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RATIONAL INTERPOLATION WITH DISSIPATIVITY CONSTRAINTS

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Abstract: *The paper presents a computational oriented overview on the rational interpolation problem with passive (positive real) constraints. The numerical difficulties associated with the standard rational interpolation procedures are discussed. Some numerical tricks to improve the numerical accuracy of the computed results are proposed. Possible applications to model order reduction for continuous (and, by using a simple bilinear transformation, of discrete) linear systems are sketched.*

Keywords: *rational interpolation, passivity, model order reduction.*

1. INTRODUCTION

Our paper deals with the general rational interpolation problem in the case when an additional goal is imposed, namely the computed rational matrix must be positive real. By using the well-known positive real lemma (see (Anderson and Vongpanitled, 1973), (Hodaka *et al.*, 2000), (Sun *et al.*, 1994)) this constraint is transferred to an adequate LTI state space model. Thus, the well known condition for the solution existence, expressed as the positive definiteness of a corresponding Lyapunov equation solution (Pick matrix in SISO case), is easily obtained. Also, this approach allowed us to develop a concise and reliable numerical algorithm to solve the problem.

The paper is organized as follows. In section 2 the general rational interpolation problem is briefly presented. The positive real constraint in the continuous time setting and the corresponding feasibility conditions are presented in section 3. Section 4 is devoted to the main numerical procedure to solve the problem and the section 5 contains a detailed algorithm which makes our approach efficient and reliable. The discrete time case is considered in section 6; in fact, all the computational procedures apply as such in this setting, possible after a bilinear transformation on the system data is performed. Some remarks concerning related topics and directions for future work conclude our presentation.

2. RATIONAL INTERPOLATION PROBLEM

To begin with, we shall give a concise statement for the general rational interpolation problem concerning a MIMO transfer matrix represented by a state space model. In short terms, we seek a

n-th order linear system $S=(A,B,C,D)$, with a $l\times m$ proper transfer matrix $G(s)=C(sI-A)^{-1}B+D$, such that the following (dynamic) cover conditions (see (Wonham, 1974), (Ionescu and Popeea, 1986)) are satisfied

$$\Lambda W=WA+HC, \tag{1}$$

$$G=WB+HD, \tag{2}$$

where typically

$$\Lambda=\begin{bmatrix} l_1 & 0 & \cdots & 0\\ 0 & l_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l_N \end{bmatrix}$$

and

$$H=\begin{bmatrix} h_1^H\\ h_2^H\\ \vdots\\ h_N^H \end{bmatrix},\quad G=\begin{bmatrix} g_1^H\\ g_2^H\\ \vdots\\ g_N^H \end{bmatrix}$$

are given (complex) $N\times N$, $N\times l$ and, respectively, $N\times m$ matrices such that the pair (Λ,H) is controllable. By letting

$$W=\begin{bmatrix} w_1^H\\ w_2^H\\ \vdots\\ w_N^H \end{bmatrix},$$

from (1) it results $l_i w_i^H = w_i^H + h_i^H C$, i.e.

$$v_i^H = h_i^H (l_i I-A)^{-1},\quad i=1:N \tag{3}$$

and (2) gives

$$g_i^H = w_i^H B + h_i^H D = h_i^H\left[C(l_i I-A)^{-1}B+D\right]$$

Therefore (1) and (2) are equivalent to the so called (N-point) *left* tangential rational interpolation problem

$$h_i^H G(l_i)=g_i^H,\quad i=1:N. \tag{4}$$

Remark 1. *By duality, the following well-known observer (or dual-cover) conditions are obtained*

$$V\Lambda=AV+BH, \tag{5}$$

$$G=CV+DH, \tag{6}$$

where

$$\Lambda=\begin{bmatrix} l_1 & 0 & \cdots & 0\\ 0 & l_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l_N \end{bmatrix}$$

and

$$H=\begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix},$$

$$G=\begin{bmatrix} g_1 & g_2 & \cdots & g_N \end{bmatrix}$$

are given (complex) $N\times N$, $m\times N$ and, respectively, $l\times N$ matrices such that the pair (H,Λ) is observable. By letting

$$V=\begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix},$$

from (5) it results

$$v_i=(l_i I-A)^{-1}Bu_i,\quad i=1:N, \tag{7}$$

therefore (5), (6) reduce to a (N-point) right tangential rational interpolation problem having the form

$$G(l_i)u_i=g_i,\quad i=1:N. \tag{8}$$

Obviously, the same equations (5), (6) describe the steady state response of the linear system $S=(A,B,C,D)$ to a persistent input

$$u(t)=\sum_{i=1}^{N} u_i e^{l_i t},$$

(see (Jora et al., 1996)). Also, putting $H=FV$ and writing (5), (6) as

$$V\Lambda=(A+BF)V,$$

$$G=(C+DF)V,$$

the well known equations for modal assignment by state feedback are obtained.

Remark 2. *In the SISO case, i.e. $m=l=1$, by taking $h_i=1$ and/or $u_i=1$ in (1)-(4) and (5)-(8) respectively, the classical rational interpolation problem is obtained, which consists of finding the transfer function $G(s)$ such that*

$$G(l_i)=g_i,\quad i=1:N, \tag{9}$$

for a given set of interpolation points

$$I=\left\{(l_i,g_i)\,\middle|\, i=1:N,\ l_i\neq l_j,\ \forall i\neq j\right\}.$$

3. RATIONAL INTERPOLATION WITH DISSIPATIVITY CONSTRAINTS

Let us consider the rational interpolation problem defined by (1), (2) where Λ is anti-stable ($\mathrm{Re}\, l_i>0$) and additionally S must be passive, i.e. $m=l$ and

1. $G(s)$ is analytic in $\operatorname{Re} s > 0$,
2. $G(s) + G^T(-s) \geq 0$, for all s such that $\operatorname{Re} s > 0$.

According to the positive real lemma, the following LMI

$$\begin{bmatrix} A^H P + PA & PB - C^H \\ B^H P - C & -(D + D^H) \end{bmatrix} \leq 0, \quad P > 0 \tag{10}$$

must be feasible. For simplicity we shall take $n = N$ and $W = I_N$ in (1), (2), hence

$$A = \Lambda - HC, \quad B = G - HD, \tag{11}$$

and consider two cases.

a) If $D = 0$ then $B = G$ and from (10) it results

$$C = B^H P = G^H P$$

and

$$(\Lambda - HC)^H P + P(\Lambda - HC) \leq 0, \quad P > 0.$$

The quadratic matrix inequality

$$\Lambda^H P + P\Lambda - PGH^H P - PHG^H P \leq 0. \tag{12}$$

reduces to the linear one

$$\Lambda Q + Q\Lambda^H - (GH^H + HG^H) \leq 0,$$

where $Q = P^{-1}$, therefore the following Lyapunov equation

$$\Lambda Q + Q\Lambda^H = GH^H + HG^H \tag{13}$$

must have a positive definite solution $Q > 0$.

Conversely, if (13) has a (unique) solution $Q > 0$ then the linear system

$$\begin{aligned} A &= \Lambda - HC, \quad B = G, \\ C &= G^H Q^{-1}, \quad D = 0 \end{aligned} \tag{14}$$

is (loseless) positive real and satisfies the interpolation conditions (1), (2).

Remark 3. *In the SISO case, when G and H are column vectors, we can take (see Remark 1) $H = [1 \ \ 1 \ \ \cdots \ \ 1]^T$ and the matrix Q is the Pick matrix Π given by*

$$\Pi = \begin{bmatrix} \dfrac{g_1 + \overline{g}_1}{l_1 + \overline{l}_1} & \cdots & \dfrac{g_1 + \overline{g}_n}{l_1 + \overline{l}_n} \\ \vdots & \vdots & \vdots \\ \dfrac{g_n + \overline{g}_1}{l_n + \overline{l}_1} & \cdots & \dfrac{g_n + \overline{g}_n}{l_n + \overline{l}_n} \end{bmatrix}. \tag{15}$$

b) Now let D be any matrix such that $D + D^H > 0$; in this case (10) is equivalent to

$$\begin{aligned} &A^H P + PA + \\ &+ (PB - C^H)(D + D^H)^{-1}(B^H P - C) < 0 \end{aligned} \tag{16}$$

where, in order to satisfy (1), (2), the pair (A, B) is given by (11). With little insight we shall take

$$C = (DH^H + G^H)P \tag{17}$$

and after some simple algebraic manipulations the same necessary and sufficient condition $Q > 0$ is obtained, where $Q = P^{-1}$ is given by (13). Clearly, if $D = 0$ then (11) and (17) reduce to (14), therefore by continuity (11) and (17) are valid for any D such that $D + D^H \geq 0$. In fact, we can take $D = G(\infty)$ if the value of $G(s)$ in the additional interpolation point $l_{N+1} = \infty$ is known.

4. COMPUTING PROCEDURE

Let the lower triangular $n \times n$ matrix M be the Cholesky factor of the positive definite matrix Q, i.e. $Q = MM^H$. From (13) we have

$$\Lambda MM^H + MM^H \Lambda^H = GH^H + HG^H \tag{18}$$

and, by left multiplying with M^{-1} and right multiplying with M^{-H}, (18) becomes

$$\tilde{\Lambda} + \tilde{\Lambda}^H = \tilde{G}\tilde{H}^H + \tilde{H}\tilde{G}^H \tag{19}$$

where

$$\tilde{\Lambda} = M^{-1}\Lambda M, \quad \tilde{G} = M^{-1}G, \quad \tilde{H} = M^{-1}H. \tag{20}$$

From (11) and (17), where $P = Q^{-1}$, we obtain

$$\begin{aligned} \tilde{A} &= M^{-1}\Lambda M - M^{-1}HCM = \tilde{\Lambda} - \tilde{H}\tilde{C}, \\ \tilde{B} &= M^{-1}G - M^{-1}HD = \tilde{G} - \tilde{H}D, \\ \tilde{C} &= (DH^H + G^H)(MM^H)^{-1}M = \\ &= D(M^{-1}H)^H + (M^{-1}G)^H = D\tilde{H}^H + \tilde{G}^H, \\ \tilde{D} &= D. \end{aligned}$$

Therefore we can compute a passive linear system whose transfer matrix interpolates the given data by using the following numerical procedure.

Algorithm 1. Given the (complex) numbers l_i, $i = 1:n$, and (complex) $n \times l$ matrices H and G, the algorithm computes (if a solution exists) a positive real linear system $\tilde{S} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, a lower triangular matrix $\tilde{\Lambda}$ and the matrices $\tilde{H}$ and $\tilde{G}$, such that the (cover) conditions (1),

(2) are satisfied for $W = I_n$, i.e. $\tilde{\Lambda} = \tilde{A} + \tilde{H}\tilde{C}$, $\tilde{G} = \tilde{B} + \tilde{H}\tilde{D}$.

1. Compute the solution Q of the Lyapunov equation (13).
2. if Q is not positive definite
   1. print 'The problem has no solution'
   2. return
3. Compute the Cholesky factor M of Q.
4. Compute the transformed data $\tilde{\Lambda}$, $\tilde{G}$, $\tilde{H}$ according to (20).
5. Take D such that $D + D^H \geq 0$.
6. Compute the positive real linear system
   $\tilde{A} = \tilde{\Lambda} - \tilde{H}\tilde{C}$, $\tilde{B} = \tilde{G} - \tilde{H}D$
   $\tilde{C} = D\tilde{H}^H + \tilde{G}^H$, $\tilde{D} = D$.

Remark 4. *If in the above algorithm we take $D = 0$, then a loseless positive real system is obtained such that*
$A^H P + PA = 0, \quad PB = C^H$,
see (10), i.e.
$\tilde{A}^H + \tilde{A} = 0, \quad \tilde{B} = \tilde{C}^H$.

To sum up, by using the algorithm 1, the interpolation conditions are satisfied and the computed system is positive real for all D such that $D + D^H \geq 0$. The system can be written as

$$\dot{x}(t) = \left[\tilde{\Lambda} - \tilde{H}(D\tilde{H}^H + \tilde{G}^H)\right]x(t) + (\tilde{G} - \tilde{H}D)u(t) =$$
$$= \tilde{\Lambda}x(t) + \tilde{G}u(t) - \tilde{H}\left[(D\tilde{H}^H + \tilde{G}^H)x(t) + Du(t)\right],$$
$$y(t) = (D\tilde{H} + \tilde{G}^H)x(t) + Du(t) =$$
$$= \tilde{G}^H x(t) + D(u(t) + \tilde{H}^H x(t)).$$

5. COMPUTATIONAL DETAILS

We now present some details for an accurate implementation of the first four statements of the above algorithm. The best method to check if the matrix Q is positive definite is to see if the Cholesky factorization $Q = MM^H$ can be carried out to completion. For obvious numerical reasons, we will avoid the explicit computation of the matrix Q (in the SISO case of the Pick matrix).

Instead we shall compute the Cholesky factor M directly from the given data. To do this, let denote $A_0 = \Lambda$, and observe that from (18), written as

$$A_0 MM^H + MM^H A_0^H = GH^H + HG^H, \quad (21)$$

the equality for the (1,1) entries gives

$$l_1 m_{11}^2 + m_{11}^2 \bar{l}_1 = \sum_{j=1}^{l} (g_{1j}\bar{h}_{1j} + h_{1j}\bar{g}_{1j}).$$

If Q is positive definite then

$$a = \frac{\sum_{j=1}^{l} \mathrm{Re}(g_{1j}\bar{h}_{1j})}{\mathrm{Re}\, l_1} > 0$$

and, hence, m_{11} is a positive real number given by

$m_{11} = \sqrt{a}$.

Conversely, if $a \leq 0$, the matrix Q is not positive definite.

The equality (21) for $(i,1)$, $i = 2:n$ entries of the above matrix equation gives

$$m_{i1} = \frac{\sum_{j=1}^{l} (g_{ij}\bar{h}_{1j} + h_{ij}\bar{g}_{1j})}{m_{11}(l_i + \bar{l}_1)}, \quad i = 2:n.$$

We have computed the first column of the Cholesky factor M.

Now, to compute, in the same manner, the second column of M, let transform the equation (21) in

$$P_1(A_0 P_1^{-1} P_1 MM^H + MM^H P_1^{-H} A_0^H) P_1^H =$$
$$= P_1(HG^H + GH^H)P_1^H,$$

or

$$A_1 M_1 M_1^H + M_1 M_1^H A_1^H = H_1 G_1^H + G_1 H_1^H,$$

where

$$A_1 = P_1 A_0 P_1^{-1}, \qquad M_1 = P_1 M,$$
$$H_1 = P_1 H, \qquad G_1 = P_1 G,$$

and $P_1 = I_n - p_1 e_1^T$, is a gaussian elementary lower triangular transformation such that

$(P_1 M)(i,1) = 0, \quad i = 2:n,$

i.e. the column vector p_1 is defined by $p_1 = \begin{bmatrix} 0 & p_{21} & \cdots & p_{n1} \end{bmatrix}^T$ with

$$p_{i1} = \frac{m_{i1}}{m_{11}}, \quad i = 2:n.$$

Because M and P_1 and $P_1^{-1} = I + p_1 e_1^T$ are triangular, the matrices A_1 and M_1 have the structure

$$A_1 = \begin{bmatrix} \times & 0 \\ \times & A_0(2:n,2:n) \end{bmatrix},$$

$$M_1 = \begin{bmatrix} \times & 0 \\ 0 & M(2:n,2:n) \end{bmatrix},$$

with $\times$ denoting generic nonzero entries.
Hence, retaining only the block $(2:n,2:n)$ equality of (25), we have

$$\hat{A}_0\hat{M}\hat{M}^H + \hat{M}\hat{M}^H\hat{A}_0^H = \hat{H}\hat{G}^H + \hat{G}\hat{H}^H, \quad (26)$$

where we have denoted

$$\hat{A}_0 = A_0(2:n,2:n), \quad \hat{M} = M(2:n,2:n),$$
$$\hat{H} = H_1(2:n,:), \quad \hat{G} = G_1(2:n,:).$$

This equation is structurally identical with (25) but of less order $n-1$. So we can compute, in the same manner as above, the first column of $\hat{M}$ which is the nonzero part of the second column of M.

By obvious induction arguments, we can continue the procedure to finally compute the Cholesky factor M and the lower triangular matrix A_{n-1}, the diagonal matrix M_{n-1} and the matrices H_{n-1}, G_{n-1} which satisfy

$$A_{n-1}M_{n-1}M_{n-1}^H + M_{n-1}M_{n-1}^H A_{n-1}^H = \\ = H_{n-1}G_{n-1}^H + G_{n-1}G_{n-1}^H, \quad (24)$$

where

$$A_{n-1} = PA_0P^{-1}, \quad M_{n-1} = PM = diag(M),$$
$$H_{n-1} = PH, \quad G_{n-1} = PG,$$

and P is a unit lower triangular matrix given by

$$P = P_{n-1}\cdots P_2P_1.$$

where

$$P_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & P_1^{(k)} \end{bmatrix},$$

with $P_1^{(k)}$ the $(n-k+1)\times(n-k+1)$ gaussian transformation matrix used at the step k.

We can go further and compute a lower triangular matrix $\tilde{A}$ and the matrices $\tilde{H}$ and $\tilde{G}$ so that the Lyapunov equation

$$\tilde{A}Q + Q\tilde{A}^H = \tilde{H}\tilde{G}^H + \tilde{G}\tilde{H}^H,$$

has the solution $Q = I_n$, i.e.

$$\tilde{A} + \tilde{A}^H = \tilde{H}\tilde{G}^H + \tilde{G}\tilde{H}^H.$$

To do this, observe that if

$$D = M_{n-1} = diag(M),$$

then

$$\tilde{A} = D^{-1}A_{n-1}D,$$
$$\tilde{H} = D^{-1}H_{n-1}, \quad \tilde{G} = D^{-1}G_{n-1}.$$

Hence

$$\tilde{a}_{ij} = \frac{A_{n-1}(i,j)m_{jj}}{m_{ii}}, \quad i = 1:n, \quad j = 1:i,$$

$$\left.\begin{aligned} \tilde{h}_{ij} &= \frac{H_{n-1}(i,j)}{m_{ii}} \\ \tilde{g}_{ij} &= \frac{G_{n-1}(i,j)}{m_{ii}} \end{aligned}\right\}, \quad i = 1:n, \quad j = 1:l.$$

Using the array A for the matrices A_k, $k = 0:n-1$ and $\tilde{A}$ and the arrays H and G for the matrices H_k, G_k, $k = 0:n-1$ and $\tilde{H}$, $\tilde{G}$, the computations are as follows.

Algorithm 2. Given the set $\{l_1, l_2, \ldots, l_n\}$ and the $n\times l$ matrices H and G, the algorithm computes the Cholesky factor M of the solution Q of the Lyapunov equation (13) if the matrix Q is positive definite. Otherwise an error message is furnished. Also, the algorithm computes the lower triangular matrix

$$A \leftarrow \tilde{A} = M^{-1}diag(l_1, l_2, \ldots, l_n)M$$

and overwrites

$$H \leftarrow \tilde{H} = M^{-1}H, \; G \leftarrow \tilde{G} = M^{-1}G.$$

1. for $i = 1:n$
    1. for $j = 1:n$
        1. $a_{ij} = 0$.
    2. $a_{ii} = l_i$
2. for $k = 1:n$
    1. $a = \dfrac{\sum_{j=1}^{l}\mathrm{Re}(g_{1j}\bar{h}_{1j})}{\mathrm{Re}\, l_k}$
    2. if $a < 0$
        1. print 'The matrix Q is not positive definite.'
        2. return
    3. $m_{kk} = \sqrt{a}$
    4. for $i = k+1:n$
        1. $m_{ik} = \dfrac{\sum_{j=1}^{l}(g_{ij}\bar{h}_{kj} + h_{ij}\bar{g}_{kj})}{m_{kk}(l_i + \bar{l}_k)}$

2. $t = \dfrac{m_{ik}}{m_{kk}}$

3. $a_{ik} = -t\, l_i$

4. for $j = 1:k$

 1. $a_{ij} = a_{ij} + t a_{kj}$

5. for $j = 1:l$

 1. $h_{ij} = h_{ij} + t h_{kj}$

 2. $g_{ij} = g_{ij} + t g_{kj}$

3. for $i = 1:n$

 1. for $j = 1: i-1$

  1. $a_{ij} = \dfrac{a_{ij} m_{jj}}{m_{ii}}$

 2. for $j = 1:l$

  1. $h_{ij} = \dfrac{h_{ij}}{m_{ii}}$

  2. $g_{ij} = \dfrac{g_{ij}}{m_{ii}}$

It is clear that this algorithm implements in a concise and reliable computational manner the first four statements of the rational interpolation procedure given in section 4. We have performed a lot of numerical experiments which confirm the good performances of all the proposed procedures.

6. DISCRETE CASE

In this case the main computational procedure remain valid if the discrete time system $S = (A, B, C, D)$ is converted to a continuous one, by using the well known bilinear transformation. Also, the computational improvements presented in section 3 have obvious correspondents.

7. CONCLUDING REMARKS

The numerical results confirm our computational oriented approach to the rational interpolation with passivity constraints. The direct computing of Cholesky factor and the other numerical improvements significantly the results accuracy.
The future author's investigations will be oriented on the possible applications of the rational interpolation problem, such as linear systems order reduction.

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