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Discrete Sliding-Mode Control of Inventory Systems with Deteriorating Stock and Remote Supply Source

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Abstract: In this paper we propose control-theoretic approach for the design of inventory management policy for systems with perishable goods. We consider periodic-review systems with uncertain, time-varying demand. The challenging issue in this type of systems is to achieve high service level with minimum costs for arbitrary demand pattern when the stock replenishment orders are procured with nonnegligible delay. In contrast to the classical, stochastic approaches, we employ a formal design methodology based on discrete sliding-mode (DSM) control. The proposed DSM controller with the sliding plane selected for a dead-beat scheme ensures the maximum service level (full demand satisfaction from the readily available resources) with smaller holding costs and reduced order-to-demand variance ratio as compared to the classical order-up-to policy.

Keywords: inventory control, sliding-mode control, perishable inventory systems, time-delay systems.

1. INTRODUCTION

An appropriate inventory management policy is crucial for efficient operation of production and logistic systems (Zipkin, 2000). Due to the similarity between the considered class of systems and engineering processes, it is a natural choice to apply control-theoretic methods in the design and analysis of strategies governing the flow of goods. However, it follows from the extensive review papers documenting the research work in the field performed in the last four decades (Nahmias, 1982; Rafaat, 1991, Goyal and Giri, 2001; Ortega and Lin, 2004; Sarimveis et al., 2008; Karaesmen et al., 2008) that certain areas of inventory control are not sufficiently addressed at the formal design level. The deficiency of application of systematic control approaches concerns in particular a large and very important class of problems related to the management of perishable commodities. Indeed, many products, such as food, drugs, gasoline, etc., lose market value over time, deteriorate due to the changes in the chemical structure or even become obsolete. The primary difficulty in developing control schemes for perishable inventories is the enlarged state space required for conducting the exact analysis of product lifetimes. The situation aggravates when the product demand varies rapidly in subsequent review periods and inventories are replenished with a nonnegligible delay, which frequently happens in modern supply chains. In such circumstances, in order to meet the service level constraints and at the same time keep stringent cost discipline, when placing an order it is necessary not only to account for the demand during procurement latency but also for the stock deterioration in that time.

There are very few successful design examples based on formal control methods for perishable inventory systems. In paper (Andijani and Al-Dajani, 1998), a linear-quadratic optimization is performed for an undelayed process. Rodrigues and Boukas (2006) developed a piecewise affine control law for a production system with deteriorating on-hand inventory and zero lead-time. On the other hand, for a continuous-time system with uncertain processing time and delay in control Boukas et al. (2000) proposed a robust controller obtained by minimizing an H_∞-norm. However, the implementation of the strategy presented in that paper requires numerical procedures for calculating the control law parameters, which limits its analytical tractability.

In this paper, we apply control-theoretic methodology to develop a new supply policy for periodic-review inventory systems with perishable goods. In the considered systems, the on-hand stock at a goods distribution center is used to fulfill unknown, time-varying demand placed by retailers (or customers). The stock deteriorates exponentially at a constant rate and is replenished with delay from a remote supply source. We assume that the lead-time delay can span multiple review periods. The design objective is to obtain high service level of the (unknown) customers' demand with minimum on-hand inventory. For this purpose, we propose discrete-time sliding-mode (DSM) control, which is well known to be efficient and robust regulation technique (Bandyopadhyay and Janardhanan, 2006; Milosavljević et al., 2006). Since the proper choice of the switching plane is the key part of the design of sliding-mode controllers (Bandyopadhyay et al., 2009, Bartoszewicz and Nowacka-Leverton, 2009, Ignaciuk and Bartoszewicz, 2011), in this work, we determine the plane parameters for a dead-beat scheme. In this way we obtain fast response to the changes in demand and the minimum stock level. In contrast to other solutions reported previously in the literature for perishable inventory systems, we adopt a formal design approach based in part on our previous results reported for the traditional inventory systems

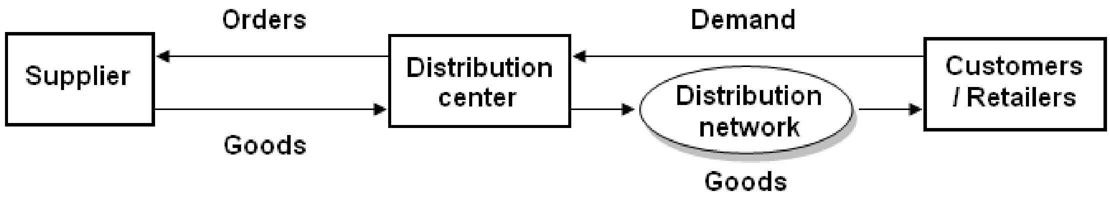

Fig. 1. Inventory system with a strategic supplier.

with nondeteriorating stock (Ignaciuk and Bartoszewicz, 2010a, b), and obtain the controller in a closed-form. The closed-form solution allows us to define a number of advantageous properties of the proposed control scheme. In particular, we show that under the proposed policy the available stock is never entirely depleted despite unpredictable demand variations, which guarantees full demand satisfaction and the maximum service level. We also specify a precise value of the storage space which should be reserved at the distribution center to always accommodate all the incoming shipments. This means that the potential necessity of expensive emergency storage outside the company premises is eliminated. Finally, we show that the order quantities generated by the presented controller are always nonnegative and bounded, which is required from the practical point of view.

The paper is organized as follows. First, in Section 2, the model of the considered inventory system with deteriorating stock and remote supply source is presented. Next, in Section 3, the design procedure is conducted, and properties of the obtained control law are discussed. The properties are illustrated in a numerical example described in Section 4. Finally, we provide conclusions in Section 5.

2. PROBLEM FORMULATION

We analyze the inventory system, faced by unknown, bounded, time-varying demand, in which the stock is replenished from a remote supply source. Such setting, illustrated in Fig. 1, is frequently encountered in production-inventory systems where a common point (distribution center), linked to a factory or external, strategic supplier, is used to provide goods for another production stage or a distribution network. The task is to design a stable control strategy which will minimize lost service opportunities (occurring when only part of the imposed demand can be satisfied from the stock available at the distribution center). The design procedure should explicitly consider the delay between placing of an order at the supplier and goods arrival at the center, and the stock reduction of perishable items during this lead-time delay.

2.1 System Model

The model of the analyzed periodic-review inventory system is illustrated in Fig. 2. The stock replenishment orders $u(\cdot)$ are issued at regular intervals kT, where T is the review period and $k = 0, 1, 2,...$. The order quantity is calculated on the basis of the current stock level $y(kT)$, the stock reference value y_{ref} and the orders history. Each non-zero order placed at the supplier is realized with lead-time $L_p > 0$ assumed to be a multiple of the review period, i.e. $L_p = n_pT$, where n_p is a positive integer. The saturating integrator in the internal loop represents the operation of accumulating the stock of perishable goods characterized by decay factor σ.

The imposed demand (the number of items requested from inventory in period k) is modeled as an *a priori* unknown, bounded function of time $d(kT)$,

$$0 \le d\left(kT\right) \le d_{\max}. \tag{1}$$

Notice that this definition of demand is quite general and it accounts for any standard distribution typically analyzed in the considered problem. If there is a sufficient number of items in the warehouse to satisfy the imposed demand, then the actually met demand $h(kT)$ (the number of items sold to the customers or sent to the retailers in the distribution network) will be equal to the requested one. Otherwise, the imposed demand is satisfied only from the arriving shipments, and the additional demand is lost (we assume that the sales are not backordered, and the excessive demand is equivalent to a missed business opportunity). Thus, we may write

$$0 \le h\left(kT\right) \le d\left(kT\right) \le d_{\max}. \tag{2}$$

For the considered system with perishable inventory the stock balance equation takes the following form

$$y\left[\left(k+1\right)T\right] = \rho y\left(kT\right) + u_R\left(kT\right) - h\left(kT\right), \tag{3}$$

where $u_R(kT)$ is the order received in period k and $\rho = 1 - \sigma$ represents the fraction of the stock which remains in the warehouse when inventory deteriorates at rate σ. For instance, if $\sigma = 0.05$, then 5% of the stock perishes in each review period and $\rho = 0.95$ or 95% of the stock remains. We assume that the warehouse is initially empty, i.e. $y(kT) = 0$ for $k < 0$, and the first order is placed at $kT = 0$. Consequently,

$$y\left(kT\right) = \sum_{j=0}^{k-1} \rho^{k-1-j} u_R\left(jT\right) - \sum_{j=0}^{k-1} \rho^{k-1-j} h\left(jT\right). \tag{4}$$

Due to the lead-time delay the first order arrives at the distribution center in period n_p, and $y(kT) = 0$ for $k \le n_p$. We assume that the goods arrive at the distribution center new and deteriorate while kept in the on-hand stock. Taking into account the initial condition $y(0) = 0$ and the fact that $u_R(kT) = u[(k - n_p)T]$, the stock level for any $k \ge 0$ may be calculated from the following equation

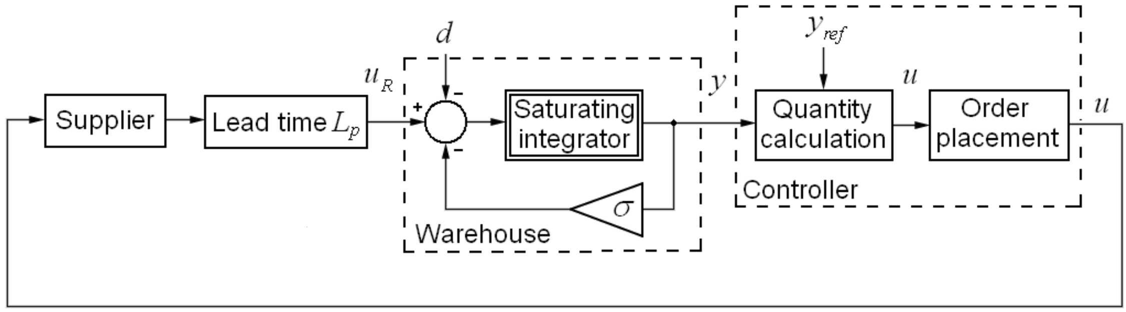

Fig. 2. System model.

$$
\begin{aligned}
y(kT) &= \sum_{j=0}^{k-1} \rho^{k-1-j} u\left[\left(j-n_p\right)T\right] - \sum_{j=0}^{k-1} \rho^{k-1-j} h(jT) \\
&= \sum_{j=-n_p}^{k-n_p-1} \rho^{k-n_p-1-j} u(jT) - \sum_{j=0}^{k-1} \rho^{k-1-j} h(jT).
\end{aligned} \tag{5}
$$

Moreover, since $u(kT) = 0$ for $k < 0$, we can write

$$
y(kT) = \sum_{j=0}^{k-n_p-1} \rho^{k-n_p-1-j} u(jT) - \sum_{j=0}^{k-1} \rho^{k-1-j} h(jT), \tag{6}
$$

which provides a closed-form expression for the on-hand stock level in the considered periodic-review inventory system with perishable goods. In order to save on notation in the remainder of the paper we will use k as the independent variable in place of kT.

2.2 State-Space Representation

In order to adapt formal design approach, it is convenient to represent the considered inventory system in the state space. We propose the following state-space representation

$$
\begin{aligned}
\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{b}u(k) + \mathbf{v}h(k), \\
y(k) &= \mathbf{q}^T \mathbf{x}(k),
\end{aligned} \tag{7}
$$

where $\mathbf{x}(k) = [x_1(k)\ x_2(k)\ \ldots\ x_n(k)]^T$ is the state vector with $x_1(k) = y(k)$ representing the stock level in period k and $x_j(k) = u(k-n+j-1)$ for any $j = 2, \ldots, n$ equal to the delayed input signal $u(\cdot)$; $\mathbf{A}$ is $n \times n$ state matrix, $\mathbf{b}$, $\mathbf{v}$, and $\mathbf{q}$ are $n \times 1$ vectors

$$
\mathbf{A} = \begin{bmatrix} \rho & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{8}
$$

and the system order $n = n_p + 1 = L_p/T + 1$ depends on review period T and lead-time L_p. The desired system state is defined as

$$
\mathbf{x_d} = \begin{bmatrix} x_{d1} \\ x_{d2} \\ \vdots \\ x_{dn-1} \\ x_{dn} \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma \\ \sigma \end{bmatrix} y_{ref} = \begin{bmatrix} 1 \\ 1-\rho \\ \vdots \\ 1-\rho \\ 1-\rho \end{bmatrix} y_{ref}, \tag{9}
$$

where y_{ref} denotes the reference stock level. Consequently, the control objective is to stabilize the first state variable (the on-hand stock) at the level y_{ref}. Since the goods perish at the rate $\sigma = 1-\rho$ while kept in the warehouse, in order to maintain the on-hand stock at the desired level once y_{ref} is reached, it needs to be refilled from the incoming shipments at the rate σy_{ref}. Therefore, using (3), all the state variables which represent the in-bound shipments $x_2, \ldots, x_n$, should converge to σy_{ref} once $y(k) = y_{ref}$, precisely as specified in (9). In a latter part of the paper we develop a control strategy which meets these design objectives. We will also show how to choose a suitable reference stock level such that a number of advantageous properties related to handling the flow of goods is achieved.

3. PROPOSED INVENTORY POLICY

In this section, we design a new supply policy for the considered inventory system with perishable goods. We adopt a formal approach based on the theory of discrete sliding-mode control. First, the design procedure is conducted, and the control law is presented a closed-form. Afterwards, the important properties of the obtained controller related to the flow of goods are formulated, and strictly proved.

3.1 Sliding-Mode Controller Design

Let us define the system error as

$$
\mathbf{e}(k) = \mathbf{x_d} - \mathbf{x}(k). \tag{10}
$$

We introduce the sliding hyperplane described by the following equation

$$
s(k) = \mathbf{c}^T \mathbf{e}(k) = 0, \tag{11}
$$

where $\mathbf{c}^T = [c_1\ c_2\ \ldots\ c_n]$ is the vector describing the sliding plane such that $\mathbf{c}^T\mathbf{b} \neq 0$. Substituting (7) into equation $\mathbf{c}^T\mathbf{e}(k+1) = 0$, the following feedback control law can be derived

$$u(k) = \left(\mathbf{c}^T\mathbf{b}\right)^{-1}\mathbf{c}^T\left[\mathbf{x_d} - \mathbf{A}\mathbf{x}(k)\right]. \tag{12}$$

Using (8) we can rewrite (12) as

$$\begin{aligned} u(k) = {} & y_{ref}c_n^{-1}\left[c_1 + (1-\rho)\sum_{j=2}^{n} c_j\right] \\ & - c_n^{-1}\left\{c_1\rho x_1(k) + \sum_{j=2}^{n} c_{j-1}x_j(k)\right\}. \end{aligned} \tag{13}$$

It is clear from (13) that the controller properties will be determined by an appropriate choice of the sliding plane parameters $c_1, c_2, \ldots, c_n$. Since typically in inventory control it is favorable to provide fast reaction to the changes in market conditions, we intend to find such parameters of the plane which will allow for the error elimination in the smallest number of steps after a demand surge (or decline). Thus, we opt for dead-beat control.

The closed-loop state matrix $\mathbf{A_c} = [\mathbf{I_n} - \mathbf{b}(\mathbf{c}^T\mathbf{b})^{-1}\mathbf{c}^T]\mathbf{A}$ with control (13) applied is determined as

$$\mathbf{A_c} = \begin{bmatrix} \rho & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\frac{c_1\rho}{c_n} & -\frac{c_1}{c_n} & -\frac{c_2}{c_n} & \ldots & -\frac{c_{n-1}}{c_n} \end{bmatrix}, \tag{14}$$

and its characteristic polynomial as

$$\begin{aligned} & \det\left(z\mathbf{I_n} - \mathbf{A_c}\right) \\ & = z^n + \frac{c_{n-1} - \rho c_n}{c_n} z^{n-1} + \ldots + \frac{c_2 - \rho c_3}{c_n} z^2 + \frac{c_1 - \rho c_2}{c_n} z. \end{aligned} \tag{15}$$

In order to obtain a dead-beat controller, $\det(z\mathbf{I_n} - \mathbf{A_c})$ should be equal to z^n, which is satisfied when

$$c_{n-1} = \rho c_n,\ c_{n-2} = \rho c_{n-1}, \ldots,\ c_2 = \rho c_3,\ c_1 = \rho c_2. \tag{16}$$

Having solved recursively this set of equations we obtain the following vector describing the parameters of the sliding plane

$$\mathbf{c}^T = \begin{bmatrix} \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \ldots & \rho & 1 \end{bmatrix} c_n. \tag{17}$$

Substituting (17) into (13), we get the control law

$$u(k) = y_{ref} - \rho^n x_1(k) - \sum_{j=2}^{n} \rho^{n-j+1} x_j(k). \tag{18}$$

From (8) the state variables x_j ($j = 2, 3, \ldots, n$) may be expressed in terms of the control signal generated at the previous $n-1$ samples as $x_j(k) = u(k-n+j-1)$. Since $x_1(k) = y(k)$ and $n = n_p + 1$, we obtain

$$u(k) = y_{ref} - \rho^{n_p+1} y(k) - \sum_{j=k-n_p}^{k-1} \rho^{k-j} u(j). \tag{19}$$

3.2 Properties of the Proposed Inventory Policy

Further in this section the properties of the proposed inventory policy will be defined in a lemma and three theorems. The lemma and the first theorem show that the order quantities established by the policy are always nonnegative and bounded, which is a crucial requirement for the practical implementation of an inventory management scheme. The second proposition specifies the warehouse capacity which needs to be provided to always accommodate the on-hand stock and the incoming shipments. Finally, the third theorem indicates how to select the reference stock value in order to ensure the maximum service level.

First, notice that $u(0) = y_{ref}$. Afterwards, for $k \geq 1$, the control signal satisfies the relation defined in the following lemma.

Lemma: If the proposed inventory policy is applied, then for any $k \geq 1$

$$u(k) = (1-\rho) y_{ref} + \rho^{n_p+1} h(k-1). \tag{20}$$

Proof: Substituting (6) into (19), we get

$$\begin{aligned} u(k) &= y_{ref} \\ &\quad - \rho^{n_p+1}\left[\sum_{j=0}^{k-n_p-1} \rho^{k-n_p-1-j} u(j) - \sum_{j=0}^{k-1} \rho^{k-1-j} h(j)\right] \\ &\quad - \sum_{j=k-n_p}^{k-1} \rho^{k-j} u(j) \\ &= y_{ref} - \rho\sum_{j=0}^{k-1} \rho^{k-1-j} u(j) + \rho^{n_p+1} \sum_{j=0}^{k-1} \rho^{k-1-j} h(j). \end{aligned} \tag{21}$$

For $k = 1$, it follows immediately from (21) that

$$u(1) = y_{ref} - \rho u(0) + \rho^{n_p+1} h(0) = (1-\rho) y_{ref} + \rho^{n_p+1} h(0), \tag{22}$$

which shows that the lemma is indeed satisfied for $k = 1$. Let us assume that (20) is true for all integers up to some $l > 1$. Using this assumption, from (21), the order quantity generated in period $l + 1$ can be expressed as

$$\begin{aligned} u(l+1) &= y_{ref} - \rho\sum_{j=0}^{l} \rho^{l-j} u(j) + \rho^{n_p+1} \sum_{j=0}^{l} \rho^{l-j} h(j) \\ &= y_{ref} - \rho y_{ref} + \rho y_{ref} - \rho\rho\sum_{j=0}^{l-1} \rho^{l-1-j} u(j) - \rho u(l) \\ &\quad + \rho\rho^{n_p+1} \sum_{j=0}^{l-1} \rho^{l-1-j} h(j) + \rho^{n_p+1} h(l) \\ &= (1-\rho) y_{ref} + \rho u(l) - \rho u(l) + \rho^{n_p+1} h(l) \\ &= (1-\rho) y_{ref} + \rho^{n_p+1} h(l). \end{aligned} \tag{23}$$

Since l is an arbitrary positive integer, we conclude that (20) actually holds for any integer $k \geq 1$. This ends the proof of the lemma. □

Theorem 1: The order quantities generated by the proposed policy are always bounded, and for any $k \geq 0$ the ordering signal satisfies the following set of inequalities

$$\sigma y_{ref} \leq u(k) \leq \max\left[y_{ref}, \sigma y_{ref} + \rho^{n_p+1} d_{\max}\right]. \quad (24)$$

Proof: It follows from the algorithm definition (19) that $u(0) = y_{ref}$, which means that the theorem is satisfied for $k = 0$. On the other hand, since $0 \leq h(\cdot) \leq d_{\max}$ and $\rho = 1 - \sigma$, we get from the lemma that for any $k > 0$

$$\sigma y_{ref} \leq u(k) \leq \sigma y_{ref} + \rho^{n_p+1} d_{\max}. \quad (25)$$

This ends the proof. □

The practical considerations of inventory management in real systems dictate the requirement of ensuring finite warehouse capacity which should be reserved at the distribution center to accommodate the stock. The next theorem demonstrates that the on-hand stock never exceeds the reference value. This means that in order to provide storage space for the goods at the center, it suffices to assign the warehouse of capacity y_{ref}.

Theorem 2: The order quantities generated by the proposed policy are always bounded, and for any $k \geq 0$ the ordering signal satisfies the following inequality

$$y(k) \leq y_{ref}. \quad (26)$$

Proof: The warehouse at the distribution center is empty for any $k \leq n_p = n - 1$. Hence, it suffices to show that the proposition is satisfied for any $k \geq n$. Let us assume that for some integer $l \geq n$, $y(l) \leq y_{ref}$. We will demonstrate that the theorem is also true for $l + 1$. Using the inventory balance equation (3), the stock level in the $l + 1$ period can be expressed as

$$\begin{aligned} y(l+1) &= \rho y(l) + u_R(l) - h(l) \\ &= \rho y(l) + u(l - n_p) - h(l). \end{aligned} \quad (27)$$

Applying (21), we get

$$\begin{aligned} y(l+1) &= \rho y(l) + y_{ref} - \rho \sum_{j=0}^{l-n_p-1} \rho^{l-n_p-1-j} u(j) \\ &\quad + \rho^{n_p+1} \sum_{j=0}^{l-n_p-1} \rho^{l-n_p-1-j} h(j) - h(l) \\ &= \rho y(l) - \rho \left[\sum_{j=0}^{l-n_p-1} \rho^{l-n_p-1-j} u(j) - \sum_{j=0}^{l-1} \rho^{l-1-j} h(j) \right] \\ &\quad + y_{ref} - \sum_{j=l-n_p}^{l-1} \rho^{l-j} h(j) - h(l). \end{aligned} \quad (28)$$

Using (6) we note that the term in the square brackets in (28) equals $y(l)$. Thus, we get

$$\begin{aligned} y(l+1) &= \rho y(l) - \rho y(l) + y_{ref} - \sum_{j=l-n_p}^{l-1} \rho^{l-j} h(j) - h(l) \\ &= y_{ref} - \sum_{j=l-n_p}^{l} \rho^{l-j} h(j). \end{aligned} \quad (29)$$

Since $h(\cdot)$ is always nonnegative, $y(l + 1) \leq y_{ref}$. Using the principle of the mathematical induction we conclude that the proposition is valid for any review period $k \geq 0$. This completes the proof. □

It comes from Theorem 2 that if for the considered inventory system the warehouse of size y_{ref} is assigned at the distribution center, then all the incoming shipments can be stored locally, and any cost associated with emergency storage is eliminated. Apart from the efficient warehouse space management, a successful inventory control strategy in modern supply chain is expected to achieve high service level. The proposition formulated below shows how the reference stock level should be selected so that all of the demand imposed on the distribution center is satisfied from the readily available resources, and the cost of the lost sales is reduced to zero.

Theorem 3: If the proposed inventory policy is applied, and the reference stock satisfies

$$y_{ref} > d_{\max} \sum_{j=0}^{n_p} \rho^j, \quad (30)$$

then for any $k \geq n_p + 1$ the stock level is strictly positive.

Proof: It follows from (2) that the realized demand is always upper bounded, i.e. for any integer $k \geq 0$ the inequality $h(k) \leq d_{\max}$ holds. Consequently, using (28) and the theorem assumption (30) we have for $k \geq n_p + 1$

$$\begin{aligned} y(k) &= y_{ref} - \sum_{j=k-1-n_p}^{k-1} \rho^{k-1-j} h(j) \\ &\geq y_{ref} - d_{\max} \sum_{j=k-1-n_p}^{k-1} \rho^{k-1-j} \\ &= y_{ref} - d_{\max} \sum_{j=0}^{n_p} \rho^j > 0. \end{aligned} \quad (31)$$

This completes the proof of Theorem 3. □

The discussed properties of the proposed inventory control policy will be illustrated in numerical tests described in Section 4.

4. NUMERICAL EXAMPLE

The properties of the proposed policy (19) are verified in a series of simulation tests. The system parameters are chosen in the following way: review period $T = 1$ day, lead-time delay $L_p = n_p T = 7$ days, inventory decay factor $\sigma = 0.08$, which implies $\rho = 1 - \sigma = 0.92$, and the maximum daily demand at the distribution center $d_{\max} = 60$ items. The simulations are run for two demand patterns illustrated in Fig. 3. The first pattern (I) reflects abrupt changes in a seasonal trend, whereas the second one (II) represents the stochastic setting of normally distributed demand with mean equal to 30 items and standard deviation equal to 20 items. In order to ensure that all of the imposed demand is realized from the readily available resources, according to (30), the reference stock level should be set bigger than 365 items. We select $y_{ref} = 370$ items.

Performance of DSM controller (19) is compared with the classical order-up-to (OUT) inventory policy. The OUT policy can be synthesized in the following way (see e.g. (Sarimveis et al., 2008))

$$u_{\text{OUT}}(k) = y_{\text{OUT}} - y(k) - \Omega(k), \quad (32)$$

which means to order up to a target level y_{OUT} whenever the total stock – equal to the on-hand stock plus open orders $\Omega(\cdot)$ – drops below y_{OUT}. Two different settings of y_{OUT} are considered: in the first simulation (curve (b) in the graphs) it is adjusted to achieve the same service level as the DSM policy, whereas in the second one (curve (c) in the graphs) the order-up-to level is set such both controllers result in the identical storage space assignment. Thus, for the OUT policy we choose the following values of y_{OUT}: in simulation (b) it is set as 630 items, and in simulation (c) it is adjusted to 380 items.

The orders generated by the DSM controller (a) and the classical inventory policy (b) and (c) are shown in Fig. 4, and the on-hand stock in Fig. 5. It is clear from the graphs that the proposed controller quickly responds to the sudden changes in the demand trend without oscillations or overshoots (I), and reduces oscillations in case (II). Moreover, the stock does not increase beyond the warehouse capacity, and it never drops to zero after the initial phase, which implies the 100% service level. The OUT policy exhibits oscillations and requires bigger storage space to accommodate the stock to achieve the same service level (curve (b) in Fig. 5), which leads to increased holding cost. On the other hand, if the safety stock level is reduced for the OUT policy to maintain the same storage space as the one imposed by our controller, the OUT service level decreases to 85% (demand (I)). In that case, large oscillations appear in the ordering signal generated by the OUT policy leading to the bullwhip effect, which is avoided by our scheme.

Figure 6 shows the evolution of the sliding variable $s(k) = \mathbf{c}^T\mathbf{e}(k)$. The vector describing the sliding plane parameters is determined (with $c_n = 1$) as

$$\mathbf{c}^T = [0.56 \quad 0.61 \quad 0.66 \quad 0.72 \quad 0.78 \quad 0.85 \quad 0.92 \quad 1]. \quad (33)$$

We can see from the graph that $s(\cdot)$ immediately decreases from its initial value $s(0) = 370$ items to a relatively narrow band $s \in [0, 34$ items) and then always remains in this band, which constitutes a clear evidence of a properly established sliding motion in discrete-time domain.

5. CONCLUSIONS

In this paper, a new supply policy for periodic-review inventory systems with deteriorating stock was designed using strict control-theoretic methodology. The proposed policy based on sliding-mode dead-beat control provides fast reaction to the changes in market conditions and stable system operation for arbitrary lead-time. It also guarantees that all of the demand is satisfied from the on-hand stock, thus eliminating the risk of missed service opportunities and the necessity for backorders.

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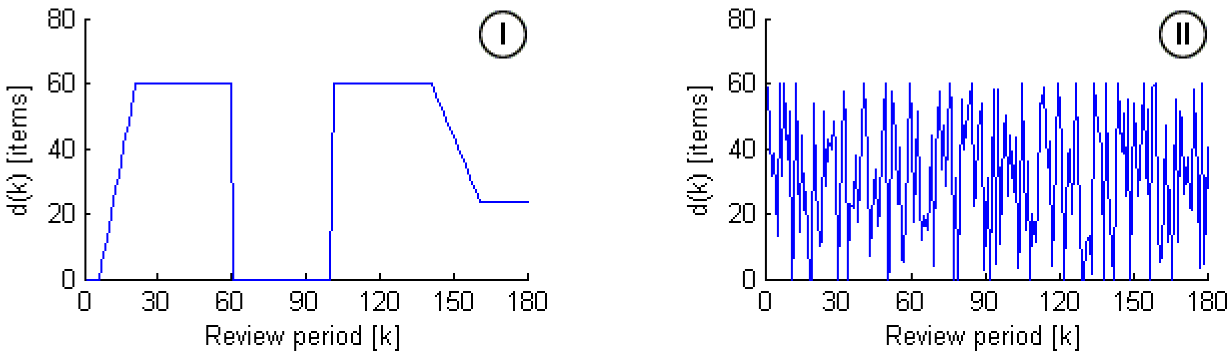

Fig. 3. Demand at the distribution center: I – seasonal trend, II – stochastic pattern (normal distribution with mean = 30 items and standard deviation = 20 items).

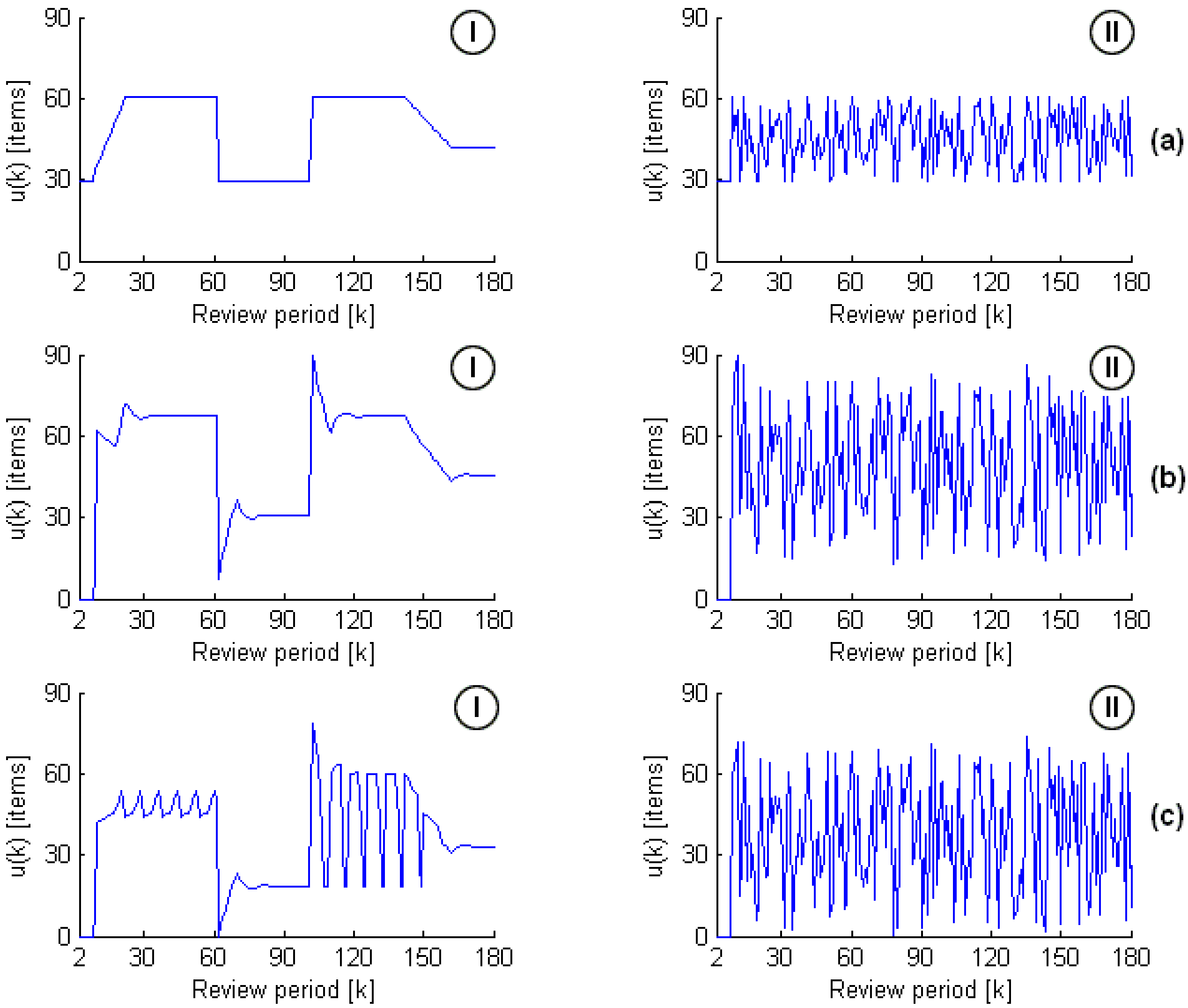

Fig. 4. Order quantities: a) policy (19), b) OUT policy with y_{OUT} = 630 items, c) OUT policy with y_{OUT} = 380 items.

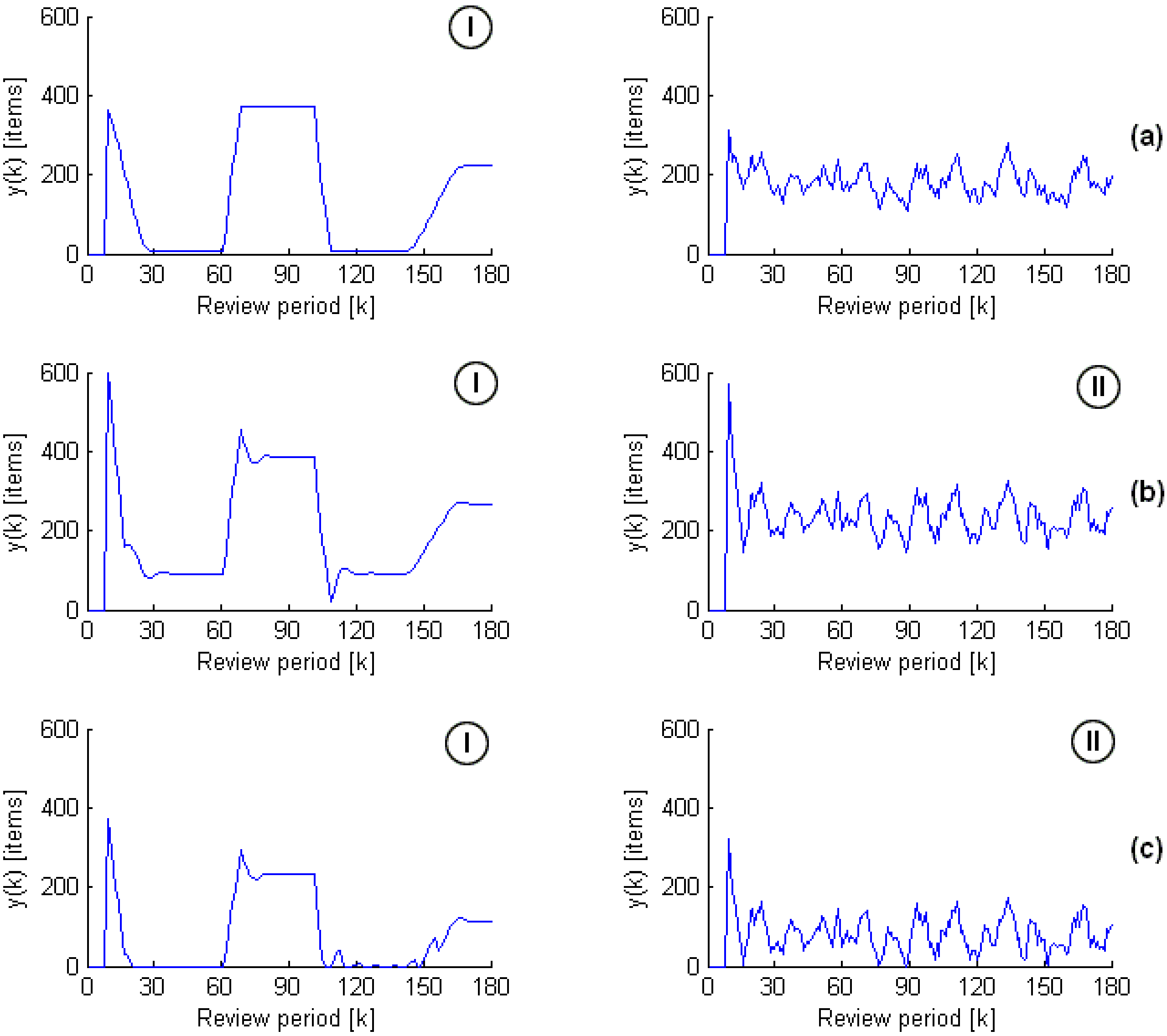

Fig. 5. Stock level: a) policy (19), b) OUT policy with y_{OUT} = 630 items, c) OUT policy with y_{OUT} = 380 items.

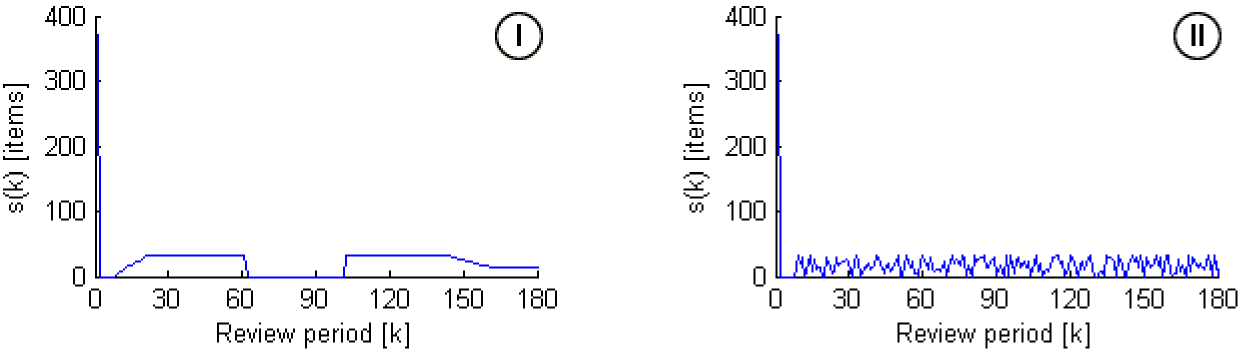

Fig. 6. Sliding variable.