SYNTHESIS OF TIME OPTIMAL CONTROL FOR A CLASS OF LINEAR SYSTEMS

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Abstract: *The paper deals with the optimal control synthesis for a class of linear time optimal control problems of decreasing order. The state-space properties and relationships between two neighboring problems within this class are studied. Based on these properties, a new approach is proposed for synthesis of time optimal control, requiring no description of the switching hyper surface. The approach consists of two main stages: first, the problems' state spaces are analyzed, and then the optimal control is obtained by using a multi-stage procedure within the considered class of problems.*

Keywords: *Time optimal control, Pontryagin's maximum principle, Synthesis of optimal systems, Linear systems.*

1. INTRODUCTION

The linear time optimal control problem has a half-a-century history. Fundamental theoretical results have been obtained and a great number of papers and monographs has been published. However, in the last decade the interest towards this problem considerably declines. It may be stated that despite the more than 40-year intensive research, the synthesis of time optimal control for high order systems is still an open problem.

An approach to go further in the solution of the time optimal synthesis problem is to refine the well-known state-space method, removing the factors that restrict its application to low order systems only. Based on some new state space properties of a class of linear systems, we propose a synthesis approach requiring no description of the switching hyper surface. This makes possible the efficient design and implementation of time optimal control for high order linear systems.

Consider the following time optimal synthesis problem for a linear system of order $\boldsymbol{k}$. The system is described by :

$$\begin{aligned}
&\dot{\boldsymbol{x}}_k = A_k \boldsymbol{x}_k + B_k u_k, \\
&\boldsymbol{x}_k = [x_1 \quad x_2 \quad . \quad . \quad . \quad x_k]^{\mathrm{T}}, \ \boldsymbol{x}_k \in R^k, \\
&A_k = \mathrm{diag}(\lambda_1, \ \lambda_2, \ . \ . \ . \ . \ , \lambda_k), \\
&\qquad \lambda_i \in R, \ \lambda_i \le 0, \ {\scriptstyle i,j=\overline{1,k}}, \ \lambda_i \ne \lambda_j \ \text{if} \ {\scriptstyle i \ne j}, \\
&B_k = [b_1 \quad b_2 \quad . \quad . \quad . \quad b_k]^{\mathrm{T}}, \ b_i \in R, \ b_i \ne 0, \ {\scriptstyle i=\overline{1,k}}, \\
&\qquad \overline{1,k} = 1, 2, \ ..., k.
\end{aligned} \tag{1}$$

The initial and the target states of the system are

$$\boldsymbol{x}_k(0) = [x_{10} \quad x_{20} \quad . \quad . \quad . \quad x_{k0}]^{\mathrm{T}} \tag{2}$$

and

$$\boldsymbol{x}_k(t_{kf}) = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k}]^{\mathrm{T}} \tag{3}$$

where t_{kf} is unspecified. The admissible control $u_k(t)$ is a piecewise continuous function that takes its values from the range

$$-u_0 \le u_k(t) \le u_0, \ u_0 = const, \ u_0 > 0 \ . \tag{4}$$

We suppose that $u_k(t)$ is continuous on the boundary of the set of allowed values (4) and in the points of discontinuity τ we have

$$u(\tau) = u(\tau + 0) \ . \tag{5}$$

The problem is to find an admissible control $u_k = u_k(\boldsymbol{x}_k)$ that transfers the system (1) from its initial state (2) to the target state (3) in minimum time, i.e. minimizing the performance index

$$J_k = \int_0^{t_{kf}} dt = t_{kf} \ . \tag{6}$$

We shall refer to this problem as **Problem *A(k)*** and to the set {Problem *A(n)*, Problem *A(n-1)*, ..., Problem *A(1)*}, $n \ge 2$, as **class of problems *A(n), A(n-1), …, A(1)*** (Penev, 1999; Penev and Christov, 2002; Penev and Christov, 2004).

The following relations exist between the systems of Problem *A(k)* and Problem *A(k-1)*, $k = \overline{n, 2}$:

$$A_k = \begin{bmatrix} A_{k-1} & 0_{((k-1)\times 1)} \\ 0_{(1\times(k-1))} & \lambda_k \end{bmatrix}, \quad B_k = \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix}, \quad \boldsymbol{x}_k(0) = \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix}. \tag{7}$$

For Problem *A(k)*, $k = \overline{n,1}$, denote:

- $u_k^o(t)$ - the optimal control which is a piecewise constant function taking the values $+u_0$ or $-u_0$ and having at most $(k-1)$ discontinuities (Athans and Falb, 1966; Boltyanskii, 1971; Feldbaum and Butkowsky, 1971; Lee and Marcus, 1967; Leitmann, 1981; Pinch, 1993; Penev, 1999; Penev and Christov, 2002; Penev and Christov, 2004; Pontryagin, et. al., 1964);
- t_{kf}^o - the minimum of the performance index;
- L_{kk-1} - the set of all state space points for which the optimal control has no more than $(k-2)$ discontinuities;
- S_k - the switching hyper surface. Note that S_k is time-invariant and includes the state space origin. As it is well-known, the switching hyper surface S_k is identical with the set L_{kk-1} (Feldbaum and Butkowsky, 1971).

2. PRELIMINARY RESULTS

We shall study first a property of the defined class of problems that indicates the existence of specific relationships between the neighboring problems within this class. It can be shown that there exists a relation between the switching hyper surface for the Problem *A(k)* and the state trajectory starting from the initial state of Problem *A(k)* and generated by the optimal control for Problem *A(k-1)*. The following result has been proved (Penev, 1999; Penev and Christov, 2002).

Theorem 1. *The state trajectory of system (1) starting from the initial point* $\boldsymbol{x}_k(0)$ *and generated by the optimal control* $u_{k-1}^o(t)$, $t \in [0, \ t_{k-1f}^o]$, *either entirely lies on the switching hyper surface* S_k, *or is above or below* S_k, *nowhere intersecting it.*

Proof. There exist two alternatives for the initial point $\boldsymbol{x}_k(0)$:

1. $\boldsymbol{x}_k(0) \in S_k$, i.e. $\boldsymbol{x}_k(0)$ lies on S_k
2. $\boldsymbol{x}_k(0) \notin S_k$, i.e. $\boldsymbol{x}_k(0)$ is above or below S_k.

Consider consequently these two cases.

Case 1: $\boldsymbol{x}_k(0) = [x_{10} \quad x_{20} \quad ... \quad x_{k0}]^{\mathrm{T}} \in S_k$. Here, the optimal control $u_k^o(t)$ has no more than $(k-2)$

discontinuities and for the points of the optimal trajectory, lying on S_k we can write

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)}B_k u_k^o(\tau)d\tau,\ t\in[0,\ t_{kf}^o],$$
$$\boldsymbol{x}_k(t_{kf}^o) = [\underbrace{0\quad 0\quad .\quad .\quad .\quad 0}_{k}]^{\mathrm{T}}. \tag{8}$$

Taking into account (7) we obtain

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)}B_k u_k^o(\tau)d\tau =$$
$$= e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}t}\begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix} +$$
$$+\int_0^t e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}(t-\tau)}\begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_k^o(\tau)d\tau =$$
$$= \begin{bmatrix} e^{A_{k-1}t} & 0 \\ 0 & e^{\lambda_k t} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix} +$$
$$+\int_0^t \begin{bmatrix} e^{A_{k-1}(t-\tau)} & 0 \\ 0 & e^{\lambda_k(t-\tau)} \end{bmatrix}\begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_k^o(\tau)d\tau =$$
$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) \\ e^{\lambda_k t}x_{k0} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{A_{k-1}(t-\tau)}B_{k-1} \\ e^{\lambda_k(t-\tau)}b_k \end{bmatrix} u_k^o(\tau)d\tau =$$
$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)}B_{k-1}u_k^o(\tau)d\tau \\ e^{\lambda_k t}x_{k0} + \int_0^t e^{\lambda_k(t-\tau)}b_k u_k^o(\tau)d\tau \end{bmatrix}, \tag{9}$$
$$t\in[0,\ t_{kf}^o].$$

It follows from (8), (9)

$$\boldsymbol{x}_k(t) = \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)}B_{k-1}u_k^o(\tau)d\tau \\ e^{\lambda_k t}x_{k0} + \int_0^t e^{\lambda_k(t-\tau)}b_k u_k^o(\tau)d\tau \end{bmatrix},$$
$$t\in[0,\ t_{kf}^o], \tag{10}$$
$$\boldsymbol{x}_k(t_{kf}^o) = [\underbrace{0\quad 0\quad .\quad .\quad .\quad 0}_{k}]^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{kf}^o) \\ 0 \end{bmatrix}.$$

Thus, the optimal control $u_k^o(t)$ of Problem *A(k)*, which is a piecewise constant function with no more than $(k-2)$ discontinuities in this case, transfers the system of Problem *A(k-*1*)* from its initial state $\boldsymbol{x}_{k-1}(0)$ to the state $\boldsymbol{x}_{k-1}(t_{kf}^o) = [\underbrace{0\quad 0\quad .\quad .\quad .\quad 0}_{k-1}]^{\mathrm{T}}$, which is the target state of Problem *A(k-*1*)*. This means that $u_k^o(t)$ is the optimal control for Problem *A(k-*1*)* as well, i.e.

$$u_k^o(t) = u_{k-1}^o(t) \quad \text{and} \quad t_{kf}^o = t_{k-1f}^o. \tag{11}$$

Therefore, from (8) and (11) we get

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)}B_k u_{k-1}^o(\tau)d\tau,$$
$$t\in[0,\ t_{k-1f}^o], \tag{12}$$
$$\boldsymbol{x}_k(t_{k-1f}^o) = [\underbrace{0\quad 0\quad .\quad .\quad .\quad 0}_{k}]^{\mathrm{T}}.$$

This completes the proof for case 1.

Case 2: $\boldsymbol{x}_k(0)\notin S_k$. In this case the optimal control $u_k^o(t)$ has exactly $(k-1)$ discontinuities. For the initial state of Problem *A(k)* we can write

$$\boldsymbol{x}_k(0) = \begin{bmatrix} x_{10} & x_{20} & . & . & . & x_{k0} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix}. \tag{13}$$

Applying the optimal control $u_{k-1}^o(t)$ of Problem *A(k-*1*)* to the system of Problem *A(k)* with initial state

$$\boldsymbol{x}_k^1(0) = \begin{bmatrix} x_{10} & . & . & . & x_{k-10} & x_{k0}^1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0}^1 \end{bmatrix},$$
$$x_{k0}^1 = -\frac{\int_0^{t_{k-1f}^o} e^{\lambda_k(t_{k-1f}^o-\tau)}b_k u_{k-1}^o(\tau)d\tau}{e^{\lambda_k t_{k-1f}^o}}, \tag{14}$$

we obtain the trajectory

$$\boldsymbol{x}_k^1(t) = e^{A_k t}\boldsymbol{x}_k^1(0) + \int_0^t e^{A_k(t-\tau)}B_k u_{k-1}^o(\tau)d\tau,$$
$$t\in[0,\ t_{k-1f}^o]. \tag{15}$$

It follows from (14), (15)

$$\boldsymbol{x}_k^1(t) = e^{A_k t}\boldsymbol{x}_k^1(0) + \int_0^t e^{A_k(t-\tau)}B_k u_{k-1}^o(\tau)d\tau =$$
$$= e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}t}\begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0}^1 \end{bmatrix} +$$
$$+\int_0^t e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}(t-\tau)}\begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_{k-1}^o(\tau)d\tau =$$
$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)}B_{k-1}u_{k-1}^o(\tau)d\tau \\ e^{\lambda_k t}x_{k0}^1 + \int_0^t e^{\lambda_k(t-\tau)}b_k u_{k-1}^o(\tau)d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u^o_{k-1}(\tau)d\tau \\ \dfrac{\int_0^{t^o_{k-1f}} e^{\lambda_k (t^o_{k-1f}-\tau)} b_k u^o_{k-1}(\tau)d\tau}{(-1)e^{\lambda_k (t^o_{k-1f}-t)}} + \int_0^t e^{\lambda_k (t-\tau)} b_k u^o_{k-1}(\tau)d\tau \end{bmatrix},$$

$$t \in [0,\ t^o_{k-1f}]. \quad (16)$$

Since $u^o_{k-1}(t)$ is the optimal control of Problem *A(k-*1*)*, then

$$\boldsymbol{x}^1_k(t^o_{k-1f}) =$$

$$= \begin{bmatrix} \boldsymbol{x}_{k-1}(t^o_{k-1f}) \\ -\int_0^{t^o_{k-1f}} e^{\lambda_k (t^o_{k-1f}-\tau)} b_k u^o_{k-1}(\tau)d\tau + \int_0^{t^o_{k-1f}} e^{\lambda_k (t^o_{k-1f}-\tau)} b_k u^o_{k-1}(\tau)d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} \boldsymbol{x}_{k-1}(t^o_{k-1f}) \\ 0 \end{bmatrix} = [\underbrace{0\ \ 0\ \ .\ \ .\ \ .\ \ 0}_{k}]^{\mathrm{T}}. \quad (17)$$

Hence, the optimal control $u^o_{k-1}(t)$, which is a piecewise constant function with no more than $(k-2)$ discontinuities, transfers the system from the initial state (14) to the state space origin in the moment t^o_{k-1f}. Therefore, the point $\boldsymbol{x}^1_k(0)$ and the trajectory starting from this point and generated by $u^o_{k-1}(t)$, $t \in [0,\ t^o_{k-1f}]$, lie entirely on the switching hyper surface S_k.

Consider now the trajectory $\boldsymbol{x}_k(t)$ with initial point (2) in form (13), generated by the optimal control $u^o_{k-1}(t)$. Taking into account (7) we can write

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)} B_k u^o_{k-1}(\tau)d\tau =$$

$$= e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix} t} \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix} + \int_0^t e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}(t-\tau)} \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u^o_{k-1}(\tau)d\tau =$$

$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u^o_{k-1}(\tau)d\tau \\ e^{\lambda_k t} x_{k0} + \int_0^t e^{\lambda_k (t-\tau)} b_k u^o_{k-1}(\tau)d\tau \end{bmatrix}, \quad (18)$$

$$t \in [0,\ t^o_{k-1f}].$$

According to (18) and (15) we have

$$\boldsymbol{x}_k(t) - \boldsymbol{x}^1_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)} B_k u^o_{k-1}(\tau)d\tau -$$

$$-\left[e^{A_k t}\boldsymbol{x}^1_k(0) + \int_0^t e^{A_k(t-\tau)} B_k u^o_{k-1}(\tau)d\tau\right], \quad (19)$$

$$t \in [0,\ t^o_{k-1f}].$$

From (19), having in mind (16) and (18), we obtain

$$\boldsymbol{x}_k(t) - \boldsymbol{x}^1_k(t) = \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u^o_{k-1}(\tau)d\tau \\ e^{\lambda_k t} x_{k0} + \int_0^t e^{\lambda_k (t-\tau)} b_k u^o_{k-1}(\tau)d\tau \end{bmatrix} -$$

$$- \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u^o_{k-1}(\tau)d\tau \\ e^{\lambda_k t} x^1_{k0} + \int_0^t e^{\lambda_k (t-\tau)} b_k u^o_{k-1}(\tau)d\tau \end{bmatrix}, \quad (20)$$

$$t \in [0,\ t^o_{k-1f}].$$

Thus

$$\boldsymbol{x}_k(t) - \boldsymbol{x}^1_k(t) = \begin{bmatrix} 0_{((k-1)\times 1)} \\ e^{\lambda_k t}(x_{k0} - x^1_{k0}) \end{bmatrix},\ t \in [0,\ t^o_{k-1f}]. \quad (21)$$

Let analyze the difference between $\boldsymbol{x}_k(t)$ and $\boldsymbol{x}^1_k(t)$ in (21). For the ***k***th co-ordinate $e^{\lambda_k t}(x_{k0} - x^1_{k0})$ we have:

1. $x_{k0} \neq x^1_{k0}$, since otherwise the initial state $\boldsymbol{x}_k(0)$ does lie on S_k, but we consider the case $\boldsymbol{x}_k(0) \notin S_k$
2. $e^{\lambda_k t}(x_{k0} - x^1_{k0})$, $t \in [0,\ t^o_{k-1f}]$, does not change its sign and never equals zero, since t^o_{k-1f} is finite.

Hence, the trajectory (15) with initial point (14), generated by $u^o_{k-1}(t)$, $t \in [0,\ t^o_{k-1f}]$, and lying entirely on S_k, is the projection on S_k of the trajectory $\boldsymbol{x}_k(t)$ with initial point $\boldsymbol{x}_k(0)$, generated by $u^o_{k-1}(t)$, $t \in [0,\ t^o_{k-1f}]$. The direction of this projection is parallel to the axis Ox_k. Moreover, the trajectory $\boldsymbol{x}_k(t)$, $t \in [0,\ t^o_{k-1f}]$, nowhere intersects the switching hyper surface S_k. Thus the theorem is proved.

We shall now study another property of the considered class of problems, which makes possible the synthesis of optimal control for Problem *A(k)*, $k = \overline{n, 2}$.

Let x_{k+}, $k = \overline{n, 2}$, be a variable taking values $+1$ or -1, i.e.

$$x_{k+} \in \{-1,\ +1\},\ k = \overline{2, n} \quad (22)$$

so that for

$$\boldsymbol{x}_k(0) = [x_{10} \quad x_{20} \quad . \quad . \quad . \quad x_{k0}]^{\mathrm{T}} = \\ = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k-1} \; x_{k0}]^{\mathrm{T}}, \\ x_{k0} : \operatorname{sign}(x_{k0}) = x_{k+}. \tag{23}$$

we have

$$u_k^o(0) = u_0 .$$

Theorem 2 (Penev, 1999). *There does not exist a piecewise constant control $u(t)$ with an amplitude u_0 and **k** non zero intervals of constancy, $1 \le k \le (n-1)$, transferring the system*

$$\dot{x}_i = \lambda_i x_i + b_i u, \; \lambda_i \in R, \; b_i \in R, \; i,j = \overline{1,n}, \\ b_i \ne 0, \; \lambda_i \ne \lambda_j \quad \text{when} \quad i \ne j \tag{24}$$

from any point of any axis $Ox_1, Ox_2, \dots, Ox_n$ in the system state space to the origin O, and vice-versa – from the origin O to a point of any axis $Ox_1, Ox_2, \dots, Ox_n$ in the state space.

From this result and the properties of the switching hyper surface S_k it follows

Corollary 1. *The unique time optimal control that transfers the system of Problem A(k), $k = \overline{n, 2}$, from every point of the positive or negative part of any state space axis $Ox_1, Ox_2, \dots, Ox_k$ to the origin O, has exactly **k** non zero intervals of constancy, and the positive, respectively the negative, part of any axis $Ox_1, Ox_2, \dots, Ox_k$ is above or below the switching hyper surface S_k.*

Thus, $x_{k+} = +1$ means that all points of the positive semi-axis Ox_k are above S_k and the optimal control value for them is $+u_0$, while the points of the negative semi-axis Ox_k are below S_k and the corresponding optimal control value is $-u_0$. In turn, $x_{k+} = -1$ means that all points of the negative semi-axis Ox_k are above S_k and the optimal control value for them is $+u_0$, while the points of the positive semi-axis Ox_k are below S_k and the optimal control value is $-u_0$.

Denote by x_{kw}, $k = \overline{n,2}$, the ***k***th coordinate of the vector

$$\boldsymbol{x}_k(t_{k-1f}^o) = e^{A_k t_{k-1f}^o} \boldsymbol{x}_k(0) + \int_0^{t_{k-1f}^o} e^{A_k (t_{k-1f}^o - \tau)} B_k u_{k-1}^o(\tau) d\tau , \tag{25}$$

where $u_{k-1}^o(t)$ and t_{k-1f}^o are respectively the optimal control and the minimum of the performance index of Problem *A(k-1)*.

Let analyze (25) taking into account (1) - (7) and the definition for x_{kw}. We have

$$\boldsymbol{x}_k(t_{k-1f}^o) = e^{A_k t_{k-1f}^o} \boldsymbol{x}_k(0) + \int_0^{t_{k-1f}^o} e^{A_k (t_{k-1f}^o - \tau)} B_k u_{k-1}^o(\tau) d\tau =$$

$$= e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix} t_{k-1f}^o} \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix} + \\ + \int_0^{t_{k-1f}^o} e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix} (t_{k-1f}^o - \tau)} \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_{k-1}^o(\tau) d\tau =$$

$$= \begin{bmatrix} e^{A_{k-1} t_{k-1f}^o} \boldsymbol{x}_{k-1}(0) + \int_0^{t_{k-1f}^o} e^{A_{k-1}(t_{k-1f}^o - \tau)} B_{k-1} u_{k-1}^o(\tau) d\tau \\ e^{\lambda_k t_{k-1f}^o} x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{k-1f}^o) \\ e^{\lambda_k t_{k-1f}^o} x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{k-1f}^o) \\ x_{kw} \end{bmatrix} = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k-1} \; x_{kw}]^{\mathrm{T}} ,$$

$$x_{kw} = e^{\lambda_k t_{k-1f}^o} x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau, \quad k = \overline{n,2} . \tag{26}$$

It follows from (26) that $\boldsymbol{x}_k(t_{k-1f}^o)$ is a point from the axis Ox_k, i.e.

$$\boldsymbol{x}_k(t_{k-1f}^o) \in Ox_k , \tag{27}$$

and $x_{nw}, x_{n-1w}, \dots, x_{2w}$ are given by

$$x_{kw} = e^{\lambda_k t_{k-1f}^o} x_{k0} + \int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau, \quad k = \overline{n,2} . \tag{28}$$

3. MAIN RESULT

In this section we shall present a new approach to the synthesis of optimal control for the initial state of Problem *A(k)* (Penev, 1999; Penev and Christov, 2002).

Theorem 3. *If the solution of Problem A(k-1), $k = \overline{n, 2}$, is found, then the optimal control of*

Problem A(k) for initial state $\boldsymbol{x}_k(0)$ can be determined as

$$u_k^o(0) = u_k(\boldsymbol{x}_k(0)) = \begin{cases} +u_0 & \text{if } x_{k+}x_{kw} > 0 \\ u_{k-1}^o(0) & \text{if } x_{k+}x_{kw} = 0 \\ -u_0 & \text{if } x_{k+}x_{kw} < 0 \end{cases}, \quad (29)$$

where x_{kw} is given by (28).

Proof. From Theorem 1 it follows that the trajectory

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k(t-\tau)}B_k u_{k-1}^o(\tau)d\tau, \quad t \in [0,\ t_{k-1f}^o], \quad (30)$$

either entirely lies on the switching hyper surface S_k, or is above or below S_k, nowhere intersecting S_k. It follows from (26) that depending on the value of x_{kw} there are two alternative cases:

Case 1: $x_{kw} = 0$. In this case

$$\boldsymbol{x}_k(t) \in S_k \equiv L_{kk-1},\ t \in [0,\ t_{k-1f}^o], \quad (31)$$

and the optimal control $u_{k-1}^o(t)$, which is a piecewise constant function with no more than $(k-1)$ non zero intervals of constancy, transfers the system of Problem *A(k)* from the initial state $\boldsymbol{x}_k(0) \in S_k$ to the origin in minimum time t_{k-1f}^o. Taking into consideration the uniqueness of the optimal control for Problem *A(k)*, it follows that $u_{k-1}^o(t)$ is also the optimal control of Problem *A(k),* i.e.

$$u_{k-1}^o(t) = u_k^o(t),\quad t_{k-1f}^o = t_{kf}^o. \quad (32)$$

Hence,

$$u_{k-1}^o(0) = u_k^o(0) \quad (33)$$

and the case

$$u_k^o(0) = u_k(\boldsymbol{x}_k(0)) = u_{k-1}^o(0) \text{ if } x_{k+}x_{kw} = 0 \quad (34)$$

in (29) is proved.

Case 2: $x_{kw} \neq 0$. Here

$$\boldsymbol{x}_k(t) \notin S_k \equiv L_{kk-1},\ t \in [0,\ t_{k-1f}^o], \quad (35)$$

and therefore $\boldsymbol{x}_k(t_{k-1f}^o) = [\underbrace{0\ \ 0\ \ .\ \ .\ \ .\ \ 0}_{k-1}\ x_{kw}]^{\mathrm{T}}$ is a point from the positive or the negative part of the axis Ox_k. Taking into account that the product $x_{k+}x_{kw}$ indicates whether the trajectory $\boldsymbol{x}_k(t)$ is above or below S_k, we have that

$x_{k+}x_{kw} > 0$ is equivalent to

$$u_k^o(0) = u_k(\boldsymbol{x}_k(0)) = +u_0 \quad (36)$$

and

$x_{k+}x_{kw} < 0$ is equivalent to

$$u_k^o(0) = u_k(\boldsymbol{x}_k(0)) = -u_0. \quad (37)$$

The cases $x_{k+}x_{kw} > 0$ and $x_{k+}x_{kw} < 0$ in (29) are thereby proved. This completes the proof of Theorem 3.

Based on this theorem, the following time optimal synthesis algorithm can be proposed.

Algorithm for synthesis of optimal control for the initial state of Problem *A(k)*, $k = \overline{n, 2}$

Step 1. Solve Problem *A(k-1)* to find $u_{k-1}^o(t)$ and t_{k-1f}^o

Step 2. Compute x_{kw} from (28)

Step 3. Determine $u_k^o(0) = u_k(\boldsymbol{x}_k(0))$ according to (29).

If $x_{kw} = 0$, the solution of Problem *A(k-1)* is also the solution of Problem *A(k)*, i.e. $u_{k-1}^o(t) = u_k^o(t)$ and $t_{k-1f}^o = t_{kf}^o$, and vice-versa: if Problem *A(k)* and Problem *A(k-1)* have the same solution, i.e. $u_{k-1}^o(t) = u_k^o(t)$ and $t_{k-1f}^o = t_{kf}^o$, then $x_{kw} = 0$.

Depending on the value of x_{kw}, there are three possibilities:

- all $\boldsymbol{x}_k(0)$ for which $x_{kw} = 0$ lie on the switching hyper surface S_k;
- all $\boldsymbol{x}_k(0)$ corresponding to $x_{kw} > 0$ are above or below S_k and the optimal control for these points is $x_{k+}u_0$;
- all $\boldsymbol{x}_k(0)$ for which $x_{kw} < 0$ are also above or below S_k, but in opposite to the area for $x_{kw} > 0$, and the corresponding optimal control is $(-1)x_{k+}u_0$.

4. CONCLUDING REMARKS

In this paper a new approach to the time optimal synthesis problem for a class of linear systems is proposed. The main feature of this approach is that the solution of the optimal control problem *A(n)* is reduced to the solution of the sub-problem *A(n-1)*. In turn, the solution of Problem *A(n-1)* requires the solution of Problem *A(n-2)*, etc. reaching Problem *A(1)*. The final solution requires turning back to Problem *A(n)* using the obtained $x_{n+}, x_{n-1+}, \ldots, x_{2+}$. The quantities $x_{n+}, x_{n-1+}, \ldots, x_{2+}$ are calculated by using the same algorithm as shown in (Penev, 1999; Penev and Christov, 2004; Penev and Christov, 2005). The proposed approach does not require the description of the switching hyper surface and thus enables the synthesis of time optimal control for high order systems of the considered class.

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