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Stability of a Class of Switched Linear Systems with Infinite Number of Subsystems

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Abstract: This paper proposes a new approach for globally uniformly asymptotically stability (GUAS) analysis. This method applies for discrete-time switched linear systems with infinite number of switching. A sufficient condition for GUAS of switched linear systems with infinite subsystem is proposed as a theorem.

Keywords: Global uniform asymptotic stability, Infinite number of subsystem, switched linear System.

1. INTRODUCTION

A switched system is a dynamical system that consists of finite or infinite number of subsystems and a logical rule that orchestrates switching between these subsystems. Mathematically, these subsystems are usually described by a collection of indexed differential or difference equations. One convenient way to classify switched systems is based on the dynamics of their subsystems, for example continuous-time or discrete-time, linear or nonlinear and so on (Lin and Antsaklis, 2009). In the recent years many researchers have been investigated on stability of switching systems. Lin and Antsaklis reviewed some results on stability and stabilizability of switched linear systems. They outlined briefly some necessary and sufficient conditions for the asymptotic stability of switched linear systems under arbitrary switching then they explored necessary and sufficient condition for the switching stabilizability of continuous-time switched linear systems.

Also they proved a necessary and sufficient condition for asymptotic stabilizability of switched linear systems (Lin and Antsaklis, 2009). Cheban et al. proposed absolute asymptotic stability of discrete linear inclusions in Banach (both finite and infinite dimensional) space also they established the relation between absolute asymptotic stability, asymptotic stability, uniform asymptotic stability and uniform exponential stability. They proved that for asymptotical compact discrete linear inclusions the concepts of asymptotic stability and uniform exponential stability are equivalent (Cheban and Mammana, 2005).

(Muller *et al.*, 2010) considered the concept of state-norm estimators for switched nonlinear systems under average dwell-time switching signals. State-norm estimators are closely related to the concept of input/output-to-state stability (IOSS). They showed that if the average dwell-time is large enough, there exists a nonswitched state-norm estimator for a switched system which each of its constituent subsystems is IOSS (Muller *et al.*, 2010). (Liberzon *et al.*, 2009) studied linear switched differential algebraic equations (DAEs), i.e., systems defined by a finite family of linear DAE subsystems and a switching signal that governs the switching between them. They showed by examples that switching between stable subsystems may lead to instability and that the presence of algebraic constraints lead to a larger variety of possible instability mechanisms compared to those observed in switched systems described by ordinary differential equations (ODEs). They prove two sufficient conditions for stability of switched DAEs based on the existence of suitable Lyapunov functions (Liberzon *et al.*, 2009).

(Agrachev *et al.*, 2010) presented new sufficient conditions for exponential stability of switched linear systems under arbitrary switching, which involve the commutator (Lie brackets) among the given matrices generating the switched system. Their proposed stability criteria was robust with respect to small perturbations of the system parameters. They investigated both disctere and continuous switched linear systems (Agrachev *et al.*, 2010). (Hien *et al.*, 2009) investigated the problem of exponential stability and stabilization of switched linear time-delay systems which the system parameter uncertainties was time-varying and unknown but they was norm-bounded. The delay in the system states was also time-varying. They designed a switching rule for the exponential stability and stabilization By using an improved Lyapunov–Krasovskii functional by using of the solution of Riccati-type equations (Hien *et al.*, 2009). (Raouf *et al.*, 2009) proposed a new sufficient condition that guarantees global exponential stability of switched linear systems based on Lyapunov-Metzler inequalities. The condition relies on the solution of a set of bilinear matrix inequalities (BMI) (Raouf *et al.*, 2009).

(Hante *et al.*, 2011) considered switched systems on Banach and Hilbert spaces governed by strongly continuous one-parameter semi groups of linear evolution operators. They provided necessary and sufficient conditions for their global exponential stability, uniform with respect to the switching signal with arbitrary switching, in terms of the existence of a Lyapunov function common to all modes (Hante *et al.*, 2011).

(Du *et al.*, 2009) proposed finite-time stability and stabilization problems for switched linear systems. They extended the concept of finite-time stability to switched linear systems, and then they represented a necessary and sufficient condition for finite-time stability of switched linear systems based on the state transition matrix of the system. They designed state feedback controllers and a class of switching signals with average dwell-time to stabilize the switched linear control systems (Du *et al.*, 2009).

(Zhai *et al.*, 2007) studied stability and L_2 gain properties for a class of switched systems that are composed of normal discrete-time subsystems. Thay showed that When all subsystems are Schur stable, a common quadratic Lyapunov function exists for all subsystems and then the switched normal system is exponentially stable under arbitrary switching.

As regards L_2 gain analysis, they introduced an expanded matrix including each subsystem's coefficient matrices, and they showed that if the expanded matrix is normal and Schur stable so that each subsystem is Schur stable and has unity L_2 gain, then the switched normal system also has unity L_2 gain under arbitrary switching (Zhai *et al.*, 2007).

(Santarelli *et al.*, 2010) designed a switched feedback controller for second order switched systems and then developed the switched state feedback control law for the stabilization of LTI systems of arbitrary dimension. They proposed switched state feedback control law by examining relevant geometric properties of the phase portraits in the case of two-dimensional systems for the stabilization of LTI systems of arbitrary dimension. The control law operates by switching between two static gain vectors in such a way that the state trajectory is driven onto a stable (n-1) dimensional hyper plane (where n represents the system dimension) then they derived a necessary and sufficient conditions to ensure stabilizability of the resulting switched system and they applied their new control condition to the problem of minimizing the maximal Lyapunov exponent of the corresponding closed-loop state trajectories (Santarelli *et al.*, 2010).

(Zhai *et al.*, 2010) proposed a unified approach to stability analysis for switched linear descriptor systems under arbitrary switching in both continuous-time and discrete-time domains. The approach is based on common quadratic Lyapunov functions incorporated with Linear Matrix Inequalities (LMIs). They show that if there exists a common quadratic Lyapunov function for the stability of all subsystems, then the switched system is stable under arbitrary switching (Zhai *et al.*, 2010).

In (Araghi *et al.*, 2009) the authors investigated the stability of switched linear systems then proposed three methods for existence of a common quadratic Lyapunov function for robust stability analysis of fuzzy Elman neural network. These methods have been considered using stabilizing state feedback control in closed loop switching system.

In (Suratgar *et al.*, 2002; Suratgar *et al.*, 2003) the authors proposed some theorems for stability analysis of TSK and linguistic fuzzy models. (Guo *et al.*, 2012) investigated the stability of a class of switched linear systems and they proposed a novel analysis method by using the 2-norm technique. Their proposed method guarantees the stability of the systems under arbitrary switching, and also provides an algorithm to find the minimum dwell time (MDT) with which switches make the switched systems stable (Guo *et al.*, 2012). In (Ratchagit *et al.*, 2012) proposed a switching design for the asymptotic stability of switched linear discrete-time systems with interval time-varying delays and he designed a switching rule for the asymptotic stability for the switched system via linear matrix inequalities. The system which he concerned was with the delay with a fast time-varying function and the lower bound was not restricted to zero. (Wang *et al.*, 2012) investigated the finite-time stability problem for a class of discrete-time switched linear systems with impulse effects. They established a sufficient condition which ensures that the state trajectory of the system remains in a bounded region of the state space over a pre-specified finite time interval. They showed that the total activation time of unstable subsystems can be greater than that of stable subsystems. In addition, the finite-time stability degree may be also greater than one (Wang *et al.*, 2012). (Su *et al.*, 2012) established a stability result for a class of linear switched systems involving Kronecker product. This problem is interesting in that the system matrix does not have to be Hurwitz at any time instant. As applications of this stability result, they proposed the solvability conditions for both the leaderless and the leader-following consensus problem for general marginally stable linear multi-agent systems under switching network topology. Their results only assume that the dynamic graph is uniformly connected (Su *et al.*, 2012). (Liu *et al.*, 2012) concerned with the stability problem of discrete-time positive switched linear systems with delays. The states of the systems under consideration are confined in the positive orthant, and the delays can be time-varying and not necessarily bounded. They established a delay independent stability criterion for switched linear systems with delays systems (Liu *et al.*, 2012). (Kermani *et al.*, 2012) investigated new stability conditions for discrete-time switched linear systems based on overvaluing systems built on vector norms and the application of Borne-Gentina criterion. This stability conditions issued from vector norms correspond to a vector Lyapunov function. In fact, the switched system to be controlled will be represented in the Companion form. A comparison system relative to a regular vector norm was used in order to get the simple arrow form of the state matrix that yields to a suitable use of Borne-Gentina criterion for the establishment of sufficient conditions for global asymptotic stability (Kermani *et al.*, 2012). Also they studied the stability and stabilization problems for continuous-time switched linear systems and they stablished a new stability conditions based on the comparison, the overvaluing principle, the application of Borne-Gentina criterion and the Kotelyanski conditions (Kermani *et al.*, 2012). In (Cimochowski *et al.*, 2012) studied the positive switched discrete-time linear systems. He eslablished a new necessary and sufficient conditions for asymptotic stability of the positive linear discrete-time switched systems with delays in states. The result of his proposal was that asymptotic stability of the switched system is equivalent to asymptotic stability of the corresponding positive discrete-time switched system without delays.

In the recent years there are many articles about stability of systems with finite number of switching but in this paper stability analysis of discrete-time switched linear systems with infinite number of switching is investigated.

2. PROBLEM STATEMENT

Consider a set Σ of matrices A_{i_k} and pick an initial point x_0, at t = 0. A switched linear system is a dynamical system of the type:

$$x_{k+1} = A_{i_k} x_k \ , A_{i_k} \in \Sigma, i_k \in I, k \in \mathbb{Z}^+ \tag{1}$$

Where I is an *infinite* index set, i_k is switching law and the state $x \in \mathbb{R}^n$ and $A_{i_k} \in \mathbb{R}^{n\times n}$.

This notation means that at every instant, the matrix A_{i_k} defining the evolution of the system can be replaced by another one from of the set Σ.

The stability of switched system when there is no restriction on the switching signals is usually called stability analysis under arbitrary switching. For this analysis, it is necessary that all the subsystems are asymptotically stable. However, even when all the subsystems of a switched system are exponentially stable, it is still possible to construct a divergent trajectory from any initial condition. Therefore, in general, the assumption of subsystems' stability is not sufficient to assure stability of switched systems under arbitrary switching, except for some special cases, such as pairwise commutative systems, symmetric or normal systems (all subsystems). Consequently, if there exists a common Lyapunov function for all the subsystems, then the stability of the switched system is guaranteed under arbitrary switching (Lin and Antsaklis, 2009).

Let us recall the following definition and lemma that will be helpful through the rest of the paper.

Definition 1. The linear switched system (1) is globally uniformly asymptotically stable (GUAS) if for any initial condition $x_0 \in \mathbb{R}^n$ and any switching law i_k, (Monovich *et al.*, 2011).

$$\lim_{k\to\infty} x_k = 0, \qquad \forall (i_k)$$

As this is supposed to hold for any initial vector x_0, it is equivalent to saying that all matrix products taken from Σ converge to the zero matrix, i.e.,

$$\lim_{k\to\infty} A_{i_k} A_{i_{k-1}} \dots A_{i_1} = 0, \ \forall (i_k)$$

The GUAS problem is closely related to determining the joint spectral radius (JSR) of the set of matrices $\Sigma = \{A_1, \dots, A_n\}$, denoted by $\rho(\Sigma)$ (Monovich *et al.*, 2011).

The formal definition of the joint spectral radius was first introduced by Rota and Strang in the 60's. In the 90's, Daubechies & Lagarias definened the generalized spectral radius, and Berger & Wang proved later these two values to be equal for bounded sets of matrices (Jungers *et al.,* 2008).

The joint spectral radius characterizes the maximal asymptotic growth rate of a point submitted to a switching linear system in descrete time. The maximal growth rate one can ensure the stability of the system, provided that this growth rate is less than one (Jungers *et al.,* 2008).

Let $\|.\|: \mathbb{R}^n \to \mathbb{R}_+$ denote the Euclidean vector norm and denote (Hartfiel, 2002)

$$\rho_k(\Sigma) = \max\left\{ \|A_{i_1} A_{i_2} \dots A_{i_k}\|^{\frac{1}{k}}, \ i_j \in \{0, 1, \dots, n\} \right\}. \tag{2}$$

Then the joint spectral radius is defined as,

$$\rho(\Sigma) = \lim_{k\to\infty} \rho_k(\Sigma).$$

And let,

$$\hat{\rho}_k(\Sigma) = \sup\left\{ \rho(A_{i_1} A_{i_2} \dots A_{i_k})^{\frac{1}{k}}, \qquad i_j \in \{0, 1, \dots, n\} \right\}.$$

The generalized spectral radius is defined as,
$\hat{\rho}(\Sigma) = \lim_{k\to\infty} \hat{\rho}_k(\Sigma)$.

For bounded set of matrices the joint spectral radius and the generalized spectral radius are equal.

In general (Hartfiel, 2002),
$\hat{\rho}_k(\Sigma)^{\frac{1}{k}} \le \hat{\rho}(\Sigma) \le \rho(\Sigma) \le \rho_k(\Sigma)^{\frac{1}{k}}$

The switched system (1) is GUAS if and only if $\rho(\Sigma) < 1$ (Monovich *et al.*, 2011).

Some results show that computing or even approximating the JSR is extremely hard (Jungers *et al.,* 2008). In this paper we propose matrix structure for the subsystems in switched linear systems that the stability of switched system is guaranteed.

Through the rest of the paper, the discrete-time switched linear system is considered as follow:

$$x(k+1) = \Sigma_k \, x(k), \ k = 1, 2, \dots \tag{3}$$

where $\Sigma_k \in \Sigma$, $\Sigma = \{\Sigma_1, \Sigma_2, \dots\}$ such that,

$$\Sigma_k = \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix} \tag{4}$$

and k is an infinite index set, k = 1,2,... , $[A_k] \in R^{n_1\times n_1}$, $[B_k] \in R^{n_1\times n_2}$ and $[C_k] \in R^{n_2\times n_2}$, the state $x \in \mathbb{R}^n, \Sigma_k \in \mathbb{R}^{n\times n}$.

If the joint spectral radius of system (3), $\rho(\Sigma) < 1$, then the dynamical system is stable, because $x_k = \Sigma^k x_0$, where $\Sigma^k \triangleq \{\Sigma_1 \dots \Sigma_k\}$ and so $\|x_k\| \le \|\Sigma^k\| |x_0| \to 0$ (Jungers *et al.,* 2008).

First we prove the following lemma.

Lemma 1. Let $\{A_k\}_{k\in\mathbb{N}}$ be a set of square matrices. If there exist $\alpha < 1$ such that for all $k \in \mathbb{N}$, $\|A_k\| \leq \alpha$, Then,

$\lim_{k\to\infty} A_k A_{k-1} \dots A_1 = 0$.

Proof:
From the submultiplicity property of norms,

$\|AB\| \leq \|A\|\|B\|$

the following equation is correct:

$\|A_k A_{k-1} \dots A_1\| \leq \|A_k\|\|A_{k-1}\| \dots \|A_1\|$

Since, $\forall\, k \in \mathbb{N}$, $\|A_k\| \leq \alpha$, therefore,

$\lim_{k\to\infty} \|A_k\|\|A_{k-1}\| \dots \|A_1\| \leq \lim_{k\to\infty} \alpha^k \to 0$

And consequently,

$\lim_{k\to\infty} \|A_k A_{k-1} \dots A_1\| \to 0$ then $\lim_{k\to\infty} A_k A_{k-1} \dots A_1 \to 0$

3. INFINITE PRODUCT OF MATRICES

Theorem1. Let $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of Matrices of the form (4) and let there exist two numbers α and γ such that $\gamma < 1$, $\alpha < 1$ and $\|A_i\| \leq \alpha$, $\|C_i\| \leq \gamma$ for some matrix norm $\|.\|$. The sequence $P_k = \Sigma_1\Sigma_2 \dots \Sigma_k$ (infinite product of matrices) converges to zero if and only if

$A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}$ converges to zero.

Proof:
To prove the sufficient condition, by construction P_k from Σ_k as follow:

$P_k = \Sigma_1\Sigma_2 \dots \Sigma_k, \quad \Sigma_k \in \Sigma, \quad k \in \mathbb{N}$

$$P_k = \begin{bmatrix} A_1A_2 \dots A_k & X_k \\ 0 & C_1C_2 \dots C_k \end{bmatrix} \tag{5}$$

Where

$$X_k = A_1A_2 \dots A_{k-1}B_k + A_1A_2 \dots A_{k-2}B_{k-1}C_k + A_1A_2 \dots A_{k-3}B_{k-2}C_{k-1}C_k + \cdots \tag{6}$$

By hypothesis of the theorem and the Lemma1 since $\|A_i\| \leq \alpha < 1$, $\|C_i\| \leq \gamma < 1$,

$\lim_{k\to\infty} A_1A_2 \dots A_k = \lim_{k\to\infty} C_1C_2 \dots C_k = 0$.

Therefore $\lim_{k\to\infty} P_k = 0$ implies that $\lim_{k\to\infty} X_k \to 0$ and therefore,

$$\lim_{k\to\infty} X_k - X_{k-1} = 0 \tag{7}$$

By some calculations, between X_k and X_{k-1}, the following relation is achieved:

$X_k = X_{k-1}C_k + A_1A_2 \dots A_{k-1}B_k$

By substraction X_{k-1}, the above equation is:

$X_k - X_{k-1} = X_{k-1}C_k + A_1A_2 \dots A_{k-1}B_k - X_{k-1}$
$= X_{k-1}(C_k - I) + A_1A_2 \dots A_{k-1}B_k$

Because of $\|C_i\| \leq \gamma < 1$, therefore $|C_k - I| \neq 0$ and $(C_k - I)$ is invertible. So,

$(X_k - X_{k-1})(C_k - I)^{-1} = X_{k-1} + A_1A_2 \dots A_{k-1}B_k(C_k - I)^{-1}$

$$(X_k - X_{k-1})(I - C_k)^{-1} = A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1} - X_{k-1} \tag{8}$$

From (7) and (8);

$\lim_{k\to\infty} X_k = \lim_{k\to\infty} A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}$

and concequently,

$$\lim_{k\to\infty} P_k = \begin{bmatrix} 0 & \lim_{k\to\infty} A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1} \\ 0 & 0 \end{bmatrix}$$

Namlely; P_k converges to zero if $A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}$ converges to zero.

To prove the necessary condition of the theorem, it must be proved that if $\lim_{k\to\infty} A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}$ converges to zero then $\lim_{k\to\infty} P_k = 0$.

To prove this, first it must be proved that:

$\lim_{k\to\infty} \|X_k\| = \lim_{k\to\infty} \|A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}\|$

By defining D_k as the difference between X_k in (6) and $A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}$,

$$D_k = X_k - A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1};\ k \in \mathbb{N} \tag{9}$$

and by defining Y_k as:

$$Y_k = A_1A_2 \dots A_kB_{k+1}(I - C_{k+1})^{-1} - A_1A_2 \dots A_{k-1}B_k(I - Ck-1 \tag{10}$$

From (9) and (10) it is obtained that,

$D_{k+1} = X_{k+1} - A_1A_2 \dots A_kB_{k+1}(I - C_{k+1})^{-1} = (D_k - Y_k)C_{k+1}$

and thus,

$\|D_{k+1}\| \leq (\|D_k\| + \|Y_k\|)\|C_{k+1}\|$, such that $\|C_i\| \leq \gamma$,

So,
$\|D_{k+1}\| \leq (\|D_k\| + \|Y_k\|)\gamma$,

$\|D_k\| \leq (\|D_{k-1}\| + \|Y_{k-1}\|)\gamma$,

$\|D_{k-i}\| \leq (\|D_{k-i-1}\| + \|Y_{k-i-1}\|)\gamma$,

By repeating the above inequalities it is obtained:

$\|D_k\|$
$\leq \|D_{k-i}\|\gamma^i + \|Y_{k-i}\|\gamma^i + \|Y_{k-i+1}\|\gamma^{i-1} + \cdots + \|Y_{k-1}\|\gamma^1$

and as a result,
$\|D_k\| \leq \|D_1\|\gamma^{k-1} + \sum_{i=1}^{k-1}\|Y_{k-i}\|\gamma^i$, $i = 1, 2, \dots, k-1$

Therefore,

$$\lim_{k\to\infty}\|D_k\| \leq \lim_{k\to\infty} \sum_{i=1}^{k-1}\|Y_{k-i}\|\gamma^i,\ 0 \leq \gamma < 1 \tag{11}$$

Because of $\lim_{k\to\infty}\|D_1\|\gamma^{k-1} = 0$.

By considering $S = \lim\ \sup_{k\to\infty}\|D_k\| < \infty$ and since $\lim_{k\to\infty} Y_k = 0$, and therefore,

$\lim\ \sup_{k\to\infty}\|D_k\| \leq \lim\ \sup_{k\to\infty} \sum_{i=1}^{k-1}\|Y_{k-i}\|\gamma^i$,
and consequently,

$S \leq 0 \Longrightarrow S = 0$.
The above equation means that, $\lim_{k\to\infty} D_k = 0$ and from (9),

$\lim_{k\to\infty} X_k = \lim_{k\to\infty} A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}, k \in \mathbb{N}$

As a result if $\lim_{k\to\infty} \|A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1}\| = 0$ then $\lim_{k\to\infty} \|X_k\| = 0$ also By hypothesis of the theorem and the lemma1,

$\lim_{k\to\infty} A_1A_2 \dots A_k = \lim_{k\to\infty} C_1C_2 \dots C_k = 0,$

Therefore,

$\lim_{k\to\infty} P_k = 0.$

So it completes the proof.

4. STABILITY OF SWITCHED LINEAR WITH INFINITE NUMBER OF SUBSYSTEM

Theorem 2. Let $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of switching system of the form (4) with, $\|A_i\| \le \alpha < 1$, $\|C_i\| \le \gamma < 1$. The discrete time switched linear system (3) with infinite number of switching system is GUAS under arbitrary switching if $\lim_{k\to\infty} A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1} = 0$.

Arbitrary switching refers to switched systems that there are no restrictions on the discrete event dynamics.

Proof: By definition1 and theorem1, any infinite product of this kind of switched linear system under these conditions converges to 0 so $\rho(\Sigma) < 1$ and the switches linear system is GUAS.

Lemma2. Suppose the discrete-time switched linear system (3) where $B_k = 0$. The discrete-time switched linear system (3) is GUAS if and only if $\{A_1, A_2, \dots A_k, \dots\}$,$\{C_1, C_2, \dots C_k, \dots\}$ are GUAS.

Proof. Let $\{\Sigma_k\}$ be of the form $\begin{bmatrix} A_k & 0 \\ 0 & C_k \end{bmatrix}$ where $[A_k] \in R^{n_1\times n_1}$ and $[C_k] \in R^{n_2\times n_2}$.

Then,

$P_k = \Sigma_1\Sigma_2 \dots \Sigma_k = \begin{bmatrix} A_1A_2 \dots A_k & 0 \\ 0 & C_1C_2 \dots C_k \end{bmatrix}$

that the state $x \in \mathbb{R}^n$, $\Sigma_k \in \mathbb{R}^{n\times n}$. So

$\lim_{k\to\infty} P_k \to 0$ if and only if,

$\lim_{k\to\infty} A_1A_2 \dots A_k \to 0$ and $\lim_{k\to\infty} C_1C_2 \dots C_k \to 0$.

So in fact the discrete- time switched linear system (3) with $B_k = 0$ is GUAS if and only if $\{A_1, A_2, \dots A_k, \dots\}$,$\{C_1, C_2, \dots C_k, \dots\}$ are GUAS.

Result1. The discrete time switched linear system (3) with infinite number of switching system with the form (4) is GUAS under arbitrary switching if there exist two numbers α and γ such that, $\|A_i\| \le \alpha < 1$, $\|C_i\| \le \gamma < 1$ and $\|B_i\| \le \beta < \infty$ is bounded.

Result2. Because of $\rho(.) \le \|.\|$, the discrete time switched linear system (3) with infinite number of switching system with the form (4) is GUAS under arbitrary switching if $\rho(A_i) < 1$, $\rho(C_i) < 1$ and $\|B_i\|$ is bounded.

Example1. Consider switched linear system (3) with two subsystems of the form (4) and let:

$$\Sigma_k = \left\{ \begin{bmatrix} -.1 & .3 & 5 \\ -.5 & .1 & 2 \\ 0 & 0 & .8 \end{bmatrix}, \begin{bmatrix} -.4 & -.2 & 4 \\ -.5 & .3 & 5 \\ 0 & 0 & .2 \end{bmatrix} \right\}, K = 1, 2.$$

Because of

$\|A_1\| = \left\| \begin{matrix} -.1 & .3 \\ -.5 & .1 \end{matrix} \right\| < 1, \|A_2\| = \left\| \begin{matrix} -.4 & -.2 \\ -.5 & .3 \end{matrix} \right\| < 1;$

$\|C_1\| = \|.8\| < 1, \|C_2\| = \|.2\| < 1$ and

$\|B_1\| = \left\| \begin{matrix} 5 \\ 2 \end{matrix} \right\| < \infty, \|B_2\| = \left\| \begin{matrix} 4 \\ 5 \end{matrix} \right\| < \infty,$

the switched linear system under arbitrary switching is GUAS.

Figure 1, shows simulation result of example 1.

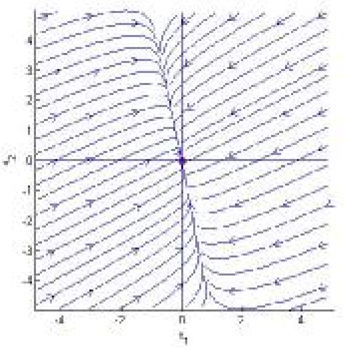


Fig. 1. Phase portrait of switched linear system with two subsystems.

Example2. Consider a discrete time switched linear system of the form 4 that,

$$\Sigma_k = \left\{ \begin{bmatrix} \frac{1}{3k} & \cos(k) & 4 \\ .5 & \frac{1}{4k} & 3\sin(k) \\ 0 & 0 & \frac{1}{2k} \end{bmatrix} \right\}, K = 1, 2, \dots$$

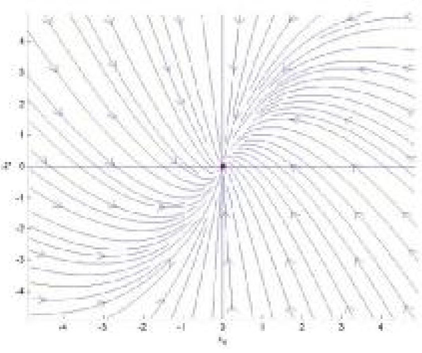
According to the result2 this infinite switched system is stable because

$$\rho(A_k) = \rho\left(\begin{bmatrix} \frac{1}{3k} & \cos(k) \\ .5 & \frac{1}{4k} \end{bmatrix} \right) < 1,$$

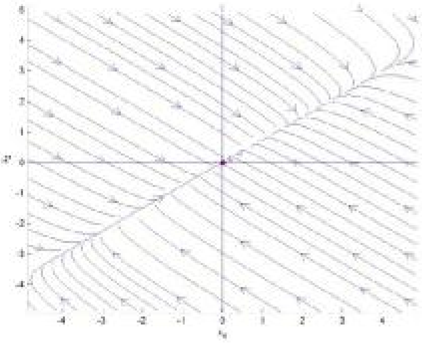
$\rho\left(\frac{1}{2k}\right) < 1, \left\| \begin{matrix} 4 \\ 3\sin(k) \end{matrix} \right\| < \infty,\ for\ k = 1,2, \dots .$

So $\lim_{k\to\infty} A_1A_2 \dots A_{k-1}B_k(I - C_k)^{-1} = 0$ for all $k \in \mathbb{N}$ and switch linear system is GUAS under arbitrary switching.

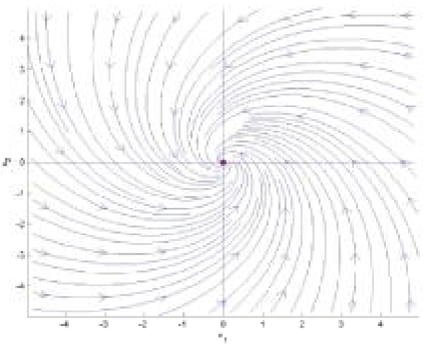
Figure 2 shows some simulation results for k=10, 100, 1000 and 2000 subsystems that switch.



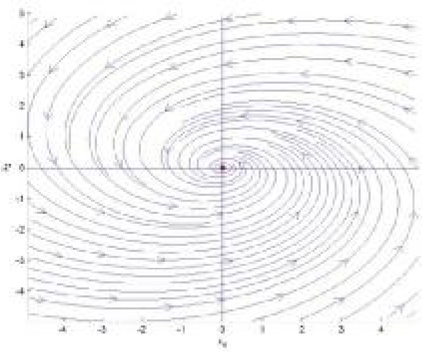
a) K=10



b) K=100



c) K=1000



d) K=2000

Fig. 2. Phase portrait and switching surfaces with 10, 100, 1000, 2000 subsystems

5. CONCLUSIONS

This paper proposed a new analytic method for globally uniformly asymptotically stability analysis of the linear switched systems with infinite number of switching. A sufficient condition based on the GUAS definition was established.

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Appendix A. Stability definitions

Consider the time-invariant dynamic system

$$\dot{x} = f(x),\ x \in \mathbb{R}^n \quad (A.1)$$

and let the initial time be $t_0 = 0$ without loss of generality.
The origin $x^* = 0$ is said to be a stable equilibrium point of (A.1) in the sense of Lyapunov, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \varepsilon, \forall t \geq 0 \quad (A.2)$$

In this case it also will be said simply that the system (A.1) is stable. Lyapunov stability does not require that trajectories starting close to the origin tend to the origin asymptotically.
The system (A.1) is called asymptotically stable if it is stable and δ can be chosen so that

$$\|x(0)\| \leq \delta \Rightarrow x(t) \to 0 \text{ as } t \to \infty \quad (A.3)$$

The set of all initial states from which the trajectories converge to the origin is called the region of attraction. If the condition (A.3) holds for all δ, i.e., if the origin is a stable equilibrium and its region of attraction is the entire state space, then the system (A.l) is called globally asymptotically stable.

If the system is not necessarily stable but has the property that all solutions with initial conditions in some neighborhood of the origin converge to the origin, then it is called locally attractive.

The system (A.1) is globally attractive if its solutions converge to the origin from all initial conditions.

Uniform stability is a concept which guarantees that the equilibrium pont is not losing stability.

Uniform asymptotic stability of the system (A.1) requires that $x^* = 0$ is uniformely stable and the convergence in equation (A.3) holds and is uniform.

The system (A.1) is called exponentially stable if there exist positive constants δ, c, and λ such that all solutions of $\dot{x} = f_\sigma(x)$ with $|x(0)| \leq \delta$ satisfy the inequality

$$|x(t)| \leq c|x(0)|e^{-\lambda t},\ \ \forall t \geq 0 \quad (A.4)$$

If this exponential decay estimate holds for all δ, the system is said to be globally exponentially stable. The constant λ in (A.4) is occasionally referred to as a stability margin (Liberzon, 2003).