Event-Based Control of the Inverted Pendulum: Swing up and Stabilization

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Abstract: Contrary to the classical (time-triggered) principle that calculates the control signal in a periodic fashion, an event-driven control is computed and updated only when a certain condition is satisfied. This notably allows to save computations in the control task while ensuring equivalent performance. In this paper, we develop and implement such strategies to control a nonlinear and unstable system, that is the inverted pendulum. We firstly propose to apply an event-based approach previously developed in Marchand et al. (2011, 2013) for the stabilization of the pendulum near its inverted position. We then study the swinging of the pendulum up to the desired position and especially design a (low computational cost) control law for this second case, based on an energy function. The switch between both strategies is also analyzed for stability reason. A real-time experimentation is realized and notably demonstrates the efficiency of the event-based schemes, even in the case where the system has to be actively actuated to remain upright. A reduction of about 98 % and 50 % of samples less than the classical scheme is achieved for the swing up and stabilization parts respectively, whereas the system performance remains the same (in terms of balancing and stabilizing time or control amplitude).

Keywords: Event-based control, cyber-physical system, inverted pendulum.

INTRODUCTION

A *cyber-physical system* is an integration of computing devices with physical processes. In practice, embedded computers and networks monitor and control some physical processes – usually with feedback loops – and these physical processes affect computations and communications, and vice versa. As a consequence, the intersection between physical and information-driven (cyber) functions represents a challenge and results in innovation, see Lee and Seshia (2011). For cyber-physical systems, the use of digital platforms emerges as an obvious trend to save space, weight and energy. However, digital implementations can result in additional challenges, like determining how frequently the control signal needs to be updated and applied such that the stability properties are still guaranteed. In the present paper in particular, we are interested in controlling the *inverted pendulum* system and we develop *event-based techniques* to reach such a reduction in the control updates.

While a pendulum is, by definition, a weight suspended from a pivot which can freely swing, an inverted pendulum is a pendulum whose mass is above its pivot point. As a result, whereas a normal pendulum is naturally stable, an inverted pendulum is inherently unstable and has to be actively balanced in order to remain upright and resistant to a disturbance. A common strategy used to achieve the expected behavior is to move the pivot point as part of a closed-loop feedback system. This problem involves a cart which is able to horizontally move and a pendulum placed on the cart such that its arm can freely move (in the same plan that the cart). The only way to balance the inverted pendulum then consists in applying an external control force to the system. This is done thanks to a DC servo-motor which provides the control force to the cart through a belt drive system. A digital controller allows to control the pendulum, simply acting on the motor. A potentiometer measures the cart position, from its rotation, while another one measures the angle of the pendulum. Their derivatives can also be deduced. The goal of the control law is to move the cart to a given position without causing the pendulum to tip over. This can be divided into two steps: ***i)*** a strategy swings the pendulum up to its upright position and, then, ***ii)*** another one stabilizes the pendulum near its (unstable) inverted position. The classical approach to realize the first part is based on using an energy function, like in Åström (1999); Åström and Furuta (2000); Yoshida (1999); Bradshaw and Shao (1996), whereas a dynamical state-feedback control calculated on the linearized model of the system can behave the second step, like in Stimac (1999); Bugeja (2003); Lam (2004); Campbell et al. (2008).

As long as the control of the inverted pendulum system is concerned, all proposed strategies were developed in a classical time-triggered and periodic fashion. Although periodicity simplifies the design and analysis, it results in a conservative usage of resources since the control law is computed and updated at the same rate regardless it is really required or not. In the present study case for instance, the controller actuates the cart during the swinging even while the energy of the pendulum naturally decreases, and yet, this is not useful. In the same idea, it is not necessary to actively control such an unstable system in order to remain upright in the stabilizing part. A discussion on these points follows in the sequel.

On the other hand, some recent works addressed resource-aware implementations of the control law using event-based sampling, where the control law is event-driven. Such a paradigm calls for resources whenever they are indeed necessary, that is for instance when the dynamics of the controlled system varies. Typical event detection mechanisms are functions of the state variation (or at least the output) of the system, like in Årzén (1999); Sandee et al. (2005); Durand and Marchand (2009); Sánchez et al. (2009b,a). Although the event-triggered control is well-motivated and allows to relax the periodicity of computations and communications, only few works report theoretical results about the stability, convergence and performance. In Åström and Bernhardsson (2002) notably, it is proved that such an approach reduces the number of sampling instants for the same final performance. Some stability and robustness proprieties are exploited in Åström and Bernhardsson (2002); Heemels et al. (2009); Lunze and Lehmann (2010); Donkers and Heemels (2010); Eqtami et al. (2010). An alternative approach consists in taking events related to the variation of a Lyapunov function – and consequently to the state too – between the current state and its value at the last sampling, like in Velasco et al. (2009), or in taking events related to the time derivative of the Lyapunov function, like in Tabuada (2007); Anta and Tabuada (2008); Marchand et al. (2011, 2013a). In the two latter references in particular, the updates ensure the strict decrease of the Lyapunov function, and so is asymptotically stable the closed-loop system.

In this paper, we propose to develop some event-based strategies to control an inverted pendulum, for both swinging up and then stabilizing its arm. The suggested stabilization technique is mainly based on the seminal works in Marchand et al. (2011, 2013a), but an event-based scheme is especially designed for the swinging part. Furthermore, such approaches have never been addressed in the literature for the inverted pendulum. Note that, whereas a single control law could be used for both control parts, we voluntarily propose two independent controllers because we think that a dedicated strategy allows to reduce much more the number of samples (and so the computational cost), in particular as regards the proposed event-based swinging method. In return, this implies to analyze the switch between both techniques for stability reason.

The rest of the document is organized as follows. In section 1, the model of the inverted pendulum is given and the event-based formulation is introduced. Also, the problem is stated and the control algorithms are intuitively presented. The main contributions are detailed in section 2. The stabilization and swing up strategies are respectively analyzed in subsections 2.1 and 2.2, and the switch from the one to the other is studied in subsection 2.3. Experimental results are presented in section 3 to highlight the capabilities of the proposed approaches and some discussions finally conclude the paper.

1. PRELIMINARIES AND PROBLEM STATEMENT

1.1 Model of the inverted pendulum

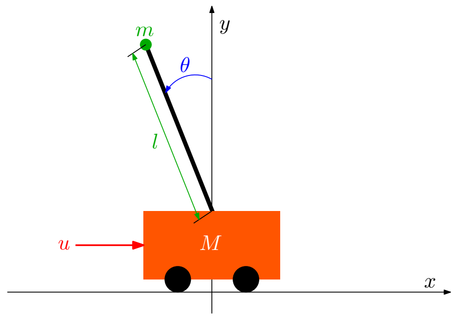


Fig. 1. Inverted pendulum system

The system of the present paper is depicted in Fig. 1, where an inverted pendulum is actuated via a cart (as explained in introduction). From this representation, the equations of motion of the pendulum and the cart are

$$I\ddot{\theta} + k\dot{\theta} - mgl\sin(\theta) - ml\ddot{p}\cos(\theta) = 0 \quad (1)$$

$$(M+m)\ddot{p} + f\dot{p} - ml\ddot{\theta}\cos(\theta) + ml\dot{\theta}^2\sin(\theta) = \rho u \quad (2)$$

where M is the mass of the cart, m is the mass of the pendulum and l is the distance from the pivot to the center of this mass, $I = J + ml^2$ where J is the moment of inertia with respect to the pivot point, g is the acceleration of gravity, f and k are the friction force and friction coefficient of the pendulum respectively. p is the position of the cart and u is a horizontal acceleration of the cart (the input), where p and u are positive if they are in the direction of the positive x-axis. Also, ρ is a parameter used to convert a voltage into a force applied on the cart. θ is the angle between the vertical and the pendulum, where θ is positive in the trigonometric direction and zero in the upright position This model is notably based on assuming that the pendulum is a rigid body and there is no limitation on the velocity of the pivot. One could refer to Åström and Furuta (2000) for further information.

Reformulating (1) and (2) gives the dynamics of the complete system

$$\begin{aligned} \ddot{p} &= \lambda_1(\theta)\Big[\kappa_1\lambda_3(\theta,\dot{\theta},\dot{p},u) + l\cos(\theta)\lambda_2(\theta,\dot{\theta})\Big] \\ \ddot{\theta} &= \lambda_1(\theta)\Big[l\cos(\theta)\lambda_3(\theta,\dot{\theta},\dot{p},u) + \kappa_2\lambda_2(\theta,\dot{\theta})\Big] \end{aligned} \quad (3)$$

$$\text{with} \left| \begin{aligned} &\lambda_1(\theta) := \frac{1}{\kappa_2 I - ml^2\cos^2(\theta)} \\ &\lambda_2(\theta,\dot{\theta}) := mgl\sin(\theta) - k\dot{\theta} \\ &\lambda_3(\theta,\dot{\theta},\dot{p},u) := \rho u - f\dot{p} - ml\dot{\theta}^2\sin(\theta) \end{aligned} \right.$$

$$\text{and} \left| \begin{aligned} &\kappa_1 := \frac{I}{m} \\ &\kappa_2 := \frac{M+m}{m} \end{aligned} \right.$$

which is a four-state system, whose states are the position of the cart p and the angle of the pendulum θ, as well as the velocity of the cart $\dot{p}$ and the angular speed $\dot{\theta}$. As a result, let

$$x := \begin{bmatrix}\theta \ \dot{\theta} \ p \ \dot{p}\end{bmatrix}^T \tag{4}$$

be the state vector of the system in the sequel.

Linearized version

Let consider the linear time-invariant dynamical system

$$\dot{x} = Ax + Bu \tag{5}$$

Such a linearized state-space representation of the pendulum close to an equilibrium point can be obtained from (3) with the state defined in (4). In fact, two equilibriums exist, that are when the pendulum is in its stable position (i.e. $\theta = \pi$) and when it is in the upright – and unstable – position (i.e. $\theta = 0$). We are interested in the latter one in the present paper, which yields the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{\kappa_2 mgl}{\kappa_3} & \dfrac{-\kappa_2 k}{\kappa_3} & 0 & \dfrac{-lf}{\kappa_3} \\ 0 & 0 & 0 & 1 \\ \dfrac{mgl^2}{\kappa_3} & \dfrac{-lk}{\kappa_3} & 0 & \dfrac{-\kappa_1 f}{\kappa_3} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{\rho l}{\kappa_3} \\ 0 \\ \dfrac{\rho\kappa_1}{\kappa_3} \end{bmatrix} \tag{6}$$

with $\kappa_3 := \kappa_2 I - ml^2$

This linearized model will be used in the sequel for the stabilizing part.

1.2 Event-based control

The model of the inverted pendulum (3) can be written as a nonlinear affine-in-the-control system

$$\dot{x}(t) = \xi\big(x(t)\big) + \psi\big(x(t)\big)u(t) \tag{7}$$
$$\text{with} \quad x(0) := x_0$$

where ξ and ψ functions are smooth and ξ vanishes at the origin, $x \in \mathbb{R}^4$ and $u \in \mathbb{R}$ in this particular case.

Definition 1.1. By *event-based feedback* we mean a set of two functions, that are ***i)*** *an event function* $\epsilon : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ that indicates if one needs (when $\epsilon \leq 0$) or not (when $\epsilon > 0$) to recompute the control law, and ***ii)*** *a feedback function* $\gamma : \mathbb{R}^4 \to \mathbb{R}$.

The solution of (7) with event-based feedback (ϵ, γ) starting in x_0 at $t = 0$ is then defined as the solution (when it exists) of the differential system

$$\dot{x}(t) = \xi\big(x(t)\big) + \psi\big(x(t)\big)\gamma\big(x(t_i)\big) \quad \forall t \in [t_i, t_{i+1}[\tag{8}$$

where the time instants t_i, with $i \in \mathbb{N}$ (determined when the event function ϵ vanishes) are considered as *events* and

$$x_i := x(t_i) \tag{9}$$

is the memory of the state value at the last event.

With this formalization, the control value is updated each time ϵ becomes negative. Usually, one tries to design an event-based feedback so that ϵ cannot remain negative (and so is updated the control only punctually). In addition, one also wants that two events are separated with a non vanishing time interval avoiding the *Zeno* phenomenon. All these properties are encompassed with the *Minimal inter-Sampling Interval (MSI)* property introduced in Marchand et al. (2011, 2013a).

Property 1.2. An event-driven feedback is said *uniformly MSI* if and only if there is some non zero minimal sampling interval for any initial condition x_0.

Remark 1.3. A uniformly MSI event-based control is a piecewise constant control with non zero sampling intervals.

In the same papers, it is also proved that nonlinear systems affine in the control – like the one of the present study case – and admitting a Control Lyapunov Function (CLF) can be globally asymptotically stabilized by means of such an event-based feedback (this seminal result is derived from the Sontag's universal formula in Sontag (1998)). A linear version was also developed in Téllez-Guzmán et al. (2012).

1.3 Intuitive presentation of the control algorithms

In this paper, event-based strategies are developed for the control of the inverted pendulum and experimentally tested, for both swinging it up and stabilizing its arm near the upright position.

In particular, the results in Marchand et al. (2011, 2013a) can be directly applied for the stabilization of the linearized model expression (5)-(6) of the inverted pendulum (see subsection 2.1). As briefly explained before, in these seminal works the event function is related to a given control Lyapunv function whose control law renders the closed-loop system globally asymptotically stable.

We then extend such a principle for the swing up control, where an energy function is used in such a way the pendulum achieves the inverted position. This extension is easily obtained since a Lyapunov function is an energy function too but, nevertheless, an event-based scheme is especially designed for the swinging up part (see subsection 2.2). The resulting algorithm computational cost is really low in this case since the control law is only updated once the pendulum changes its direction of rotation in its balancing.

Finally, the switch from balancing to stabilizing is studied and shows that the transition is stable by construction of both control techniques (see subsection 2.3).

2. CONTROL OF THE INVERTED PENDULUM

2.1 Event-based stabilization near the upright position

Since we are interested here in the stabilization of the pendulum near its (unstable) equilibrium position, the event-based feedback developed in Marchand et al. (2011, 2013a) can be restricted to the stabilization of a linear system in this subsection. The adaptation of the previous work in such a particular case is trivial. Let consider the linear time-invariant dynamical system (5). A positive definite matrix P solution of the Riccati equation

$$PA + A^TP - 4PBR^{-1}B^TP = -Q \tag{10}$$

where Q and R are also positive definite, exists since (A, B) is a stabilizable pair. Then

$$V_1(x) := x^T P x \tag{11}$$

is a CLF for system (5) since for all $x \neq 0$, $u = -2R^{-1}B^T P x$ renders $\dot{V}_1$ strictly negative. It is then known that it is possible to design a feedback control that asymptotically stabilizes the system (5). The following theorem is a particular case of the *event-based universal formula* proposed in Marchand et al. (2011, 2013a) for linear systems:

Theorem 2.1. (Event-based stabilization of linear system). Taking the CLF V_1 in (11) for system (5), where P is a positive definite matrix solution of the Riccati equation (10), then the event-based feedback (ϵ_1, γ_1) defined by

$$\gamma_1(x) := -2R^{-1}B^T P x \tag{12}$$

$$\begin{aligned}\epsilon_1(x, x_i) := (\sigma - 1)x^T \big(PA + A^T P\big)x \\ -4x^T PBR^{-1}B^T P\big(\sigma x - x_i\big)\end{aligned} \tag{13}$$

$$\text{with} \quad \sigma \in]0, 1[$$

where x_i is defined in (9) and σ is a tunable parameter, is uniformly MSI and asymptotically stable.

Proof. The proof was given in Marchand et al. (2011, 2013a) for nonlinear affine in the control systems. The particular case of linear systems is hence trivial.

Remark 2.2. The idea behind the construction of the event function ϵ_1 in (13) is to compare the time derivative of the Lyapunov function (10) ***i)*** in the event-based case, that is applying $x(t_i)$ in the feedback control law (12), and ***ii)*** in the classical case, that is applying $x(t)$ instead of $x(t_i)$ in the feedback. The event function ϵ_1 is the weighted difference between both, where σ is the weighted value. By construction, an event is enforced when the event function vanishes to zero, that is hence when the stability of the event-based scheme does not behave as the one in the classical case. One can refer to Marchand et al. (2011, 2013a) for further details.

Remark 2.3. The control parameters

- $\sigma \in]0, 1[$ changes the frequency of events: the higher σ, more frequent are the events;
- $R > 0$ changes how fast is the control signal: the smaller R, larger is the control signal and smaller is the output of the controlled system. This parameter was identified as an event-based LQR parameter in Téllez-Guzmán et al. (2012), where the (infinite horizon) quadratic cost functional to minimize is given by

$$J = \int_0^\infty \Big(x^T Q x + u^T R u\Big) dt$$

Finally, this theorem can be directly applied for the stabilization of the inverted pendulum near its upright position using its linearized state-space representation (5)-(6). Indeed, choosing a positive definite matrix P satisfying (10) for A and B defined in (6) and applying the feedback control given in (12)-(13) will render the inverted pendulum stable near its (unstable) upright position $\theta = 0$.

2.2 Event-based swing up by energy control

Whereas the previous subsection details the stabilization of the inverted pendulum near its upright position, another control strategy is before required in order to swing the pendulum up to this equilibrium. This was notably presented in Åström and Furuta (2000) – using an energy function – for the classical (time-triggered) case that we propose to adapt here as an event-based strategy.

Classical strategy

Let us consider here only the equation of motion of the pendulum (1) where the friction forces are neglected. This leads

$$I\ddot{\theta} - mgl\sin(\theta) - mlv\cos(\theta) = 0 \tag{14}$$

where $v := \dot{p}$ is the velocity of the pivot (the input in this restricted case). Note that v is positive if it is in the direction of the positive x-axis. The *uncontrolled* pendulum state space can hence be represented as a cylinder (since the origin of the system is assumed to be fixed, because $v = 0$, and the pendulum of the study can only move in two dimensions). In this case, the system has two equilibriums corresponding to $\theta = \pi$, $\dot{\theta} = 0$ (stable position) and $\theta = 0$, $\dot{\theta} = 0$ (unstable position). The energy of the uncontrolled pendulum is

$$E(\theta, \dot{\theta}) = \frac{1}{2}I\dot{\theta}^2 + mgl\big(\cos(\theta) - 1\big) \tag{15}$$

which is defined to be zero when the pendulum is stationary in the upright position. One way to swing the pendulum up to this upward position then consists in giving it an energy that corresponds to the upright position. However, this cannot be done in one swing due to limitations of the actuator. Actually, to perform energy control it is necessary to understand how the energy is influenced by the acceleration of the pivot. Computing the time derivative of E, and substituting $\ddot{\theta}$ from (14), yields

$$\dot{E} = I\dot{\theta}\ddot{\theta} - mgl\dot{\theta}\sin(\theta) = mlv\dot{\theta}\cos(\theta) \tag{16}$$

As a result, controlling the energy is easy since the system is a simple integrator with varying gain, however the controllability is lost when the right-hand side of (16) vanishes. This occurs for $\theta = \pm\frac{\pi}{2}$ or $\dot{\theta} = 0$, that is when the pendulum is horizontal or when it reverses its velocity. Also, to increase energy the acceleration of the pivot $\dot{v}$ should be positive when the quantity $\dot{\theta}\cos(\theta)$ is positive, and inversely. A control strategy can then be found using the Lyapunov method, as suggested in Åström and Furuta (2000).

Theorem 2.4. (Swing up a pendulum by energy control). Taking the Lyapunov function

$$V_2(\theta, \dot{\theta}) := \frac{1}{2}\Big(E(\theta, \dot{\theta}) - \varepsilon\Big)^2 \tag{17}$$

for system (14), where E is defined in (15) and ε is a given (desired) energy value, then the control law

$$v = -\alpha\big(E - \varepsilon\big)\dot{\theta}\cos(\theta) \tag{18}$$

$$\text{with} \quad \alpha \in \mathbb{R}^+$$

where α is a tunable parameter, drives the energy towards its desired value ε.

Proof. Substituting (18) in (16), and substituting this result in the derivative of the Lyapunov function (17) with respect to time gives

$$\dot{V}_2 = \dot{E}\big(E-\varepsilon\big) = -\alpha ml\Big(\big(E-\varepsilon\big)\dot{\theta}\cos(\theta)\Big)^2 \tag{19}$$

The Lyapunov function (17) hence decreases as long as $\dot{\theta} \neq 0$ and $\cos(\theta) \neq 0$. Moreover, since the pendulum cannot maintain a stationary position with $\theta = \pm\frac{\pi}{2}$ then the control law v drives the energy towards ε.

Remark 2.5. Once the energy of the pendulum is close "enough" to the desired value ε, the control switches to the strategy depicted in subsection 2.1 in order to stabilize the inverted pendulum near its upright position. This is discussed in subsection 2.3. Nevertheless, the value of the desired energy ε can be defined by the designer from (15) for a given angle and rate of change of the angle, afterwards denoted θ_ε and $\dot{\theta}_\varepsilon$.

Event-based proposal

In this paper we propose to adapt the previous classical control strategy in such a way the pendulum swings up to its upright position when applying an event-based feedback. As explained in subsection 2.1, an event-based strategy means to keep constant the control signal between two events, as follows

$$v = \gamma_2(\theta_i, \dot{\theta}_i) \quad \forall t \in [t_i, t_{i+1}[\tag{20}$$

where

$$\begin{aligned} \theta_i &:= \theta(t_i) \\ \dot{\theta}_i &:= \dot{\theta}(t_i) \end{aligned} \tag{21}$$

making the analogy with the principle detailed in (9), and γ_2 is the feedback function defined next.

Let analyze in detail how varies the Lyapunov function (17) used to swing the pendulum up to its inverted position. As already explained in the proof of Theorem 2.4, the Lyapunov function decreases as long as $\dot{\theta} \neq 0$ and $\cos(\theta) \neq 0$. Therefore, why not to enforce events only when these conditions are met (since one does not really need to update the control law while one of these two conditions is achieved because the energy is naturally decreasing). This is the main idea of our proposal. Making the assumption that only the one or the other changes at a given time, an event can hence be simply detected when changes the function

$$\operatorname{sgn}\big(\dot{\theta}\cos(\theta)\big) \tag{22}$$

$$\text{with} \quad \operatorname{sgn}(z) := \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0 \end{cases}$$

Based on this idea and on Theorem 2.1, we propose the following theorem:

Theorem 2.6. (Event-based swing up of the pendulum). Making the assumption that only $\dot{\theta}$ or $\cos(\theta)$ changes at a given time and taking the Lyapunov function V_2 in (17) for system (14), where E in (15) describes the energy of the system, then the event-based feedback (ϵ_2, γ_2) defined by

$$\gamma_2(\theta, \dot{\theta}) := -\alpha\big(E-\varepsilon\big)\operatorname{sgn}\big(\dot{\theta}\cos(\theta)\big) \tag{23}$$

$$\epsilon_2(\theta, \dot{\theta}, \theta_i, \dot{\theta}_i) := \big|\operatorname{sgn}\big(\dot{\theta}\cos(\theta)\big) - \operatorname{sgn}\big(\dot{\theta}_i\cos(\theta_i)\big)\big| \tag{24}$$

where θ_i, $\dot{\theta}_i$ and α are defined in (21) and (18) respectively, is uniformly MSI and drives the energy towards its desired value ε.

Proof. The proof for the energy driving is trivial and based on proof of Theorem 2.4. Substituting (23) in (16) and then in the time derivative of the Lyapunov function (17) gives

$$\dot{V}_2 = -\alpha ml\big(E-\varepsilon\big)\dot{\theta}\cos(\theta)\big(E_i-\varepsilon\big)\operatorname{sgn}\big(\dot{\theta}_i\cos(\theta_i)\big) \tag{25}$$

where $E_i := E(\theta_i, \dot{\theta}_i)$. The Lyapunov function (17) decreases as long as $\dot{\theta} \neq 0$, $\cos(\theta) \neq 0$, $\operatorname{sgn}\big(\dot{\theta}\cos(\theta)\big) = \operatorname{sgn}\big(\dot{\theta}_i\cos(\theta_i)\big)$ and $\operatorname{sgn}\big(E-\varepsilon\big) = \operatorname{sgn}\big(E_i-\varepsilon\big)$. As before, the pendulum cannot maintain a horizontal position, which solves the problem for the two first conditions. Also, the problem of the third one is solved thanks to the event function ϵ_2 since an event is enforced when it occurs. As regards the latter one, $E - \varepsilon < 0$ could only occur when the energy is towards the upright position (if ε was defined with respect to the switching condition, this is discussed latter in subsection 2.3), and so is switched the control strategy for the stabilization of the pendulum. As a consequence, the event-based feedback proposed in (23)-(24) drives the energy towards its desired value ε.

As regards the uniformly MSI property of the event-based feedback (23)-(24), one knows that an event is enforced when the pendulum reverses its velocity by construction and, consequently, two events cannot successively occur due to the system inertia. This ends the proof.

Remark 2.7. The event-based swing up control strategies can also be easily adapted to take into account the maximum acceleration of the pivot, as detailed in Åström and Furuta (2000) for the classical scheme.

2.3 Switch from balancing to stabilizing

In previous subsections, we detailed how to ***i)*** *swing the inverted pendulum up to its upright position* and then ***ii)*** *stabilize it near this unstable position.* The switch between both is done when the angle is in a given region, which can be summarized by

$$(\epsilon, \gamma) = \begin{cases} (\epsilon_1, \gamma_1) & \text{if} \quad |\theta| \leq \Theta \\ (\epsilon_2, \gamma_2) & \text{elsewhere} \end{cases} \tag{26}$$

where Θ is a tunable parameter. Actually, its value as to be defined with respect to the value of θ_ε used to define the energy to achieve during the swing up strategy, i.e. ε obtained by (15) for a given θ_ε and $\dot{\theta}_\varepsilon$. Such a solution is to choose $\dot{\theta}_\varepsilon = 0$ and θ_ε as the desired angle for switching. However, the switch could not occur when $\Theta = \theta_\varepsilon$, due to frictions and some other perturbations. Also, if Θ is lower than θ_ε then the balancing will not swing the pendulum up to this angle and so never will occur the switch. As a result, Θ has to be higher than and close "enough" to θ_ε, since with a high Θ the strategy would switch whereas the rate of the angle of the pendulum is still important.

In order to facilitate the switch between balancing and stabilizing strategies, the designer has to guarantee that the rate of change of the angle is small enough and so is kept the pendulum in this region when using the stabilizing control strategy. In other words, whereas stabilizing feedback control proposed in (12)-(13) renders the time derivative of the Lyapunov function (11) strictly negative – and so is decreasing the energy of the whole system – is also ensured the decrease of the pendulum only ?

Let define

$$x_\theta := \left[\theta\ \dot{\theta}\right]^T \quad \text{and} \quad x_p := [p\ \dot{p}]^T \tag{27}$$

Using this notation, the linearized system of the inverted pendulum becomes (when neglecting the friction forces)

$$\frac{d}{dt}\begin{pmatrix} x_\theta \\ x_p \end{pmatrix} = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}\begin{pmatrix} x_\theta \\ x_p \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} w \tag{28}$$

with $A_1 = \begin{bmatrix} 0 & 1 \\ a_1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ a_3 & 0 \end{bmatrix}$

and $B_1 = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$

where w is the new control input and $a_i, b_i > 0$ can be found back from (5)-(6). Note that from these same expressions, one can write

$$B_1 := \beta B_2 \tag{29}$$

where β thus defines the ratio between both matrices.

Theorem 2.8. (Stability of the switch). Taking the Lyapunov function

$$V_3(x_\theta, x_p) := \frac{1}{2}\left(x_\theta^T P_\theta x_\theta + x_p^T P_p x_p\right) \tag{30}$$

for the linearized system (28), where x_θ and x_p are defined in (27), where P_θ and P_p are some positive definite matrices solution of the Riccati equations defined as follows

$$\begin{aligned} P_\theta A_1 + A_1^T P_\theta - 4P_\theta B_1 R^{-1} B_1^T P_\theta = -Q_\theta \\ P_p A_2 + A_2^T P_p - 4P_p B_2 R^{-1} B_2^T P_p = -Q_p \end{aligned} \tag{31}$$

where Q_θ, Q_p and R are positive definite, and taking the new control law defined by

$$w := -\frac{a_3}{b_2}\theta + \frac{2}{1+\beta} R^{-1} B_1^T \left(P_\theta x_\theta + P_p x_p\right) + u \tag{32}$$

where the control law for u is given in (12) and the ratio β is defined in (29), then the switch from event-based balancing defined in Theorem 2.6 to event-based stabilizing defined in Theorem 2.1, using the switching condition (26), is stable.

Proof. The derivative of the Lyapunov function (30) with respect to time is

$$\dot{V}_3 = x_\theta^T P_\theta \left(A_1 x_\theta + B_1 w\right) + x_p^T P_p \left(A_2 x_p + A_3 x_\theta + B_2 w\right)$$

which, substituting (32) and (12), yields

$$\begin{aligned} \dot{V}_3 = & \frac{1}{2} x_\theta^T \left(P_\theta A_1 + A_1^T P_\theta - 4P_\theta B_1 R^{-1} B_1^T P_\theta\right) x_\theta \\ & + \frac{1}{2} x_p^T \left(P_p A_2 + A_2^T P_p - 4P_p B_2 R^{-1} B_2^T P_p\right) x_p \\ & - \frac{b_1}{2b_2} x_\theta^T P_\theta A_3 x_\theta < 0 \end{aligned} \tag{33}$$

when taking the Lyapunov function in (11) as defined by

$$P := \begin{bmatrix} P_\theta & 0 \\ 0 & P_p \end{bmatrix} \tag{34}$$

The result in (33) means that the decrease of V_3 implies the decrease of the energy of the pendulum. Also, the stability of the event-based feedback (12)-(13) using (32) for system (28) is still ensured.

Remark 2.9. The principle intuitively remains true taking into account the friction forces f and k since they can only slow down the motion of the pendulum.

Remark 2.10. The condition (34) benefits by doing more simple the computing in practice – reducing by four the number of product operations in the Lyapunov function – and, consequently, the event function (13). The one for balancing in (23) requires small computing too. As a consequence, the whole event-based proposal can be said *low cost.*

3. EXPERIMENTAL RESULTS

In this last section, we implement and test our proposal on a practical inverted pendulum, depicted in Fig. 2. The system runs in real-time but data are sent/received via *Matlab/Simulink* thanks to the *xPC Target* hardware-in-the-loop environment. As already explained, two steps are required: ***i)*** *a first controller swings the pendulum up to its upright position* (the control strategy is based on an energy function, as explained in section 2.2) and, then, ***ii)*** *another strategy stabilizes the pendulum near this unstable equilibrium* (state-feedback control, see section 2.1). Actually, the complete identification of the system and the classical (time-triggered) control strategies were already done for the present inverted pendulum study case. One could refer to Murueta Fortiz (2009) for further details. The different parameters of the model are $M = 2.57\,kg$, $m = 1.47\,kg$, $l = 0.028\,m$, $J = 0.024\,kg.m^2$, $g = 9.8\,m.s^{-2}$, $f = k = 10^{-4}\,kg.m^2.s^{-1}$ and $\rho = 3$, leading to the linearized system

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 16.3 & -6\cdot 10^{-3} & 0 & -4\cdot 10^{-5} \\ 0 & 0 & 0 & 1 \\ 0.2 & -4\cdot 10^{-5} & 0 & -2.5\cdot 10^{-5} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.24 \\ 0 \\ 0.76 \end{bmatrix}$$

The simulation results of both parts are represented in Fig. 3(a) when using classical (time-triggered) control laws. The four top plots show the dynamics of the system states, that are the position and the velocity of the cart, the angle and the angular velocity of the pendulum. The bottom plot shows the control signal. Note that the number of samples required to perform the bench is also indicated. The (constant) sampling period is $10\,ms$ here. The two parts can be clearly identified in Fig. 3(a). Thus, ***i)*** from 0 to about $20\,s$ the angle of the pendulum oscillates

Fig. 2. Inverted pendulum system used for the experiments

– moving the cart (as one can see looking at the position) – until achieving the upright position, that is $\theta = 2j\pi, j \in \mathbb{N}$. ***ii)*** Once the equilibrium point is achieved (or almost achieved), that is when $|\theta|$ becomes lower than $\Theta = 0.2\,rad$ with respect to (26), then the control switches in order to now stabilize the inverted pendulum in this position and, finally, the cart has to move to a given position (i.e. the origin) without causing the pendulum to tip over.

The event-based proposals are then tested. The simulation results are depicted in Fig. 3(b). One can see that the results (in swinging and maintaining the position) are quite similar with respect to the classical strategy and the pendulum is stabilized in about the same time. Nevertheless, the number of updates is now divided by 2.5 for the same performance. The different control steps are now analyzed.

Event-based balancing: On one hand, the system runs with the event-driven proposal detailed in Theorem 2.6, that is for the swinging of the inverted pendulum. The control parameters were calculated for

$$P = \begin{bmatrix} 0.12 & 0 & 0 & 0 \\ 0 & 0.87 & 0 & 0 \\ 0 & 0 & 3.63 & 0 \\ 0 & 0 & 0 & 186.86 \end{bmatrix}$$

The results are detailed in Fig. 4(a), where an extra plot represents the sign of the function (22) used for enforcing events. One could notice that an important reduction of the number of samples is achieved (about 98 % less) with similar performance since the pendulum achieves its inverted position in almost the same time.

Event-based stabilization: On another hand, the system runs with the event-driven proposal presented in Theorem 2.1, that is for the stabilization of the pendulum close to its inverted position. The control parameters are $\alpha = 0.8$ and $\varepsilon = 0.12\,J$. A zoom of the whole simulation (Fig. 3(b)) is performed between 45 and 55 s in Fig. 4(b), where an extra plot represents the system energy (that is the Lyapunov function). An important reduction of the number of samples is also achieved (about 55 % less) with similar performance since the cart achieves the desired position ($p = 0$) in almost the same time. Moreover, whereas the control is kept constant during two events (which can be several times the time-triggered sampling period) – like at time 46.5 s – an unstable system can be stabilized anyway.

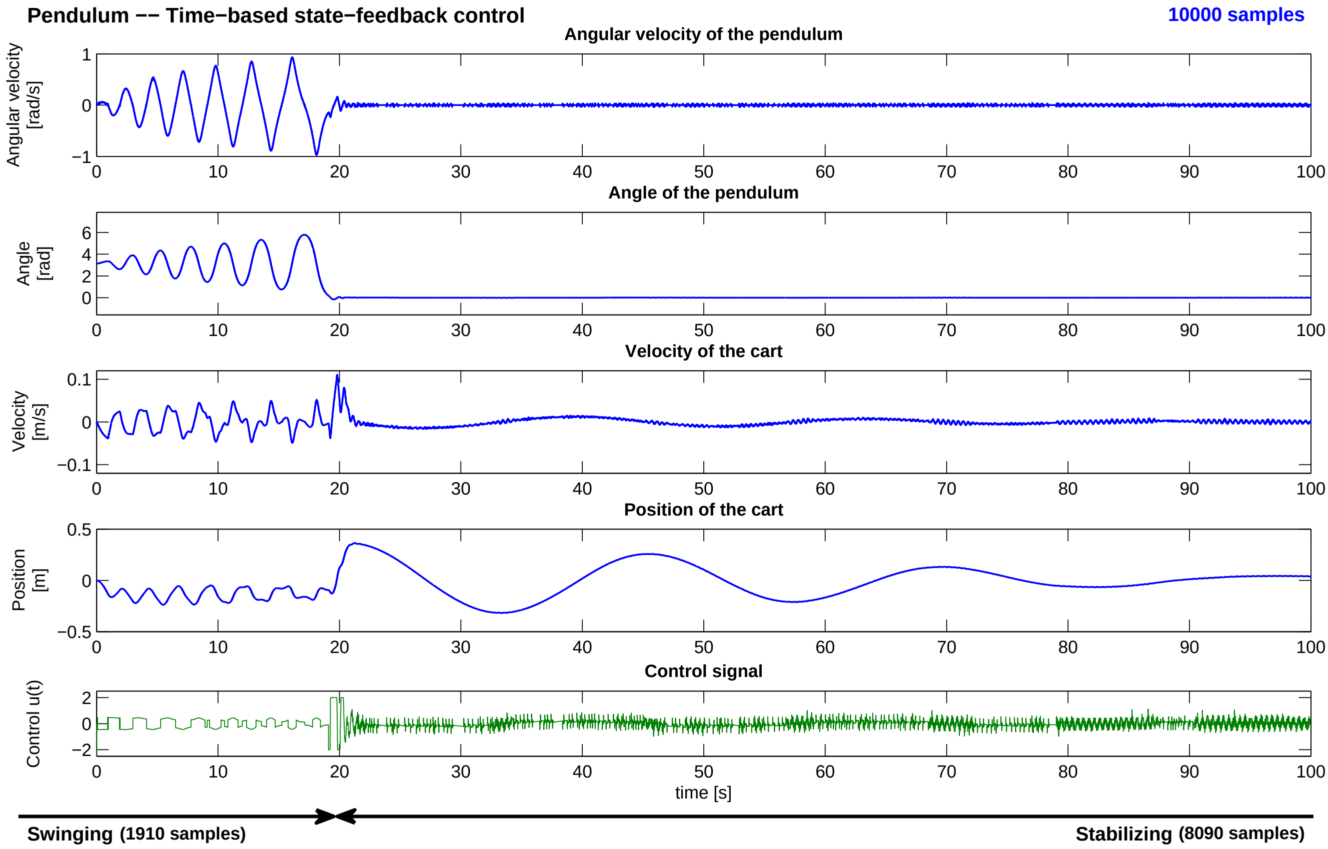
Switch: The switch between balancing and stabilization is done for $\Theta = 0.2\,rad$. It can be seen in Fig. 3(b) at about 18 s when the angle arises about 2π and so changes the control strategy.

One can firstly remark that the actuator is less often requested thanks to the proposed event-based framework. Indeed, whereas the control law is continuously and periodically updated in the classical scheme, there exist some time intervals where the control signal is not updated thanks to the event-driven approaches (this can be well observed in Fig. 4 in particular). Furthermore, in the present practical case we designed *discrete* event functions with the same (constant) sampling period than for the classical control scheme. In this sense, in the worst case the control updates can only be as frequent as in the classical scheme. As a consequence, such a scheme will not only reduce the number of samples (and so the control computational cost) but also generally increase the lifetime expectancy of the motor, and these, for the same performance.

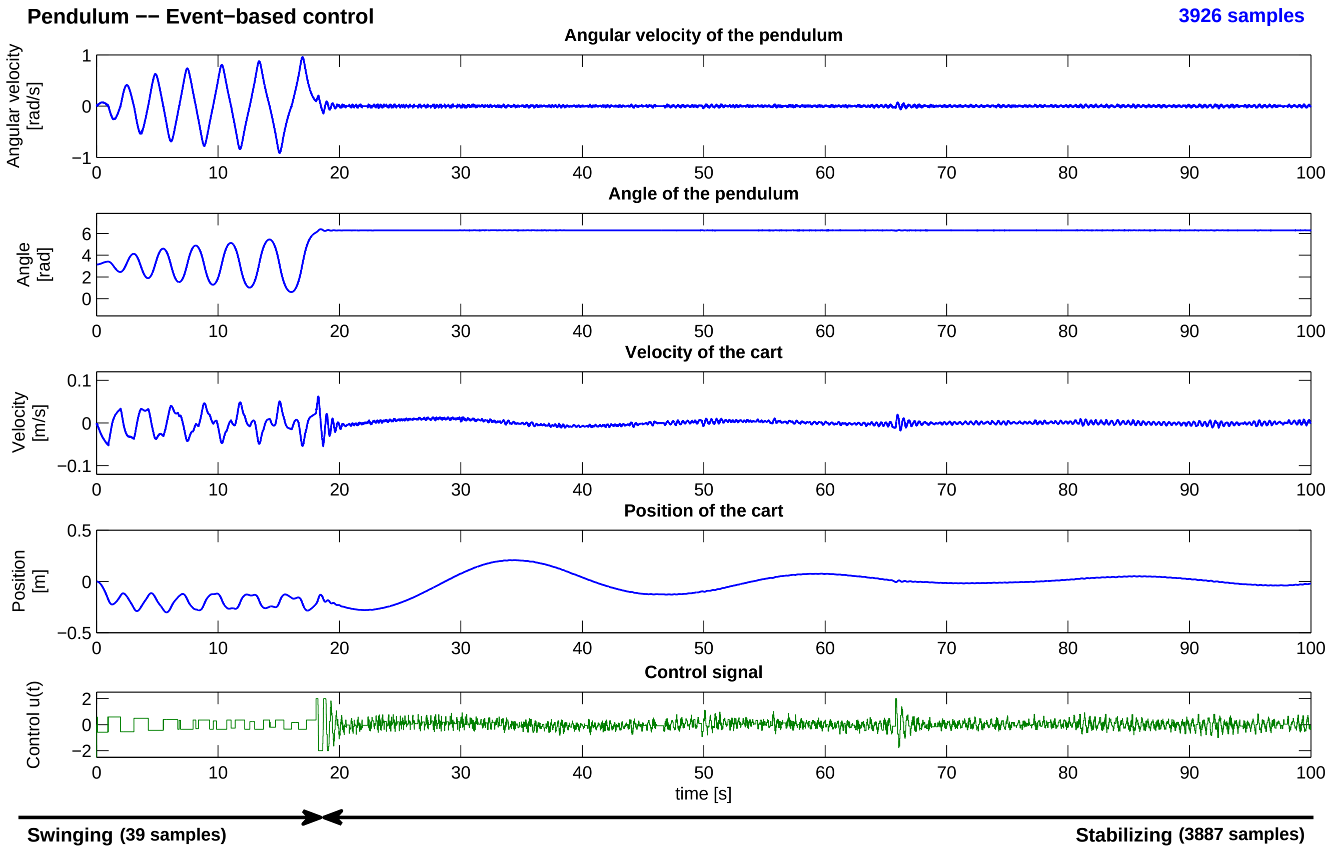
The fact to design discrete event functions makes the implementation of the event-based control algorithm is not more complex than for a classical controller. In return, one can observe some small perturbations, like at the time instant 66 s in Fig. 3(b). One can imagine that this is due to such a discrete implementation. Certainly, a theoretical updating instant which occurs just after a sampling instant will only be detected at the next sample, and so could become unstable the pendulum during this detection time if the control law was not recently updated. This point remains to analyze.

CONCLUSIONS AND FUTURE WORKS

The main contribution of this paper is to propose some event-based control strategies for a highly nonlinear and unstable system, that is the inverted pendulum. The principle consists in only updating the control signal when required from a stability point of view. Some strategies were presented to control both the swing of the pendulum up to its upright position and its stabilization near this unstable equilibrium. The first setup is based on an energy function which allows to drive the pendulum towards the inverted position; the second is an event-based state feedback which event function is built from a Lyapunov function. The switch between both strategies is also studied for stability reason. The proposals are tested on a real-time testbed, where the number of samples is clearly reduced (about

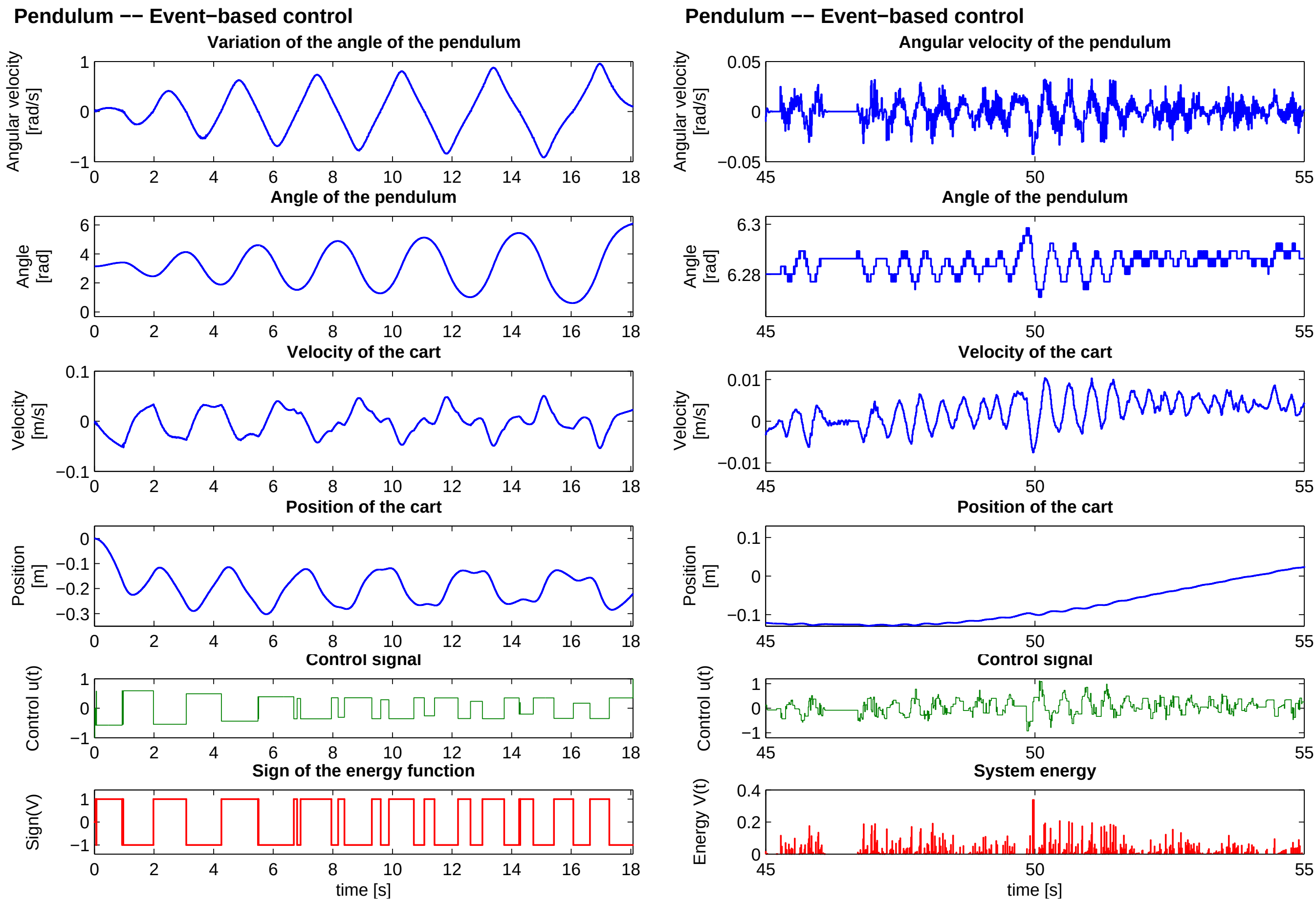


(a) Time-triggered control strategies.



(b) Event-based control proposals.

Fig. 3. Swing up and stabilization of the inverted pendulum.



(a) Swing of the inverted pendulum up to the inverted position.

(b) Stabilization of the pendulum near its unstable position.

Fig. 4. Zooms for the event-based proposals.

98 % and 50 % less than in the classical scheme when respectively swinging and stabilizing the pendulum) with similar final performance. As a result, the encouraging results strongly confirm the interest for developing event-based control strategies.

Next step is to develop other nonlinear event-based strategies for the control of cyber-physical systems, in the spirit of Marchand et al. (2011, 2013a). In particular we are working on ***i)*** *simple event functions*, like in Marchand et al. (2013b), in order to reduce even much more the control computational cost, and ***ii)*** *networked control systems*, like in Durand (2013), where a reduction of the updates allows to decrease the communications too.

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