{A}^+, \boldsymbol{A}^0, \boldsymbol{R} \in \mathbb{R}^{n\times n}, \quad (1)$$

where $\boldsymbol{A}^- \le \boldsymbol{A}^+$, $\boldsymbol{R} \ge 0$, are componentwise inequalities, is called an "*interval matrix*". The notation $\mathcal{A}$ preserves this meaning throughout the paper.

A continuous-time linear system with parameter uncertainties, of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t),\ \boldsymbol{A} \in \mathcal{A},\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0,\ t, t_0 \in \mathbb{R}_+,\ t \ge t_0, \quad (2)$$

is called an "*interval system*", "*interval matrix system*" or "*dynamical interval system*". The usage of an interval system (2) assumes that the entries of $\boldsymbol{A}$ are fixed (not time-varying), but the knowledge of their values is limited to intervals, instead of precise numbers.

For applications, the eigenvalue range of the interval matrix $\mathcal{A}$ (1) presents a great interest, since it creates an algebraic portrait corresponding to the different dynamics that may be exhibited by the interval system (2). In guaranteeing the stability of the interval system (2), the crucial role is played by the *right end-point* of the eigenvalue range of the interval matrix $\mathcal{A}$ (1), defined as

$$I(\mathcal{A}) = \max_{\boldsymbol{A}\in\mathcal{A}} \max_{k=1,\dots,n} \mathrm{Re}\{\lambda_k(\boldsymbol{A})\}, \quad (3)$$

where $\lambda_k(\boldsymbol{A})$, $k = 1,\dots,n$, denote the eigenvalues of $\boldsymbol{A}$. Obviously, for any interval dynamic system of form (2) with $I(\mathcal{A}) < 0$, the positive quantity $-I(\mathcal{A})$ represents the stability margin.

Taking the importance of the information offered by $I(\mathcal{A})$ into account, papers such as (Juang and Shao, 1989), (Wang and Lin, 1991), (Rohn, 1992), (Rohn, 1998), (Kolev and Petrakieva, 2005), (Leng *et al*., 2008), (Hladík *et al*., 2010), (Pastravanu and Matcovschi, 2010) (Matcovschi *et al*., 2010), proposed techniques for computing *approximations* of $I(\mathcal{A})$, since, generally speaking, $I(\mathcal{A})$ is not directly calculable.

Given an arbitrary interval matrix $\mathcal{A}$, any approximation of a $I(\mathcal{A})$ represents a *right bound of the eigenvalue range* of $\mathcal{A}$.

The objective of the current paper consists in developing a study on the evaluation of right bounds for the eigenvalue ranges of interval matrices with *general structure*. In this paper we do not discuss particular types of interval matrices (such as symmetric, skew-symmetric, etc) for which there exist specialized results that allow the exploration of the eigenvalue ranges.

We consider two classes of methods providing right bounds, as detailed below by (*i*) and (*ii*).

(*i*) $I(\mathcal{A})$ is approximated by a value $I_e(\mathcal{A})$ that satisfies the inequality

$$I(\mathcal{A}) \le I_e(\mathcal{A}), \qquad (4)$$

and is calculable from a mathematical expression, especially constructed as an estimation of $I(\mathcal{A})$ by majorization. Therefore $I_e(\mathcal{A})$ is a *right outer bound* of the eigenvalue range.

(ii) $I(\mathcal{A})$ is approximated by a value $I_c(\mathcal{A})$ which is computable as the solution of a global optimization problem with constraints given by the interval coefficients of $\mathcal{A}$.

Corresponding to the class of methods (*i*), our paper considers three *estimation principles* that result from three articles frequently cited in literature, namely (Rohn, 1998), (Kolev and Petrakieva, 2005), and (Hladík *et al*., 2010). These estimation principles provide different expressions for the calculation of a right outer bound $I_e(\mathcal{A})$, reason for which we use a supplementary subscript to specify the paternity. Thus, the estimations $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$ and $I_{e,HDT}(\mathcal{A})$ correspond, respectively, to the three articles mentioned above, preserving the citation order.

Corresponding to the class of methods (*ii*), our paper considers the use of the **ga** function from the Global Optimization Toolbox (The MathWorks. Inc. 2010a). This is a genetic-algorithm-based optimizer which is able to handle the nonsmooth cost function defined by the greatest real part of the eigenvalues of $\boldsymbol{A} \in \mathcal{A}$ and performs a global search for the extremum within the range defined by the interval-type coefficients. When referring to the approximation $I_c(\mathcal{A})$ computed via the **ga** solver, we include the information about the optimizer in our notations, by placing "ga" as a second subscript, i.e. $I_{c,ga}(\mathcal{A})$.

For a set of relevant interval matrices (most of them selected from literature), we compare the results provided by the estimation principles (i.e. $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$, $I_{e,HDT}(\mathcal{A})$) with the solutions of the global optimization approach (i.e. $I_{c,ga}(\mathcal{A})$). Both estimation principles and numerical optimization present advantages and drawbacks. Briefly speaking, the estimation principles rely on relatively simple mathematical expressions and the effects of the computational errors are rather low. However the degree of approximation introduced by these expressions (without any computational error) may be significant, meaning that $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$, $I_{e,HDT}(\mathcal{A})$ $I_e(\mathcal{A})$ are rough majorants of $I(\mathcal{A})$ (unknown). On the other hand, the global optimization may find precise values for the right end points $I(\mathcal{A})$ of many interval matrices, but, in general, the accuracy is highly dependent on the software performance. Under such circumstances, we consider that our construction (founded on estimation principles versus global optimization) is able to support a fruitful *comparative analysis.*

It is worth saying that we have found a strong motivation for this research in some comments formulated by paper (Hladík *et al*., 2010). The authors try to develop a comparison between their approach and Rohn's work, with respect to the bounds of real eigenvalues of interval matrices. Finally, the comparison cannot decide on a "winner", since Rohn's results are better for some examples, whereas, for other examples, the estimations obtained in accordance with (Hladík *et al*., 2010) are superior.

Unlike the comparative study in (Hladík *et al*., 2010), we focus on the right bounds (meaning both real and complex eigenvalues). The interest for the right bounds has already been explained above, as offering quantitative information about the stability margin in the case of Hurwitz stability investigation. In our analysis we also include the estimation principle proposed by (Kolev and Petrakieva, 2005), because the mathematical background is totally different from (Rohn, 1998) and (Hladík *et al*., 2010). Further details on these differences will be pointed out after some brief presentations of the three methods.

The remainder of the text is organized in seven sections playing the roles described below. Sections 2 - 4 offer brief overviews of the estimation principles of the right outer bounds $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$, $I_{e,HDT}(\mathcal{A})$, derived from the articles (Rohn, 1998), (Kolev and Petrakieva, 2005), (Hladík *et al*., 2010), respectively. Section 5 presents the numerical computation of $I_{c,ga}(\mathcal{A})$ via global optimization. Section 6 develops a comparative analysis of the results provided by the estimation principles ($I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$, $I_{e,HDT}(\mathcal{A})$) versus the solutions of the global optimization approach ($I_{c,ga}(\mathcal{A})$), for an illustrative set of interval matrices. Section 7 formulates some concluding remarks on the importance of our work.

2. ESTIMATION PRINCIPLE FOR $I_{e,R}(\mathcal{A})$

The procedure proposed in the paper (Rohn, 1998) refers to interval matrices with general structure. No additional assumption is requested for the use of this procedure (unlike the procedure presented by the following section, whose applicability is restricted by the fulfillment of some specific assumptions).

For any matrix $\boldsymbol{A} \in \mathcal{A}$, $\boldsymbol{A} = \boldsymbol{A}^0 + \boldsymbol{U}$, with $-\boldsymbol{R} \le \boldsymbol{U} \le \boldsymbol{R}$, the real parts of the eigenvalues are upper bounded as shown below:

$$\begin{aligned}&\mathrm{Re}\{\lambda_k(\boldsymbol{A})\} \le \max_{\|\boldsymbol{x}\|_2=1}\left(\tfrac{1}{2}\boldsymbol{x}^T[\boldsymbol{A}+\boldsymbol{A}^T]\boldsymbol{x}\right)\\ &= \max_{\|\boldsymbol{x}\|_2=1}\left(\tfrac{1}{2}\boldsymbol{x}^T[(\boldsymbol{A}^0+\boldsymbol{U})+((\boldsymbol{A}^0)^T+\boldsymbol{U}^T)]\boldsymbol{x}\right)\\ &\le \max_{\|\boldsymbol{x}\|_2=1}\left(\tfrac{1}{2}\boldsymbol{x}^T[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]\boldsymbol{x}\right)+\\ &\quad+\max_{\|\boldsymbol{x}\|_2=1}\left(\tfrac{1}{2}\boldsymbol{x}^T[\boldsymbol{U}+\boldsymbol{U}^T]\boldsymbol{x}\right),\quad k=1,\dots,n.\end{aligned} \tag{5}$$

Since the matrix $\boldsymbol{U}$ satisfies the componentwise inequality $|\boldsymbol{U}|\le\boldsymbol{R}$, we can write

$$\max_{\|\boldsymbol{x}\|_2=1}\left(\tfrac{1}{2}\boldsymbol{x}^T[\boldsymbol{U}+\boldsymbol{U}^T]\boldsymbol{x}\right)\le\max_{\|\boldsymbol{x}\|_2=1}\left(\tfrac{1}{2}\boldsymbol{x}^T[\boldsymbol{R}+\boldsymbol{R}^T]\boldsymbol{x}\right). \tag{6}$$

Thus we get the majorization

$$\begin{aligned}&\mathrm{Re}\{\lambda_k(\boldsymbol{A})\} \le \max_{\|\boldsymbol{x}\|_2=1}\left(\boldsymbol{x}^T\tfrac{1}{2}[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]\boldsymbol{x}\right)+\\ &\quad+\max_{\|\boldsymbol{x}\|_2=1}\left(\boldsymbol{x}^T\tfrac{1}{2}[\boldsymbol{R}+\boldsymbol{R}^T]\boldsymbol{x}\right),\quad k=1,\dots,n.\end{aligned} \tag{7}$$

Denote by $\lambda_{\max}\left(\tfrac{1}{2}[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]\right)$ and $\lambda_{\max}\left(\tfrac{1}{2}[\boldsymbol{R}+\boldsymbol{R}^T]\right)$ the greatest eigenvalue of the symmetrical matrix $\tfrac{1}{2}[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]$ and, respectively, $\tfrac{1}{2}[\boldsymbol{R}+\boldsymbol{R}^T]$. In accordance with the Courant-Fischer theorem, these two notations have the following meaning:

$$\begin{aligned}&\lambda_{\max}\left(\tfrac{1}{2}[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]\right)=\max_{\|\boldsymbol{x}\|_2=1}\left(\boldsymbol{x}^T\tfrac{1}{2}[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]\boldsymbol{x}\right),\\ &\lambda_{\max}\left(\tfrac{1}{2}[\boldsymbol{R}+\boldsymbol{R}^T]\right)=\max_{\|\boldsymbol{x}\|_2=1}\left(\boldsymbol{x}^T\tfrac{1}{2}[\boldsymbol{R}+\boldsymbol{R}^T]\boldsymbol{x}\right).\end{aligned} \tag{8}$$

Thus, relying on paper (Rohn, 1998), we have:

Theorem 1.
The value

$$I_{e,R}(\mathcal{A})=\lambda_{\max}\left(\tfrac{1}{2}[\boldsymbol{A}^0+(\boldsymbol{A}^0)^T]\right)+\lambda_{\max}\left(\tfrac{1}{2}[\boldsymbol{R}+\boldsymbol{R}^T]\right) \tag{9}$$

is a right outer bound of the eigenvalue range of the interval matrix $\mathcal{A}$ defined by (1). ■

3. ESTIMATION PRINCIPLE FOR $I_{e,KP}(\mathcal{A})$

The procedure in (Kolev and Petrakieva, 2005) relies on two assumptions, which limit the range of applicability as commented below.

Assumption 1.
Any matrix $\boldsymbol{A}$ belonging to the interval matrix $\mathcal{A}$ (1) has a real (simple or multiple) eigenvalue, denoted by $\lambda_{\max}(\boldsymbol{A})$, that dominates the spectrum of $\boldsymbol{A}$, i.e.

$$\mathrm{Re}\{\lambda_k(\boldsymbol{A})\}\le\lambda_{\max}(\boldsymbol{A}),\ k=1,\dots,n. \quad ■ \tag{10}$$

The second assumption will be presented after the following paragraph that introduces some notations, with the same meaning as in (Kolev and Petrakieva, 2005).

The procedures starts with the computation of the pair $(\lambda^0,\boldsymbol{x}^0)$, where $\lambda^0=\lambda_{\max}(\boldsymbol{A}^0)$ is the dominant eigenvalue of $\boldsymbol{A}^0$, and $\boldsymbol{x}^0$ is its associated eigenvector, i.e. $\boldsymbol{A}^0\boldsymbol{x}^0=\lambda^0\boldsymbol{x}^0$. The eigenvector $\boldsymbol{x}^0=\left[x_1^0\dots x_n^0\right]^T$ is taken normalized (scaled), such that one of its components is 1. Consider the last component normalized, i.e. $x_n^0=1$.

Assumption 2.
The normalization of the last component applies to each eigenvector $\boldsymbol{x}(\boldsymbol{A})\in\mathbb{R}^n$ associated with $\lambda_{\max}(\boldsymbol{A})$, for all $\boldsymbol{A}\in\mathcal{A}$. ■

For arbitrary $\boldsymbol{A}\in\mathcal{A}$, $\boldsymbol{A}=\left[a_{ij}\right]_{i,j=\overline{1,n}}$, introduce the vector $\boldsymbol{y}=[y_1\dots y_n]^T\in\mathbb{R}^n$, with $y_i=x_i(\boldsymbol{A})$, $i=1,\dots,n-1$, and $y_n=\lambda_{\max}(\boldsymbol{A})$. Thus, the equality

$$\boldsymbol{A}\cdot\boldsymbol{x}(\boldsymbol{A})=\lambda_{\max}(\boldsymbol{A})\cdot\boldsymbol{x}(\boldsymbol{A}), \tag{11}$$

which defines $\lambda_{\max}(\boldsymbol{A})$ and its associated eigenvector $\boldsymbol{x}(\boldsymbol{A})$, can be written as

$$\begin{aligned}&\sum_{j=1}^{n-1}a_{ij}y_j-y_ny_i+a_{in}=0,\ \ i=1,\dots,n-1,\\ &\sum_{j=1}^{n-1}a_{nj}y_j-y_n+a_{nn}=0,\end{aligned} \tag{12}$$

where

$$a_{ij}=a_{ij}^0+u_{ij},\ -R_{ij}\le u_{ij}\le R_{ij},\ \ i,j=1,\dots,n, \tag{13}$$

and

$$\begin{aligned}&y_i=y_i^0+v_i,\ \ v_i^-\le v_i\le v_i^+,\ \ i=1,\dots,n,\\ &y_1^0=x_1^0/x_n^0,\ \dots,\ y_{n-1}^0=x_{n-1}^0/x_n^0,\ \ y_n^0=\lambda^0.\end{aligned} \tag{14}$$

In (13), the values a_{ij}^0, R_{ij}, $i,j=1,\dots,n$, are known as the entries of the constant matrices $\boldsymbol{A}^0,\boldsymbol{R}\in\mathbb{R}^{n\times n}$ used in defining the interval matrix (1). In (14), the values y_i^0, $i=1,\dots,n$, are known (as explained by the second row), but the margins v_i^-, v_i^+, $i=1,\dots,n$, are unknown.

Now, it is obvious that the knowledge of the precise value for v_n^+ would allow the computation of the right end point of the eigenvalue range as

$$I(\mathcal{A})=y_n^0+v_n^+=\lambda^0+v_n^+. \tag{15}$$

Paper (Kolev and Petrakieva, 2005) aims to find a majorant V_n^+ of v_n^+ ($V_n^+ \geq v_n^+$) such that the value $y_n^0 + V_n^+$ represents an outer estimation of $I(\mathcal{A})$.

Consider the matrix $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ and the vector $\boldsymbol{v} \in \mathbb{R}^n$, with the elements u_{ij}, $i, j = 1,\ldots,n$, and v_i, $i = 1,\ldots,n$. Then system (12) where a_{ij}, y_i are explicitly written by the help of (13) and, respectively (14), becomes a nonlinear system with the compact form

$$\tilde{\boldsymbol{A}}^0 \boldsymbol{v} = \boldsymbol{b}(\boldsymbol{U},\boldsymbol{v}). \tag{16}$$

In (16), the elements of the vector $\boldsymbol{v} \in \mathbb{R}^n$ are variables (unknowns), the elements of matrix $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ are constrained parameters

$$-R_{ij} \leq u_{ij} \leq R_{ij},\ i, j = 1,\ldots,n, \tag{17}$$

and $\boldsymbol{b}(\boldsymbol{U},\boldsymbol{v}) : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function. $\tilde{\boldsymbol{A}}^0 \in \mathbb{R}^{n\times n}$ is a constant matrix whose elements are

$$\tilde{a}_{ij}^0 = \begin{cases} a_{ij}^0, \text{ for } i = 1,...,n,\ j = 1,...,n-1,\ j \neq i, \\ a_{ii}^0 - y_n^0, \text{ for } i, j = 1,...,n-1,\ j = i, \\ -y_i^0, \text{ for } i = 1,...,n-1,\ j = n, \\ -1, \text{ for } i = j = n. \end{cases} \tag{18}$$

If matrix $\tilde{\boldsymbol{A}}_0$ is nonsingular, system (16) is equivalent to the system $\boldsymbol{v} = (\tilde{\boldsymbol{A}}_0)^{-1}\boldsymbol{b}(\boldsymbol{U},\boldsymbol{v})$. Let $\boldsymbol{r} = [r_1 \ldots r_n]^T \in \mathbb{R}^n$, $r_i > 0$, $i = 1,\ldots,n$, be an arbitrary positive vector. If $-\boldsymbol{r} \leq \boldsymbol{v} \leq \boldsymbol{r}$, then the components of the vector $(\tilde{\boldsymbol{A}}_0)^{-1}\boldsymbol{b}(\boldsymbol{U},\boldsymbol{v}) = \Theta = [\theta_1 \ldots \theta_n]^T \in \mathbb{R}^n$ have their absolute values majorized by:

$$|\,\theta_i\,| \leq \delta_i + \sum_{j=1}^{n-1} d_{ij} r_j + r_n \sum_{j=1}^{n-1} c_{ij} r_j,\ i = 1,\ldots n. \tag{19}$$

In inequalities (19), the coefficients δ_i, d_{ij}, c_{ij}, $i, j = 1,\ldots,n$, have non-negative values that are calculated from the known values a_{ij}^0, R_{ij}, $i, j = 1,\ldots,n$, and y_i^0, $i = 1,\ldots n$, in accordance with relations (15) – (19) in (Kolev and Petrakieva, 2005).

Majorizations (19) hold for any values of the parameters u_{ij} satisfying constraints (17). Starting from (19), let us constructs the nonlinear system

$$r_i = \delta_i + \sum_{j=1}^{n-1} d_{ij} r_j + r_n \sum_{j=1}^{n-1} c_{ij} r_j,\ i = 1,\ldots n, \tag{20}$$

which has the same right-hand side as (19) and r_i, $i = 1,\ldots,n$, are unknowns. The vector form of system (20) is

$$\boldsymbol{r} = \boldsymbol{\delta} + \boldsymbol{D}\,\boldsymbol{r} + \boldsymbol{C}\,\boldsymbol{g}(\boldsymbol{r}) \tag{21}$$

with

$$\begin{aligned} &\boldsymbol{C} = \left[c_{ij}\right]_{i,j=\overline{1,n}} = |\,(\tilde{\boldsymbol{A}}^0)^{-1}\,|, \\ &\boldsymbol{\delta} = [\delta_1 .. \delta_n]^T = \boldsymbol{C}\boldsymbol{R}\,|\,\boldsymbol{x}^0\,|, \\ &\boldsymbol{D} = \left[d_{ij}\right]_{i,j=\overline{1,n}} = \boldsymbol{C}\breve{\boldsymbol{R}}, \end{aligned} \tag{22}$$

where matrix $\breve{\boldsymbol{R}}$ is the same as $\boldsymbol{R}$ except for the last column whose elements are zeros. The components of the nonlinear function $\boldsymbol{g}(\boldsymbol{r})$ are $g_i(\boldsymbol{r}) = r_i r_n$, $i = 1,...,n-1$, and $g_n(\boldsymbol{r}) = 0$.

If $\boldsymbol{r} \in \mathbb{R}^n$, $\boldsymbol{r} > 0$, is a positive solution of system (21), then the symmetrical rectangular set $-\boldsymbol{r} \leq \boldsymbol{v} \leq \boldsymbol{r}$ permits the usage of *Brouwer's fixed point theorem* for the function $\boldsymbol{f_U}(\boldsymbol{v}) : [-\boldsymbol{r},\boldsymbol{r}] \to [-\boldsymbol{r},\boldsymbol{r}]$, $\boldsymbol{f_U}(\boldsymbol{v}) = (\tilde{\boldsymbol{A}}_0)^{-1}\boldsymbol{b}(\boldsymbol{U},\boldsymbol{v})$, for any values of the parameters u_{ij} satisfying constraints (17). Thus, all the solutions $\boldsymbol{v} \in \mathbb{R}^n$ of system (12) (equivalent to the system $\boldsymbol{v} = (\tilde{\boldsymbol{A}}_0)^{-1}\boldsymbol{b}(\boldsymbol{U},\boldsymbol{v})$) are fixed points of $\boldsymbol{f_U}(\boldsymbol{v})$.

Thus, relying on paper (Kolev and Petrakieva, 2005), we have:

Theorem 2.
If Assumptions 1, 2 are satisfied, and the nonlinear system (21) has a positive solution $\boldsymbol{r} = [r_1 \ldots r_n]^T$, $r_i > 0$, $i = 1,\ldots,n$, then the value

$$I_{e,KP}(\mathcal{A}) = y_n^0 + r_n = \lambda^0 + r_n \tag{23}$$

is a right outer bound of the eigenvalue range of the interval matrix $\mathcal{A}$ defined by (1). ■

4. ESTIMATION PRINCIPLE FOR $I_{e,HDT}(\mathcal{A})$

From the results presented in (Hladík *et al.*, 2010) we derive a procedure for calculating the right outer bound of the eigenvalues of an interval matrix with general form. An arbitrary matrix belonging to the interval $\boldsymbol{A} \in \mathcal{A}$ is regarded as $\boldsymbol{A}^0$ perturbed by an additive perturbation $\boldsymbol{U} = \boldsymbol{A} - \boldsymbol{A}^0$. This perturbation satisfies the componentwise inequality $|\boldsymbol{U}| \leq \boldsymbol{R}$, which implies $\|\boldsymbol{U}\|_2 \leq \sigma_{\max}(\boldsymbol{R})$, where $\sigma_{\max}(\boldsymbol{R}) = \|\boldsymbol{R}\|_2$ denotes the greatest singular value of the matrix $\boldsymbol{R}$.

We first consider the case when the matrix $\boldsymbol{A}^0$ is diagonalizable on $\mathbb{C}$, i.e. there exists $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ such that

$$\boldsymbol{V}^{-1}\boldsymbol{A}^0\boldsymbol{V} = \mathrm{diag}\{\lambda_1(\boldsymbol{A}^0),\ldots,\lambda_n(\boldsymbol{A}^0)\}. \tag{24}$$

Denote by $\kappa_2(\boldsymbol{V}) = \|\boldsymbol{V}^{-1}\|_2\,\|\boldsymbol{V}\|_2$ the condition number with respect to the quadratic norm of the matrix $\boldsymbol{V} \in \mathbb{C}^{n\times n}$. In accordance with Proposition 2.2 and 2.3 in (Hladík *et al.*, 2010), for every eigenvalue of $\boldsymbol{A} \in \mathcal{A}$, generically denoted as $\lambda(\boldsymbol{A})$, there exists a subscript $k = 1,\dots,n$, such that

$$|\lambda(\boldsymbol{A}) - \lambda_k(\boldsymbol{A}^0)| \le \kappa_2(\boldsymbol{V})\, \|\boldsymbol{A} - \boldsymbol{A}^0\|_2 \le \kappa_2(\boldsymbol{V})\, \sigma_{\max}(\boldsymbol{R}). \quad (25)$$

If the matrix $\boldsymbol{A}^0$ is not diagonalizable on $\mathbb{C}$, we consider its Jordan canonical form $\boldsymbol{J}$, i.e. there exists $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ satisfying

$$\boldsymbol{V}^{-1}\boldsymbol{A}^0\boldsymbol{V} = \boldsymbol{J}. \quad (26)$$

Let p be the maximal dimension of the Jordan's blocks in $\boldsymbol{J}$. Introduce the following two positive quantities

$$\Theta_2 = \sqrt{\frac{p(p+1)}{2}}\, \kappa_2(\boldsymbol{V})\, \sigma_{\max}(\boldsymbol{R}), \qquad \Theta = \max\left\{\Theta_2, \Theta_2^{1/p}\right\}. \quad (27)$$

In accordance with Proposition 2.4 and 2.5 in (Hladík *et al*., 2010), for every eigenvalue of $\boldsymbol{A} \in \mathcal{A}$, generically denoted as $\lambda(\boldsymbol{A})$, there exists a subscript $k = 1,\dots,n$, such that

$$|\lambda(\boldsymbol{A}) - \lambda_k(\boldsymbol{A}^0)| \le \Theta. \quad (28)$$

It is obvious that for $p = 1$ (i.e. $\boldsymbol{A}^0$ has a diagonal canonical form), inequality (28) is equivalent to inequality (25). In other words, inequality (28) with $p \ge 1$ incorporates inequality (25) as a particular case corresponding to $p \ge 1$.

Thus, relying on paper (Hladík *et al*., 2010), we have:

Theorem 3.
The value

$$I_{e,HDT}(\mathcal{A}) = \max_{k=1,\dots,n} \{\mathrm{Re}(\lambda_k(\boldsymbol{A}_0))\} + \Theta \quad (29)$$

is a right outer bound of the eigenvalue range of the interval matrix $\mathcal{A}$ defined by (1). ■

5. NONLINEAR OPTIMIZATION APPROACH AS AN ALTERNATIVE TO THE ESTIMATION PRINCIPLES

As already mentioned in the introductory section, even if we calculate the estimations $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$, $I_{e,HDT}(\mathcal{A})$ for a given interval matrix, the right end point $I(\mathcal{A})$ still remains unknown. For some interval matrices, the knowledge of one or several estimations can be sufficient for supporting an application (as for instance the negative value of an estimation guarantees the Hurwitz stability). Generally speaking, as resulting from Sections 2 – 4, the mathematical expressions of $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$, $I_{e,HDT}(\mathcal{A})$ are obtained as majorizations of the unknown quantity $I(\mathcal{A})$. Therefore we propose the numerical computation of $I(\mathcal{A})$ as an alternative to the values calculable by the aforementioned methods.

The computational approach to the right end point $I(\mathcal{A})$ can be stated as a global optimization problem that maximizes the cost function

$$J(\boldsymbol{U}) = \max_{k=1,\dots,n} \mathrm{Re}\{\lambda_k(\boldsymbol{A}^0 + \boldsymbol{U})\}, \quad (30)$$

with $\boldsymbol{U} = \left[u_{ij}\right]$, $i, j = 1,\dots,n$, subject to the interval-type constraints (17). The implementation needs a solver able to cope with the nonsmooth dependence of the cost function (30) on the variables $u_{ij} \in \mathbb{R}$, $i, j = 1,\dots,n$. This requirement is not satisfied by most of the numerical software packages that provide high quality optimization routines relying on standard / conventional minimization algorithms. Therefore we oriented towards genetic-algorithm-based optimizers that, at least at the theoretical level, are insensitive with respect to the discontinuities of the derivatives. Moreover the theoretical support of this class of algorithms also ensures a global search of the extremum. This point of view was also encouraged by the following remark of D. Hertz in (Hertz, 2009) "Genetic algorithms are promising for solving such type of problems.".

Our final choice was the function **ga** from the Global Optimization Toolbox for MATLAB (The MathWorks. Inc. 2010a), whose most recent version (namely 3.1) was released in 2010. This choice ensures full compatibility (in the sense of computation accuracy) with the calculation of the estimations $I_{e,R}(\mathcal{A})$, $I_{e,KP}(\mathcal{A})$ and $I_{e,HDT}(\mathcal{A})$, whose mathematical expressions were implemented in MATLAB too. In the introductory section, we proposed the notation $I_{c,ga}(\mathcal{A})$ for the approximation of $I(\mathcal{A})$ obtained by using the **ga** solver.

6. COMMENTED EXAMPLES - COMPARATIVE ANALYSIS

The current section aims to create a relevant comparative analysis of the right bounds calculated by the two classes of methods previously discussed. Most of the tested interval matrices have been selected from publications with similar research topics. We have also devised our own tests for addressing aspects insufficiently investigated by other works.

The first subsection briefly presents the software tools used for developing the proposed comparative study, the second subsection focuses on illustrative tests and the third discusses the meaning of the numerical results.

6.1. Software tools

Besides the approach to global optimization via the **ga** solver (commented in Section 5), we have to consider some supplementary *computational tasks*, requiring specialized tools, as detailed below.

(a) Test the validity of Assumptions 1, 2 associated with Theorem 2. We have developed an instrument with visual facilities that portraits the location of the eigenvalues in the complex plane for a large number of concrete matrices $\boldsymbol{A} \in \mathcal{A}$. For a given interval matrix $\mathcal{A} = [\boldsymbol{A}^-, \boldsymbol{A}^+]$ we randomly generate 100,000 matrices, uniformly distributed in $\mathcal{A}$ and we plot their eigenvalues. This graphical representation allows testing the validity of Assumption 1. If Assumption 1 is satisfied, the eigenvectors corresponding to the dominant eigenvalues of the randomly generated matrices are further used for testing Assumption 2.

(b) ***Find a reasonable approximation of*** $I(\mathcal{A})$ ***from the direct computation of the eigenvalues for a large number of concrete matrices randomly generated*** $\boldsymbol{A}_{rnd} \in \mathcal{A}$**.** The tool presented at **(a)** calculates the value

$$I_{c,rnd}(\mathcal{A}) = \max_{\boldsymbol{A}_{rnd} \in \mathcal{A}} \max_{k=1,\dots,n} \mathrm{Re}\{\lambda_k(\boldsymbol{A}_{rnd})\} \tag{31}$$

which represents a *right inner bound* of the eigenvalue range of $\mathcal{A}$, since the inequality

$$I_{c,rnd}(\mathcal{A}) \le I(\mathcal{A}) \tag{32}$$

is satisfied regardless of the concrete matrices randomly generated. For 100,000 matrices uniformly distributed in $\mathcal{A}$, the approximation $I_{c,rnd}(\mathcal{A})$ is accurate enough to draw the attention on a (rather unlikely) failure or poor result of the **ga** solver.

(c) ***Find the solution(s) to the nonlinear algebraic system*** (19) ***associated with Theorem 2***. We use the **fsolve** solver from the Optimization Toolbox for MATLAB (The MathWorks. Inc. 2010b), with adequate initial guesses.

(d) ***Investigate the diagonal / Jordan canonical form of the center matrix*** $\boldsymbol{A}^0$ ***before applying Theorem 3.*** Whenever matrix $\boldsymbol{A}^0$ is diagonalizable, we use the eigenvectors provided by the **eig** function from the MATLAB kernel. If $\boldsymbol{A}^0$ has eigenvalues with algebraic multiplicity $q \ge 2$ (e.g. Example 5, Subsection 6.2), we get the Jordan form by using the **jordan** function from the Symbolic Math Toolbox for MATLAB (The MathWorks. Inc. 2010c). Since the Jordan form of a numerical matrix is extremely sensitive to numerical errors, we have represented the elements of each matrix $\boldsymbol{A}^0$ by ratios of small integers.

6.2. Illustrative Tests

This subsection considers seven examples we have chosen as very relevant from the large set of tests performed during the research period. For each example, the interval matrix is labelled with the number of the example, i.e. $\mathcal{A}_k$ for $k = 1,\dots,7$. Thus, $\mathcal{A}_k$ is defined by the matrices $\boldsymbol{A}_k^0, \boldsymbol{R}_k \in \mathbb{R}^{n\times n}$ as shown in equation (1).

For each interval matrix $\mathcal{A}_k$, $k = 1,\dots,7$:

● the following information is provided: the portrait of the eigenvalues of $\mathcal{A}_k$ generated randomly by the tool described in 6.1(a) is presented in Figure *k*; the value of the eigenvalues of $\boldsymbol{A}^0$, which are also marked by the symbol "$\times$" in the associated graphical representation; the right inner bound $I_{c,rnd}(\mathcal{A}_k)$ calculated by the tool described in 6.1(b).

● the global optimization method (described in Section 5) is applied, yielding the value $I_{c,ga}(\mathcal{A}_k)$.

● the three estimation principles (described in Sections 2 - 5) are applied, yielding the right out bounds $I_{e,R}(\mathcal{A}_k)$, $I_{e,KP}(\mathcal{A}_k)$ (if the estimation principle works), $I_{e,HDT}(\mathcal{A}_k)$.

In each example we comment only on the usage of Theorems 2 and 3, since their applicability depends on the fulfillment of some conditions. In all tested examples there was no failure of the **ga** optimizer. The other numerical elements presented by our study (eigenvalues of $\boldsymbol{A}^0$; the portrait of the eigenvalues of $\mathcal{A}$ generated randomly by the tool described in 6.1(a); the right inner bound $I_{c,rnd}$ calculated by the tool described in 6.1(b); the right outer bound $I_{e,R}$ calculated by Theorem1) do not require discussions on the computational aspects.

Table 1 collects the values $I_{c,rnd}$, $I_{c,ga}$, $I_{e,R}$, $I_{e,KP}$ (if the estimation principle works), $I_{e,HDT}$ for all the considered examples. Each row of this table is associated with a tested interval matrix and contains a value written in bold placed in one of the three last columns; this value represents the best (i.e. the smallest) right outer bound provided by the estimation principles.

Example 1. (Kolev and Petrakieva, 2005, Example 1)

The first interval matrix we consider, denoted by $\mathcal{A}_1$, is given by (1) with

$$\boldsymbol{A}_1^0 = \begin{bmatrix} -3.8 & 1.6 \\ 0.6 & -4.2 \end{bmatrix},\ \boldsymbol{R}_1 = \begin{bmatrix} 0.17 & 0.17 \\ 0.17 & 0.17 \end{bmatrix}. \tag{33}$$

The eigenvalues of the center matrix $\boldsymbol{A}_1^0$ are $\lambda_1^0 = -3$ and $\lambda_2^0 = -5$. By computations we verify that Assumptions 1 and 2 in Theorem 2 are satisfied. The general form of the nonlinear system (21) yields the second order system

$$\begin{aligned} r_1 &= 0.765 + 0.255 r_1 + 0.5 r_1 r_2, \\ r_2 &= 0.357 + 0.119 r_1 + 0.3 r_1 r_2, \end{aligned} \tag{34}$$

that has only one pair of positive solutions, $r_1 = 0.741743$ and $r_2 = 0.572708$. According to Theorem 2, we estimate the outer bound of the eigenvalue range of the interval matrix $\mathcal{A}_1$ as $I_{e,KP}(\mathcal{A}_1) = \lambda_1(\boldsymbol{A}_1^0) + r_2 = -2.427292$.

Since matrix $\boldsymbol{A}_1^0$ in (33) is diagonalizable, for computing the estimation $I_{e,HDT}(\mathcal{A}_1)$ we apply Theorem 3 with $p = 1$.

Example 2. (Kolev and Petrakieva, 2005, Example 3)

The second interval matrix we consider, denoted by $\mathcal{A}_2$, is given by (1) with $\boldsymbol{A}_2^0, \boldsymbol{R}_2 \in \mathbb{R}^{8\times 8}$ having the values of the nonzero elements given in relations (35a) and (35b), respectively. The eigenvalues of randomly selected matrices from $\mathcal{A}_2$ are represented graphically in Figure 2(a) with a zoom on the region containing the dominant eigenvalues of $\mathcal{A}_2$ presented in Figure 2(b). The eigenvalues of the center matrix $\boldsymbol{A}_2^0$ given by (35a) are $\lambda_1^0 = -12.555999$, $\lambda_{2,3}^0 = -21.991 \pm j469.27$, $\lambda_{4,5}^0 = -69.115 \pm j\,13823$, $\lambda_6^0 = -564.94$ and $\lambda_{7,8}^0 = -1256.6 \pm j25103$.

By computations we verify that Assumptions 1 and 2 in Theorem 2 are satisfied. The general form of the nonlinear system (21) yields the system

$$
\begin{aligned}
r_1 &= 0.0501r_1 + 0.0191r_2 + 1.628\cdot 10^{-4} r_1 r_8 \\
&\quad +22.783\cdot 10^{-4} r_2 r_8 \\
r_2 &= 0.0014r_1 + 0.0505r_2 + 22.783\cdot 10^{-4} r_1 r_8 \\
&\quad +0.65\cdot 10^{-4} r_2 r_8 \\
r_3 &= 0.1r_3 + 0.0063 1r_4 + 0.658\cdot 10^{-6} r_3 r_8 \\
&\quad +0.723\cdot 10^{-4} r_4 r_8 \\
r_4 &= 0.9083\cdot 10^{-4} r_3 + 0.1r_4 + 0.723\cdot 10^{-4} r_3 r_8 \\
&\quad +0.658\cdot 10^{-7} r_4 r_8 \\
r_5 &= 0.05r_5 + 0.052r_6 + 0.396\cdot 10^{-5} r_5 r_8 \\
&\quad +0.398\cdot 10^{-4} r_6 r_8 \\
r_6 &= 0.25\cdot 10^{-4} r_5 + 0.05r_6 + 0.398\cdot 10^{-4} r_5 r_8 \\
&\quad +0.199\cdot 10^{-7} r_6 r_8 \\
r_7 &= 0.588r_7 + 0.0018r_7 r_8 \\
r_8 &= 1.353 + 0.011r_1 + 0.071r_2 + 2.09\cdot 10^{-4} r_3 + 0.338r_4 \\
&\quad +20.643\cdot 10^{-4} r_5 + 11.582r_6 + 12.784r_7 + 0.0013r_1 r_8 \\
&\quad +0.4\cdot 10^{-4} r_2 r_8 + 1.665\cdot 10^{-4} r_3 r_8 + 0.151\cdot 10^{-6} r_4 r_8 \\
&\quad +0.0033r_5 r_8 + 0.164\cdot 10^{-5} r_6 r_8 + 0.0297r_7 r_8.
\end{aligned} \tag{36}
$$

System (34) has positive solutions with $r_8 = 1.35300$. According to Theorem 2, we estimate the outer bound of the eigenvalue range of the interval matrix $\mathcal{A}_2$ as $I_{e,KP}(\mathcal{A}_2) = \lambda_1(\boldsymbol{A}_2^0) + r_8 = -11.202999$.

Since matrix $\boldsymbol{A}_2^0$ is diagonalizable, for computing the estimation $I_{e,HDT}(\mathcal{A}_2)$ we apply Theorem 3 with $p = 1$.

Example 3. (Hladik et al.*, 2010, Example 2.7)*

The third interval matrix we consider, denoted by $\mathcal{A}_3$, is given by (1) with

$$
\boldsymbol{A}_3^0 = \begin{bmatrix} -4.5 & -8.5 & 14.5 & 4.8 & -1.1 \\ 17.5 & 17.5 & 1.5 & 4.5 & 10.5 \\ 17.1 & -3.1 & 2.0 & 12.5 & 6.2 \\ 18.5 & 2.5 & 18.5 & 5.5 & 6.5 \\ 13.5 & 18.5 & 9.5 & -17.5 & 10.5 \end{bmatrix},\quad
\boldsymbol{R}_3 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.2 & 0.1 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.1 & 0.4 & 0.1 & 0.5 & 0.2 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}. \tag{37}
$$

The eigenvalues of the center matrix $\boldsymbol{A}_3^0$ in (37) are $\lambda_1^0 = 20.7214$, $\lambda_{2,3}^0 = 15.1215 \pm j15.9556$, $\lambda_4^0 = -4.0671$ and $\lambda_5^0 = -15.8973$.

By computations we verify that Assumptions 1 and 2 in Theorem 2 are satisfied. The general form of the nonlinear system (21) yields the system

$$
\begin{aligned}
r_1 &= 0.1017+ \\
&\quad +0.0455r_1 + 0.0493r_2 + 0.0455r_3 + 0.0399r_4 \\
&\quad +0.0355r_1 r_5 + 0.0077r_2 r_5 + 0.0126r_3 r_5 + 0.0249r_4 r_5 \\
r_2 &= 0.1542+ \\
&\quad +0.0659r_1 + 0.0814r_2 + 0.0659r_3 + 0.0670r_4 \\
&\quad +0.0654r_1 r_5 + 0.0486r_2 r_5 + 0.0516r_3 r_5 + 0.0006r_4 r_5 \\
r_3 &= 0.0646+ \\
&\quad +0.0239r_1 + 0.0307r_2 + 0.0239r_3 + 0.0310r_4 \\
&\quad +0.0068r_1 r_5 + 0.0066r_2 r_5 + 0.0230r_3 r_5 + 0.0266r_4 r_5 \\
r_4 &= 0.1186+ \\
&\quad +0.0484r_1 + 0.0646r_2 + 0.0484r_3 + 0.0550r_4 \\
&\quad +0.0500r_1 r_5 + 0.0047r_2 r_5 + 0.0540r_3 r_5 + 0.0002r_4 r_5 \\
r_5 &= 3.3356+ \\
&\quad +1.4236r_1 + 1.5430r_2 + 1.4236r_3 + 1.3578r_4 \\
&\quad +0.7497r_1 r_5 + 0.8130r_2 r_5 + 0.3978r_3 r_5 + 0.5746r_4 r_5.
\end{aligned} \tag{38}
$$

$$
\begin{array}{llllll}
a_{12}^0 = 439.82 & a_{21}^0 = -439.82 & a_{22}^0 = -43.983 & a_{34}^0 = 13823 & a_{43}^0 = -13823 & \\
a_{44}^0 = -138.23 & a_{56}^0 = 25133 & a_{65}^0 = -25133 & a_{66}^0 = -2513.3 & a_{77}^0 = -565.49 & \\
a_{81}^0 = 0.00115 & a_{82}^0 = -0.5750 & a_{84}^0 = 2.3010 & a_{86}^0 = 82.637 & a_{87}^0 = 16.427 & a_{88}^0 = -12.556;
\end{array} \tag{35a}
$$

$$
\begin{array}{llllll}
r_{12}^0 = 21.99 & r_{21}^0 = 21.991 & r_{22}^0 = 6.8170 & r_{34}^0 = 1382.3 & r_{43}^0 = 1382.3 & \\
r_{44}^0 = 74.644 & r_{56}^0 = 1256.6 & r_{65}^0 = 1256.6 & r_{66}^0 = 1181.2 & r_{77}^0 = 325.16 & \\
r_{81}^0 = 2.449\cdot 10^{-4} & r_{82}^0 = 0.0419 & r_{84}^0 = 0.5680 & r_{86}^0 = 15.716 & r_{87}^0 = 3.1240 & r_{88}^0 = 1.3530.
\end{array} \tag{35b}
$$

System (38) has a positive solution with $r_5 = 5.708225$. According to Theorem 2, we estimate the outer bound of the eigenvalue range of the interval matrix $\mathcal{A}_3$ defined by (1) &(37) as $I_{e,KP}(\mathcal{A}_3) = \lambda_1(\boldsymbol{A}_3^0) + r_5 = 26.4296$.

Since matrix $\boldsymbol{A}_3^0$ is diagonalizable, for computing the estimation $I_{e,HDT}(\mathcal{A}_3)$ we apply Theorem 3 with $p = 1$.

Example 4. (Hladik et al., *2010, Example 2.8 for $\varepsilon = 0.01$)*

The fourth interval matrix we consider, denoted by $\mathcal{A}_4$, is given by (1) with

$$\boldsymbol{A}_4^0 = \begin{bmatrix} 4 & 6 & 13 & 1 \\ -4 & -5 & -16 & -4 \\ 1 & 2 & 6 & 1 \\ 0 & -2 & -10 & -1 \end{bmatrix}, \ \boldsymbol{R}_4 = 0.01 \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \quad (39)$$

The eigenvalues of the center matrix $\boldsymbol{A}_4^0$ from (39) are $\lambda_1^0 = \lambda_2^0 = 1 - 2j$ and $\lambda_3^0 = \lambda_4^0 = 1 + 2j$.

The analysis of Figure 4 leads to the conclusion that $\mathcal{A}_4$ does not satisfy Assumption 1, therefore Theorem 2 cannot be applied.

The Jordan form of matrix $\boldsymbol{A}_4^0$ consists of two 2×2 blocks, therefore for computing the estimation $I_{e,HDT}(\mathcal{A}_4)$ we apply Theorem 3 with $p = 2$.

Example 5. (Hladik et al., *2010, Example 2.8 for $\varepsilon = 1$)*

The fifth interval matrix we consider, denoted by $\mathcal{A}_5$, is given by (1) with

$$\boldsymbol{A}_5^0 = \boldsymbol{A}_4^0, \ \boldsymbol{R}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \quad (40)$$

The analysis of Figure 5 leads to the conclusion that $\mathcal{A}_5$ does not satisfy Assumption 1, therefore Theorem 2 cannot be applied.

Since the center matrix $\boldsymbol{A}_5^0$ coincides with $\boldsymbol{A}_4^0$, the same as in Example 4, for computing the estimation $I_{e,HDT}(\mathcal{A}_5)$ we apply Theorem 3 with $p = 2$.

Example 6. (Ahn and Chen, 2007)

The next interval matrix we consider, denoted by $\mathcal{A}_6$, is given by (1) with

$$\boldsymbol{A}_6^0 = \begin{bmatrix} 1.50 & -0.01 & 3.40 \\ 7.10 & -3.40 & -1.30 \\ 2.10 & 0.01 & -7.00 \end{bmatrix}, \boldsymbol{R}_6 = \begin{bmatrix} 0.075 & 0.005 & 1.70 \\ 3.550 & 1.700 & 0.65 \\ 1.050 & 0.005 & 3.50 \end{bmatrix}. (41)$$

The eigenvalues of the center matrix $\boldsymbol{A}_6^0$ are $\lambda_1^0 = 2.2632$, $\lambda_2^0 = -3.4031$ and $\lambda_3^0 = -7.7601$.

By computations we verify that Assumptions 1 and 2 in Theorem 2 are satisfied. The general form of the nonlinear system (21) yields the system

$$\begin{aligned} r_1 &= 3.7794 + 0.4691 r_1 + 0.0043 r_2 + 0.1006 r_1 r_3 \\ &\quad + 0.0009 r_2 r_3 \\ r_2 &= 9.5992 + 1.2787 r_1 + 0.3020 r_2 + 0.0698 r_1 r_3 \\ &\quad + 0.1756 r_2 r_3 \\ r_3 &= 1.0678 + 0.0979 r_1 + 0.0018 r_2 + 0.2106 r_1 r_3 \\ &\quad + 0.0002 r_2 r_3 \end{aligned} \quad (42)$$

Since no positive solution to (42) can be found, Theorem 2 cannot be applied.

Since matrix $\boldsymbol{A}_6^0$ is diagonalizable, for computing the estimation $I_{e,HDT}(\mathcal{A}_6)$ we apply Theorem 3 with $p = 1$.

Example 7.

The last interval matrix we consider, denoted by $\mathcal{A}_7$, is given by (1) with

$$\boldsymbol{A}_7^0 = \begin{bmatrix} -4.5 & 0 & 0 \\ 1.5 & -2 & 0 \\ 3 & 8 & -1.75 \end{bmatrix}, \ \boldsymbol{R}_7 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2 & 2 & 0.25 \end{bmatrix}. \quad (43)$$

The eigenvalues of the center matrix $\boldsymbol{A}_7^0$ are $\lambda_1^0 = -1.75$, $\lambda_2^0 = -2$ and $\lambda_3^0 = -4.5$.

Any matrix $\boldsymbol{A} = \left[a_{ij}\right]_{i,j=\overline{1,3}}$ in the interval matrix $\mathcal{A}_7$ is lower triangular, with the diagonal elements satisfying $a_{11} \in [-5,-4]$, $a_{22} \in [-3,-1]$ and $a_{33} \in [-2,-1.75]$. Since the eigenvalues of a triangular matrix coincide with its diagonal elements, in this particular example, we can find the exact value of the right end-point of the eigenvalue range of the interval matrix $\mathcal{A}_7$ as $I(\mathcal{A}_7) = -1$.

By computations we verify that Assumptions 1 and 2 in Theorem 2 are satisfied. The general form of the nonlinear system (21) yields the system

$$\begin{aligned} r_1 &= 0.1818 r_1 + 0.3636 r_1 r_3, \\ r_2 &= 3.0909 r_1 + 4 r_2 + 2.1818 r_1 r_3 + 4 r_2 r_3, \\ r_3 &= 0.25 + 18 r_1 + 22 r_2 + 12 r_1 r_3 + 20 r_2 r_3. \end{aligned} \quad (44)$$

Table 1. Right bounds of the test interval matrices considered in Section 6.

Interval matrix	$I_{c,rnd}(\mathcal{A})$	$I_{c,ga}(\mathcal{A})$	$I_{e,R}(\mathcal{A})$	$I_{e,KP}(\mathcal{A})$	$I_{e,HDT}(\mathcal{A})$
$\mathcal{A}_1$ (33)	–2.645559	–2.645729	**–2.541966**	–2.427292	–2.449868
$\mathcal{A}_2$ (35)	–11.203142	–11.201939	1.9790e+003	**–11.202999**	2.0685e+003
$\mathcal{A}_3$ (37)	22.840857	23.362817	35.499876	**26.429629**	29.310144
$\mathcal{A}_4$ (39)	1.259458	1.293722	13.844500	Ass. 1 not satisfied	**2.061208**
$\mathcal{A}_5$ (40)	8.450673	9.718839	**17.804500**	Ass. 1 not satisfied	107.120850
$\mathcal{A}_6$ (41)	3.703909	3. 779617	**8.157379**	no positive solution to (42)	18.552955
$\mathcal{A}_7$ (43)	$I(\mathcal{A}_7) = -1$	–1.000001	**3.204283**	no positive solution to (44)	140.7023

The only solution to system (44) that has nonnegative elements is $r_1 = r_2 = 0$, $r_3 = 0.25$. Since no solution with positive elements can be found for system (44), Theorem 2 cannot be applied.

6.3. Effectiveness of the approximation methods

For the class of methods estimating right outer bounds, we can see that the best approximation is given by Theorem 1 for four examples, by Theorem 2 for two examples and by Theorem 3 for an example. In other words, there exists no method definitely superior to the others. Generally speaking each method seems to be more effective for a type of problems, as summarized below: Theorem 1 for matrices $\boldsymbol{R}$ with large entries, Theorem 2 for the special cases when the whole eigenvalue range is globally dominated by a simple real eigenvalue, Theorem 3 for matrices $\boldsymbol{R}$ with small entries. The use of Theorem 2 is drastically limited by the two associated assumptions, as well as by the hypothesis requesting positive solutions for a nonlinear algebraic system.

For the approximation of the right end point of the eigenvalue range by global optimization, all our tests have shown that the **ga** function represented a good decision in choosing the optimization software. We have got no failure, and each reported solution was very close to the right inner bound calculated from a large set of randomly generated matrices.

Finally it is worth mentioning the important role played by our instrument built to visualize the eigenvalue portrait based on the randomly generated matrices. First, it was indispensable for testing Assumptions 1, 2 associated with Theorem 2. Then it was helpful in understanding some correlations between the eigenvalue location and the performance of the estimation methods. It also provided reference values for assessing the results of the global optimization.

CONCLUSIONS

Our paper explores two classes of methods that provide approximations for the right end point of the eigenvalue ranges of interval matrices. We have performed a large number of tests and the most relevant ones have been discussed in the previous section of this article.

Relying on these tests, we can say as an overall remark / recommendation:

- If the approximation is formulated / requested in the sense of the best right outer bound, then all three estimation principles have to be applied for selecting the minimum value.
- The most accurate approximation can be obtained via global optimization, by using the **ga** solver. Thus, running **ga** in mutual validation with the calculation of a right inner bound from randomly generated matrices may ensure a reliable computational approach.

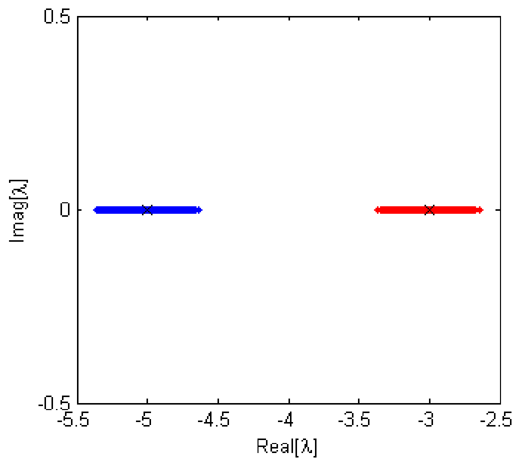
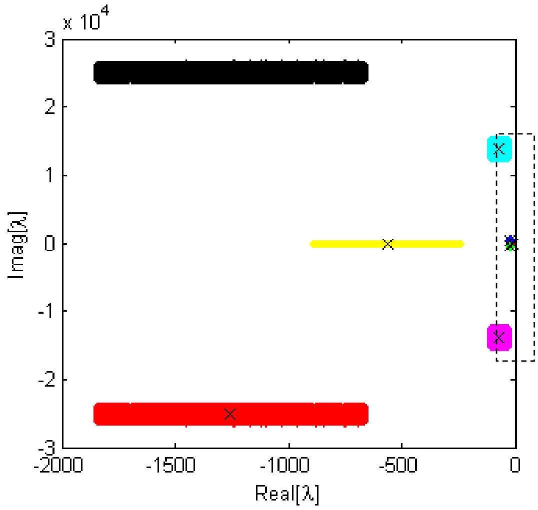


Figure 1. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_1$ given by (33).

(a) portrait of the eigenvalues



(b) zoom in the region containing the dominant eigenvalues.

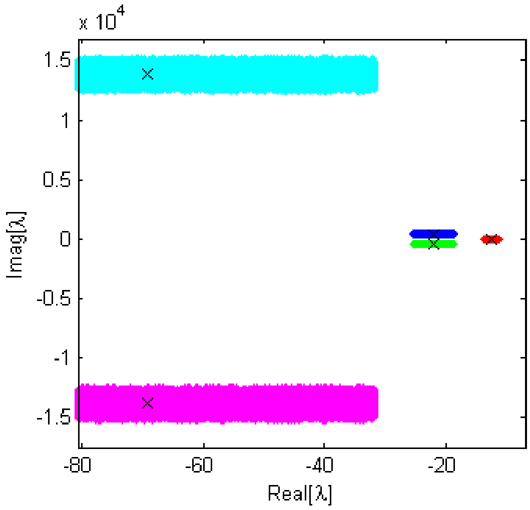


Figure 2. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_2$ given by (35).

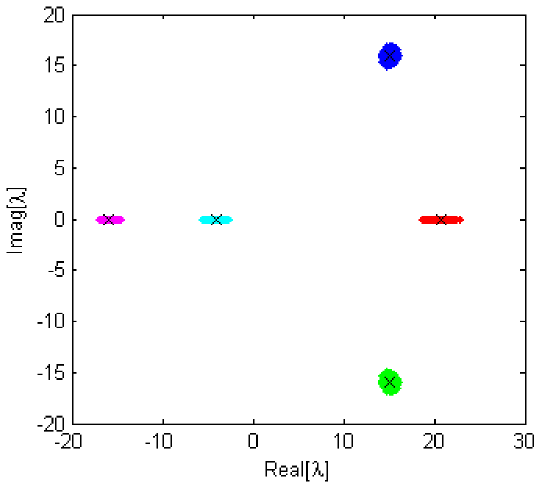


Figure 3. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_3$ given by (37).

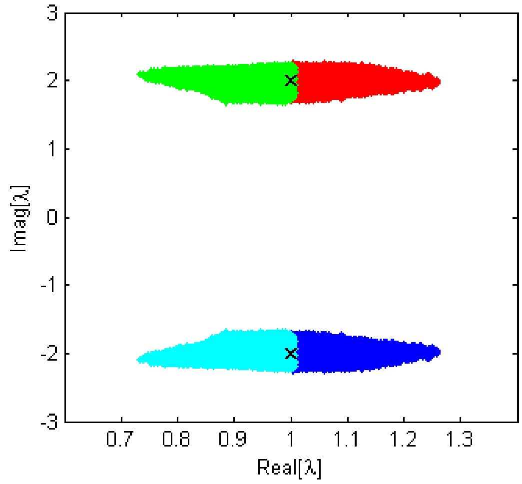


Figure 4. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_4$ given by (39).

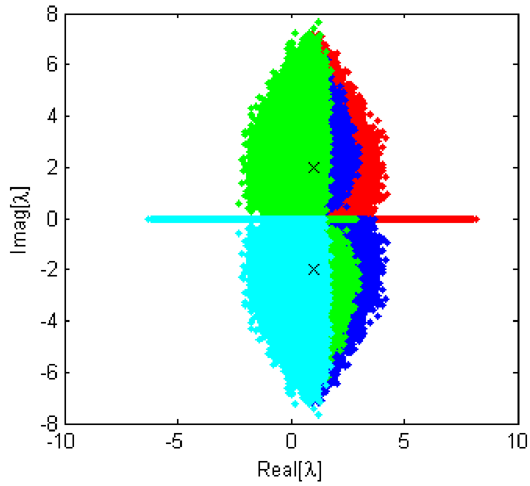


Figure 5. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_5$ given by (40).

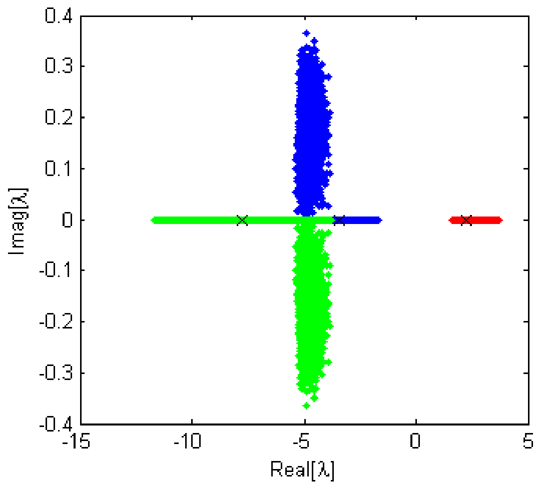


Figure 6. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_6$ given by (41).

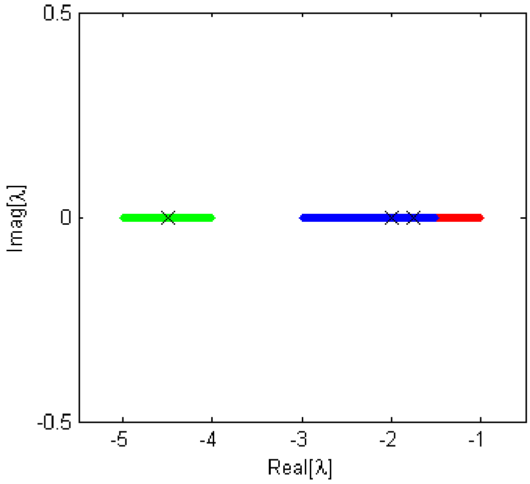


Figure 7. Eigenvalues of randomly selected matrices in the interval matrix $\mathcal{A}_7$ given by (43).

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