Novel Tuning Expressions for Fractional Order ($[PD]^\beta$ and $[PI]^\alpha$) Controllers Using a Generalized Plant Structure

Ameya Anil Kesarkar and N. Selvaganesan*

Department of Avionics, Indian Institute of Space Science and Technology (IIST), Thiruvananthapuram, India-695547.
(e-mail: n_selvag@iist.ac.in)*

Abstract: The available tuning expressions in the literature for three-parameter $[PI]^\alpha$ and $[PD]^\beta$ controllers are specific to a given class of plant transfer function. If one intends to design such controllers for a new plant, then the tuning expressions meeting required specifications need to be derived accordingly. This is a tedious and work intensive process. Instead, this paper presents novel tuning expressions of such controllers, which can be readily used for any class of integer or fractional order LTI plants. The usefulness of such expressions is further demonstrated by considering several plant examples belonging to different classes including a real time case of precision modular servo experimental set-up available at our Control System Lab. The results confirm correctness of the proposed framework.

Keywords: fractional order controller, generalized plant structure, $[PI]^\alpha$, $[PD]^\beta$, Oustaloup approximation.

1. INTRODUCTION

The mathematical topic of fractional calculus Oldham (1974) has recently found applications in the control engineering resulting in an area popularly known as 'Fractional Order Control' Podlubny I. (1999), Xue et al (2007). Fractional order control deals with designing of the controllers which are governed by fractional order differential equations Chen et al. (2009). The compact form expressions of such controllers possess easily tunable characteristics, which help in meeting stringent loop performance Padula (2011), Monje et al (2004).

The popularly used three-parameter-fractional-controllers are PI^α, $[PI]^\alpha$, PD^β, and $[PD]^\beta$. Tuning of these controllers for a specific class of integer order plant, such as position and velocity servo system Li et al (2010), Li (2008), Luo (2009), Wang et al. (2009a), FOPDT (First Order Plus Dead Time) Wang et al. (2009b) is found in the literature. Similar work in the context of fractional order plants has also been covered in the literature Luo et al. (2010).

The designed controllers in the above works meet the required gain crossover frequency, phase margin, and isodamping conditions (We refer them as Wang et al. (2009b) specifications hereafter). However, these specifications don't guarantee the closed loop stability in general. Hence, closed loop stability check is essential after the controller design.

One can further see that in these works a particular Linear Time Invariant (LTI) plant is considered and the corresponding analytical expressions for the controller parameters are derived. For example, the motion control plant is considered in Li et al (2010), Li (2008), Luo (2009), and Wang et al. (2009a) to derive the expressions for PI^α, $[PI]^\alpha$, PD^β, and $[PD]^\beta$. Similarly, such corresponding derivations are made for FOPDT plant and fractional order velocity servo system in Wang et al. (2009b) and Luo et al. (2010) respectively.

Instead of deriving for each class of plants accordingly, if the controller expressions meeting Wang et al. (2009b) specifications and applicable to all class of plants are made available, it will considerably save the control engineer's time and efforts. Such unification was first attempted in our previous work Kesarkar (2011a). However, the plant structure considered therein doesn't handle integer order plants with complex poles and/or zeros and also the fractional order plants.

This issue was resolved in our subsequent work Kesarkar (2011b) where we defined a generalized plant structure which can accommodate any integer or fractional order plant transfer function and consequently derived the expressions for PI^α and PD^β controllers.

In the current paper, this work is extended further to derive the expressions for $[PI]^\alpha$ and $[PD]^\beta$ controllers thereby completing such unification for all three-parameter-fractional-controllers. The contribution of this paper is as follows:

(1) The novel unified tuning expressions are derived for $[PI]^\alpha$ and $[PD]^\beta$ controllers to meet Wang et al. (2009b) specifications.
(2) Several examples are considered of the plants belonging to different classes and the correctness of our proposed unified expressions is demonstrated.

Overview of the paper: Section 2 presents introduction to fractional calculus and fractional order control. In section 3, our generalized plant structure in Kesarkar (2011b) is explained and the novel expressions for $[PI]^\alpha$ and $[PD]^\beta$ are derived. In section 4, the illustration of our derivations is made by considering plants such as fractional thermal process, DC motor position servo system, precision modular servo system, fractional horsepower dynamometer, and DC motor velocity servo system. Section 5 presents the concluding remarks.

2. BASICS OF FRACTIONAL CALCULUS AND FRACTIONAL ORDER CONTROL

2.1 Fractional Calculus

Conventional calculus deals with integer order differentiation and integration. Generalization of conventional calculus so as to consider differentiation and integration of any order (not necessarily integer) leads to fractional calculus Oldham (1974).

In fractional Calculus, the fundamental differ-integration operator ${}_aD_t^\alpha$ (where a and t are the limits of the operation) is defined as Chen et al. (2009):

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \displaystyle\int_a^t (d\tau)^{-\alpha} & \alpha < 0 \end{cases} $$

Where α is the order of the operation, generally $\alpha \in \mathbb{R}$ but α could also be a complex number.

Out of many definitions of fractional differ-integration in FC, the popular ones are Xue et al (2007):

- Grunwald-Letnikov (G-L) Definition:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) $$

- Riemann-Liouville (R-L) Definition:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

$(n-1) \leq \alpha < n$ where n is integer and a is real number.
$\left[\frac{t-a}{h}\right] \to integer$

- Caputo Definition:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

Fractional Order Transfer Function Model Laplace transform of the defined fractional-order operator is Xue et al (2007):

$$ L({}_aD_t^\alpha f(t)) = s^\alpha F(s) \text{ (with zero initial conditions.)} $$

LTI fractional order system with input u, and output y has following model Xue et al (2007):

$$ a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + ... + a_0 D^{\alpha_0} y(t) = $$
$$ b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + ... + b_0 D^{\beta_0} u(t) $$

where, a_i, α_i $(i = 0, 1, \ldots, n)$, b_k, β_k $(k = 0, 1, \ldots, m)$ are real constants. n and m are positive integers.

Therefore, Laplace transform on both sides (assuming zero initial conditions) results into the following transfer function:

$$ \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + ... + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + ... + a_0 s^{\alpha_0}} $$

2.2 Fractional Order Controllers

In control engineering, the application of fractional calculus can be either in system modelling or controller design. The typical fractional order controllers ($C(s)$) are as follows:

- Fractional order proportional-integral controller, which is of two types Wang et al. (2009b):
  · PI^α
  $$ C(s) = K_p \left(1 + \frac{K_i}{s^\alpha}\right) \quad (1) $$
  · $[PI]^\alpha$
  $$ C(s) = K_p \left(1 + \frac{K_i}{s}\right)^\alpha \quad (2) $$
  (Integer PI has the form: $C(s) = K_p\left(1 + \frac{K_i}{s}\right)$)
- Fractional order proportional-derivative controller, which is of two types Luo (2009):
  · PD^β
  $$ C(s) = K_p\left(1 + K_d s^\beta\right) \quad (3) $$
  · $[PD]^\beta$
  $$ C(s) = K_p\left(1 + K_d s\right)^\beta \quad (4) $$
  (Integer PD has the form: $C(s) = K_p\left(1 + K_d s\right)$)
- Fractional order proportional-integral-derivative controller Podlubny I. (1999):
  · $PI^\alpha D^\beta$
  $$ C(s) = K_p \left(1 + \frac{K_i}{s^\alpha} + K_d s^\beta\right) $$
  (Integer PID has the form: $C(s) = K_p\left(1 + \frac{K_i}{s} + K_d s\right)$)

In this paper, for the tuning purpose only three-parameter-fractional-controllers (i.e. PI^α, $[PI]^\alpha$, PD^β, and $[PD]^\beta$) are taken into consideration. Since $PI^\alpha D^\beta$ has 5 parameters, it is not selected.

3. PROBLEM FORMULATION

In our previous work Kesarkar (2011b), we defined the following generalized plant structure that can accommodate any class of fractional and integer order LTI plants:

$$ \begin{aligned} P(s) &= K \frac{(a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \cdots + a_m s^{\alpha_m})}{(b_0 s^{\beta_0} + b_1 s^{\beta_1} + \cdots + b_n s^{\beta_n})} e^{-Ls} \\ &= K \frac{\sum_{i=0}^{m} (a_i s^{\alpha_i})}{\sum_{k=0}^{n} (b_k s^{\beta_k})} e^{-Ls} \end{aligned} \quad (5) $$

In general, K, a_i, α_i $(i = 0, 1, 2, \ldots, m)$, b_k, β_k $(k = 0, 1, 2, \ldots, n)$, L are real constants. m and n are integers. L

denotes time delay or dead time of the plant. K is positive without loss of generality.

When α_i, β_k assume integer values, (5) represents integer order plant. Since α_i ($i = 0, 1, 2, \ldots, m$) and β_k ($k = 0, 1, 2, \ldots, n$) are not necessarily integers, the transfer function given by (5) can accomodate any integer or fractional order plant. For instance,

- $m = 0$, $a_0 = 1$, α_0, $n = 1$, $b_0 = T$, $\beta_0 = 1$, $b_1 = 1$, $\beta_1 = 0$ leads to FOPDT plant stucture,
$$P(s) = \frac{K}{Ts + 1} e^{-Ls}$$
- $m = 0$, $a_0 = 1$, α_0, $n = 2$, $b_0 = T_1 \cdot T_2$, $\beta_0 = 2$, $b_1 = (T_1 + T_2)$, $\beta_1 = 1$, $b_2 = 1$, $\beta_2 = 0$ leads to SOPDT (Second Order Plus Dead Time) plant stucture,
$$P(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)} e^{-Ls}$$
- For Position Servo plant as shown below, the generalized structure parameters are: $m = 0$, $a_0 = 1$, α_0, $n = 1$, $b_0 = a$, $\beta_0 = 2$, $b_1 = 1$, $\beta_1 = 1$, $L = 0$.
$$P(s) = \frac{K}{s(as + 1)}$$
An example of this plant class will be considered later in Section 4 for illustration.
- $m = 0$, $a_0 = 1$, α_0, $n = 1$, $b_0 = a$, $\beta_0 = 0.5$, $b_1 = 1$, $\beta_1 = 0$, $L = 0$ leads to Half-order Fractional Velocity Servo structure,
$$P(s) = \frac{K}{as^{0.5} + 1}$$
The fractional horsepower dynamometer example Luo et al. (2010) to be considered later in Section 4 belongs to such plant class.

3.1 Control Loop and Set of Specifications

The unity feedback control loop is shown in Fig. 1. $R(s)$, $E(s)$, $U(s)$, and $Y(s)$ denote laplace transform of reference input, error, controller output, and plant output respectively.

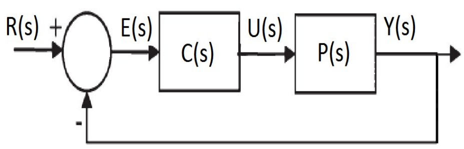

Fig. 1. Unity feedback control Loop

To derive the tuning expressions for all three-parameter-fractional-controllers (i.e. PI^α, $[PI]^\alpha$, PD^β, and $[PD]^\beta$), the following Wang et al. (2009b) specifications are considered:

- Phase margin (ϕ_m)
$$\angle[C(j\omega_c)P(j\omega_c)] = -\pi + \phi_m \tag{6}$$
- Gain crossover frequency (ω_c)
$$|C(j\omega_c)P(j\omega_c)| = 1 \tag{7}$$
- Robustness to gain variation (Isodamping)
$$\left(\frac{d(\angle[C(j\omega)P(j\omega)])}{d\omega}\right)_{\omega=\omega_c} = 0 \tag{8}$$
The isodamping condition Chen (2005) ensures constant phase margin irrespective of plant gain (K) variations. The effect of such robustness can be seen in closed loop step response as the constant maximum peak overshoot in spite of gain variations.

Remark 1. This paper focusses on the generalized tuning expressions with respect to Wang et al. (2009b) specifications. However, one can adopt the same approach to develop corresponding unified expressions for any other set of three specifications.

3.2 Expressions for Three-Parameter-Fractional Controllers

In our previous work Kesarkar (2011b), we presented the generalized tuning expressions for PI^α and PD^β controllers as follows:

***PI^α* and *PD^β* Controllers** The case of PI^α and PD^β is handled together by considering the following general controller form:
$$C(s) = K_1 (1 + K_2 s^\gamma) \tag{9}$$
There are two cases:

(1) $\gamma > 0$: PD^β controller of the form (3) with, $K_p = K_1$, $K_d = K_2$, and $\beta = \gamma$.

(2) $\gamma < 0$: PI^α controller of the form (1) with, $K_p = K_1$, $K_i = K_2$, and $\alpha = -\gamma$.

Substitution of P(s) and C(s) expressions (as given in (5) and (9), respectively) in (6)-(8) yields the following expressions for controller parameters K_1, K_2, and γ:

$$K_2 = \frac{-M\omega_c^{-\gamma}}{M\cos\left(\frac{\pi}{2}\gamma\right) - \sin\left(\frac{\pi}{2}\gamma\right)} \tag{10}$$

$$K_2 = \frac{-H \pm \sqrt{H^2 - 4N^2\omega_c^{2\gamma}}}{2N\omega_c^{2\gamma}} \tag{11}$$

$$K_1 = \frac{1}{K}\sqrt{\frac{\left(p_2^2+q_2^2\right)}{\left(p_1^2+q_1^2\right)\left(\left[1+K_2\omega_c^\gamma\cos\left(\frac{\pi}{2}\gamma\right)\right]^2+\left[K_2\omega_c^\gamma\sin\left(\frac{\pi}{2}\gamma\right)\right]^2\right)}} \tag{12}$$

where,

$$M = \tan\left(-tan^{-1}\left(\frac{q_1}{p_1}\right) + tan^{-1}\left(\frac{q_2}{p_2}\right) + L\omega_c - \pi + \phi_m\right)$$

$$p_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\cos\left(\frac{\pi}{2}\alpha_i\right)\right), q_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)$$

$$p_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\cos\left(\frac{\pi}{2}\beta_k\right)\right), q_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\sin\left(\frac{\pi}{2}\beta_k\right)\right)$$

$$H = \left(2N\omega_c^\gamma\cos\left(\frac{\pi}{2}\gamma\right) - \gamma\sin\left(\frac{\pi}{2}\gamma\right)\omega_c^{\gamma-1}\right)$$

$$N = \frac{-p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right)+q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2+q_1^2}$$
$$+L+\frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right)-q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2+q_2^2}$$

On solving (10), (11), and (12) simultaneously, we get K_1, K_2 and γ.

In the present paper, we extend this work to develop such novel tuning expressions for $[PI]^\alpha$ and $[PD]^\beta$ controllers as follows:

$[PI]^\alpha$ ***Controller:*** The substitution of P(s) and C(s) expressions (given in (5) and (2) respectively) in (6)-(8) yields the following expressions for controller parameters K_p, K_i, and α (See APPENDIX 1 for derivation.):

$$K_i = -\omega_c \tan\left(\frac{-tan^{-1}\left(\frac{q_1}{p_1}\right)+tan^{-1}\left(\frac{q_2}{p_2}\right)+L\omega_c-\pi+\phi_m}{\alpha}\right) \quad (13)$$

$$K_i = \frac{\alpha \pm \sqrt{\alpha^2-4N^2\omega_c^2}}{2N} \quad (14)$$

$$K_p = \frac{1}{K}\sqrt{\frac{(p_2^2+q_2^2)}{(p_1^2+q_1^2)\left(1+\left(\frac{K_i}{\omega_c}\right)^2\right)^\alpha}} \quad (15)$$

where,

$$p_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\cos\left(\frac{\pi}{2}\alpha_i\right)\right), q_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)$$
$$p_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\cos\left(\frac{\pi}{2}\beta_k\right)\right), q_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\sin\left(\frac{\pi}{2}\beta_k\right)\right)$$

$$N = \frac{-p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right)+q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2+q_1^2}$$
$$+L+\frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right)-q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2+q_2^2}$$

On solving (13), (14), and (15) simultaneously, we get K_p, K_i, and α.

$[PD]^\beta$ ***Controller:*** The substitution of P(s) and C(s) expressions (given in (5) and (4) respectively) in (6)-(8) yields the following expressions for controller parameters K_p, K_d, and β (See APPENDIX 2 for derivation.):

$$K_d = \frac{\tan\left(\frac{-tan^{-1}\left(\frac{q_1}{p_1}\right)+tan^{-1}\left(\frac{q_2}{p_2}\right)+L\omega_c-\pi+\phi_m}{\beta}\right)}{\omega_c} \quad (16)$$

$$K_d = \frac{\beta \pm \sqrt{\beta^2-4N^2\omega_c^2}}{2N\omega_c^2} \quad (17)$$

$$K_p = \frac{1}{K}\sqrt{\frac{(p_2^2+q_2^2)}{(p_1^2+q_1^2)\left(1+(K_d\omega_c)^2\right)^\beta}} \quad (18)$$

where,

$$p_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\cos\left(\frac{\pi}{2}\alpha_i\right)\right), q_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)$$
$$p_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\cos\left(\frac{\pi}{2}\beta_k\right)\right), q_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\sin\left(\frac{\pi}{2}\beta_k\right)\right)$$

$$N = \frac{-p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right)+q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2+q_1^2}$$
$$+L+\frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right)-q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2+q_2^2}$$

On solving (16), (17), and (18) simultaneously, we get K_p, K_d, and β.

Remark 2. It is important to note that Wang et al. (2009b) specifications only ensure the required positive phase margin at a given gain crossover frequency and don't guarantee closed loop stability in general. For instance, if there occur multiple gain crossover frequencies, such restrictive specifications cannot ensure all the phase margins to be positive. Hence, the generalized derivations presented in this section are useful only for those plants which lead to closed loop stablilty. Therefore, closed loop stability check is essential after designing the controller for Wang et al. (2009b) specifications.

4. ILLUSTRATION WITH EXAMPLES

In this section, the design of $[PI]^\alpha$ and $[PD]^\beta$ controllers using the generalized tuning expressions proposed in section 3.2 is demonstrated by considering several plant examples belonging to different classes.

Example 1. $[PI]^\alpha$ Design for Fractional Horsepower Dynamometer

The fractional horsepower dynamometer Luo et al. (2010) has following transfer function:

$$P(s) = \frac{1}{0.4s^{0.5}+1}$$

The above plant is compared with the generalized plant structure (5) and (13), (14), and (15) are solved simultaneously for $\omega_c = 10rad/s$, $\phi_m = 70°$ to get the following controller:

$$C(s) = 0.2097\left(1+\frac{97.8062}{s}\right)^{1.007}$$

For solving (13) and (14) simultaneously, we follow here a graphical approach in which K_i is plotted against α as per (13) and (14). This leads to two intersecting curves. The point of intersection gives the solution for K_i and α. Further, K_p is obtained using (15). Fig. 2 shows the graphical approach adopted here.

For assessing the performance of designed controller, Oustaloup Oustaloup et al. (2000), Vinagre et al. (2000) approximation of the fractional order term is considered.

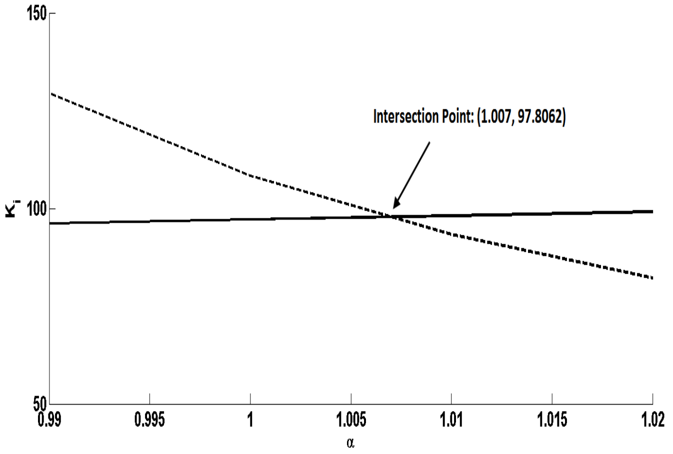

Fig. 2. Graphical approach to obtain K_i and α (fractional horsepower dynamometer)

The order of approximation is taken as 3 and it is valid over [0.001, 1000] rad/s.

Fig. 3 shows the bode plot for $[PI]^\alpha$ compensated plant for fractional horsepower dynamometer. It is seen from the figure that the designed controller meets the desired phase margin and gain crossover frequency specifications.

It is important to check the closed loop stability. One way to verify such is to obtain pole-zero map of the closed loop transfer function. If all the poles lie on the left side of $j\omega$-axis, the closed loop system is stable. One can also get the closed loop stability status from the open loop bode response obtained using MATLAB Matlab (R2010a) toolbox (as seen in Fig. 3, for instance). For the fractional horsepower dynamometer case, thus, it is confirmed that the closed loop system is stable with the designed controller.

Fig. 4 shows the closed loop unit step response with the tuned controller for ± 20% variation in plant gain K around its nominal value. This shows that the isodamping condition is also met.

Example 2. $[PI]^\alpha$ Design for DC Motor Velocity Servo System

The DC motor velocity servo system Wang et al. (2009a) has following transfer function:

$$P(s) = \frac{1}{0.4s + 1}$$

For $\omega_c = 10 rad/s$, $\phi_m = 70°$, the resulting controller is:

$$C(s) = 2.7482 \left(1 + \frac{18.1507}{s}\right)^{0.5567}$$

The corresponding bode and closed loop step response plots are presented in Fig. 5 and 6 respectively. It can be seen from these responses that the designed controller meets the required specifications. Also, the closed loop system is stable as observed in Fig. 5.

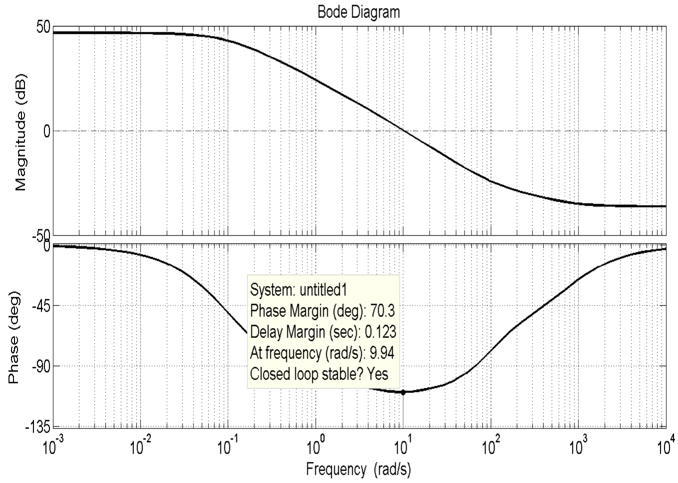

Fig. 3. Bode plot for $[PI]^\alpha$ compensated plant (fractional horsepower dynamometer)

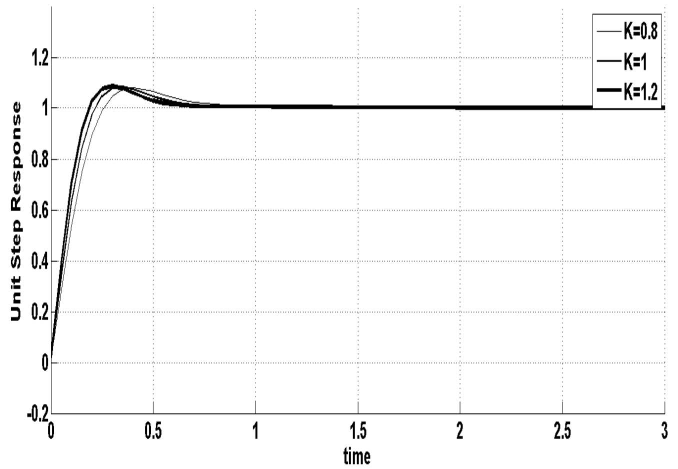

Fig. 4. Closed loop unit step response with $[PI]^\alpha$ controller (fractional horsepower dynamometer)

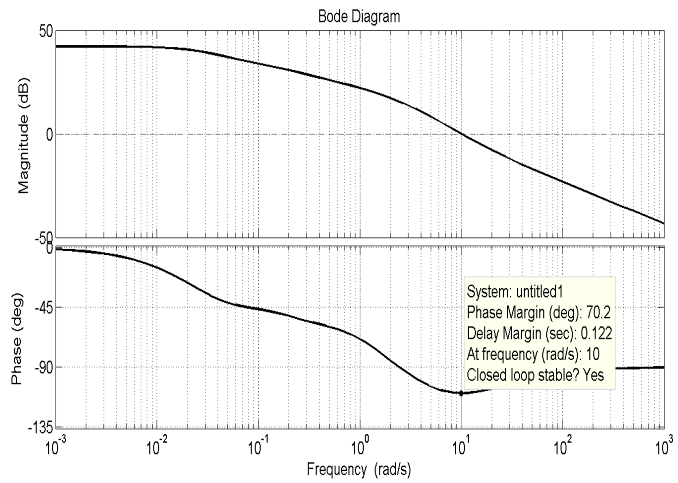

Fig. 5. Bode plot for $[PI]^\alpha$ compensated plant (DC motor velocity servo system)

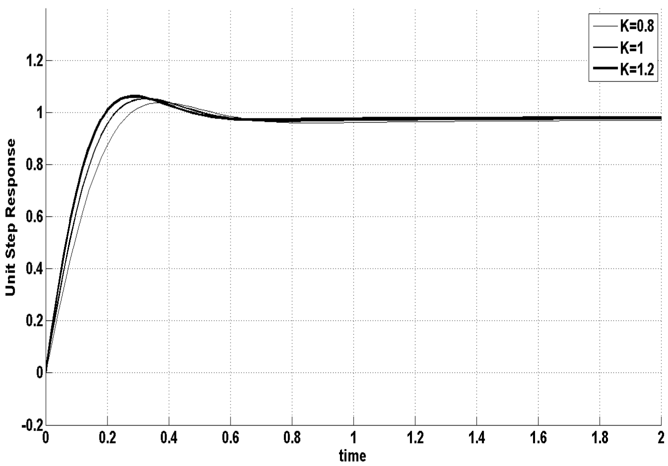

Fig. 6. Closed loop unit step response with $[PI]^\alpha$ controller (DC motor velocity servo system)

Example 3: $[PI]^\alpha$ Design for Precision Modular Servo System

An experimental set-up of precision modular servo System Reference-Manual-33-927S developed by Feedback Instruments, UK available at our Control Systems Lab is shown in Fig. 7. The set-up consists of DC Motor, Digital Encoder, Power Supply, Pre Amplifier, Servo Amplifier, and Analogue Control Interface units.

The complete mathematical model of the precision modular servo System is nonlinear due to presence of elements such as saturation limits in the Pre Amplifier and Servo Amplifier stages, friction in the Motor, static backlash due to clearance in the belt that connects Motor shaft to Digital Encoder.

For demonstrating the controller design with our unified expression technique, linear portion of the system in the form of its transfer function is considered. It relates pre amplifier input voltage to the voltage equivalent of DC Motor shaft angular position as follows:

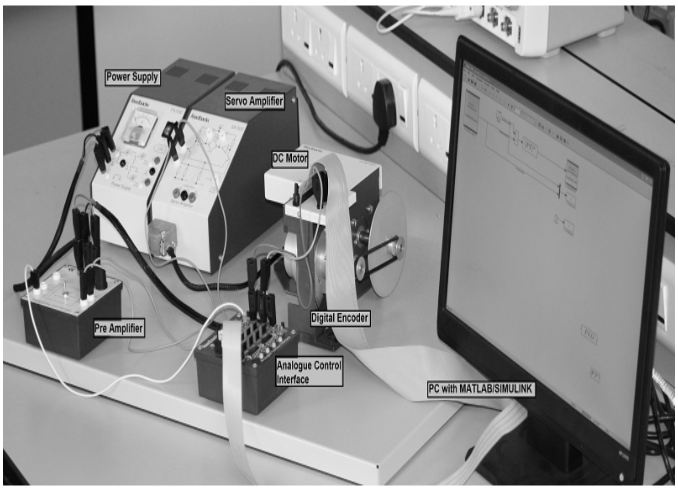

Fig. 7. Precision Modular Servo Real time Setup

$$P(s) = \frac{KK_t}{s(JLs^2 + (RJ + dL)s + (dR + K_bK_t))}$$

The numeric details of the plant parameters Reference-Manual-33-927S are given in Table 1.

Table 1. List of Plant Parameters

Symbol	Description	Value	Unit
J	Moment of Inertia	140×10^{-7}	kgm^2
K_t	Torque Constant	0.052	Nm/A
K_b	Electromotive Force Constant	0.057	Vs/rad
d	Linear Approximation of Viscous Friction	10^{-6}	Nms/rad
R	Resistance	2.5	Ω
L	Inductance	2.5	mH
K	Amplifier Gain	9.6	-

Therefore,

$$P(s) = \frac{1.4263 \times 10^7}{s^3 + 1000s^2 + 8.476 \times 10^4 s}$$

For $\omega_c = 10rad/s$, $\phi_m = 70°$, the resulting controller is:

$$C(s) = 0.0524\left(1 + \frac{13.7567}{s}\right)^{0.2459}$$

The corresponding bode plot and closed loop step response are presented in Fig. 8 and 9 respectively. It can be seen from these responses that the designed controller meets the required specifications. Also, the closed loop system is stable.

Real Time Testing: For the transfer function model of precision modular servo set-up, loop shaping is performed to meet Wang et al. (2009b) specifications as shown previously. It is important to notice that the nonlinearity in the plant hasn't been considered in the controller design stage. Nevertheless, such an approach is well suited for such plant cases where the nonlinearity effects on the transient response are mild.

In general, after design, the controller is tested with the real set-up. Subsequently, controller parameters are manually adjusted if the performance deviates from the desired one due to nonlinearities.

For the current case, we show here the real time testing results obtained with the designed controller ($C(s) = 0.0524\left(1 + \frac{13.7567}{s}\right)^{0.2459}$) using Hardware-in-loop configuration. For this purpose, a step reference input of magnitude 10 is given and the corresponding response is obtained as presented in Fig. 10.

It is observed from Fig. 10 that due to presence of nonlinearity, sustained oscillations are produced in the steady state of step response. Performance analysis of the transient response and fine tuning of the controller parameters is not discussed here owing to the limited scope of the current paper.

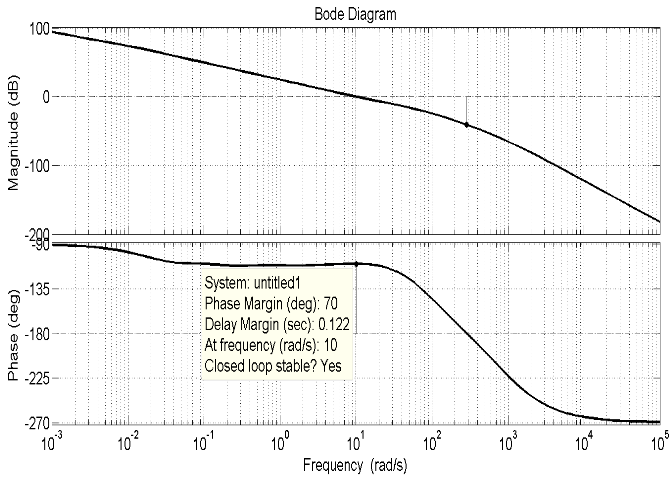

Fig. 8. Bode plot for $[PI]^\alpha$ compensated plant (Precision Modular Servo system)

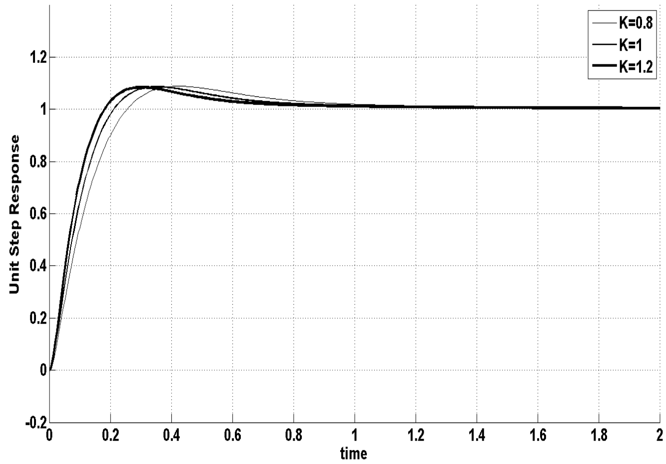

Fig. 9. Closed loop unit step response with $[PI]^\alpha$ controller (Precision Modular Servo system)

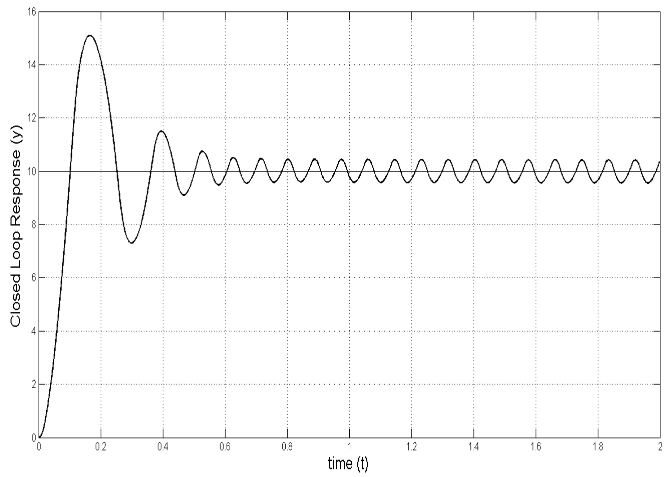

Fig. 10. Real time closed loop step response with $[PI]^\alpha$ controller (Precision Modular Servo system)

Example 4. $[PD]^\beta$ Design for Fractional Order Thermal Process

The following transfer function describes a thermal process Petras et al (2002) heated by an electrical radiator with the temperature measured by a pyrometer:

$$P(s) = \frac{1}{39.69s^{1.26} + 0.598}$$

For $\omega_c = 0.5rad/s$, $\phi_m = 70°$, (16), (17), and (18) are solved simultaneously to get the following controller:

$$C(s) = 16.2769\,(1 + 0.6484s)^{0.0824}$$

Fig. 11 shows the bode Plot for $[PD]^\beta$ compensated thermal process. From the plot, it is observed that the required gain crossover frequency and phase margin are met. Also, phase plot is locally flat around ω_c. This ensures constant phase margin in spite of plant gain variations. Thus, the isodamping condition is also satisfied. Further, closed loop system is stable as seen from Fig. 11.

Fig. 12 shows the closed loop unit step response with the tuned controller for ± 20% variation in thermal process gain K around its nominal value. The constant maximum peak overshoot in spite of process gain variations confirms the isodamping condition in time domain.

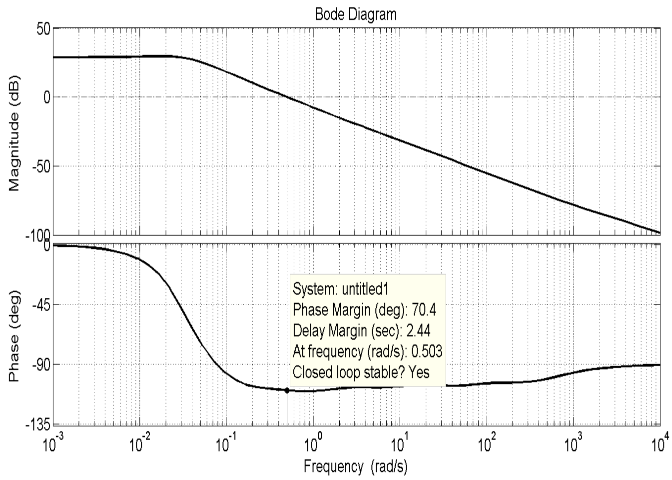

Fig. 11. Bode plot for $[PD]^\beta$ compensated plant (thermal process)

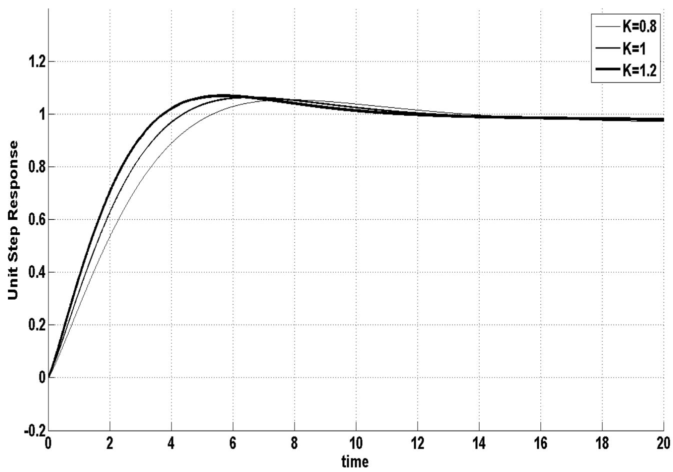

Fig. 12. Closed loop unit step response with $[PD]^\beta$ controller (thermal process)

Example 5. $[PD]^\beta$ Design for DC Motor Position Servo System

The DC motor position servo system has following transfer function Li (2008), Luo (2009):

$$P(s) = \frac{1}{s(0.4s+1)}$$

For $\omega_c = 10 rad/s$, $\phi_m = 70°$, (16), (17), and (18) are solved simultaneously to get the following controller:

$$C(s) = 16.7780\,(1 + 0.2992s)^{0.7826}$$

Fig. 13 shows the bode plot for $[PD]^\beta$ compensated DC motor position servo system. Fig. 14 shows the closed loop unit step response with the tuned controller for ± 20% variation in plant gain K around its nominal value. It can be seen from these responses that the desired conditions are met. Also, closed loop stability is maintained as seen from Fig. 13.

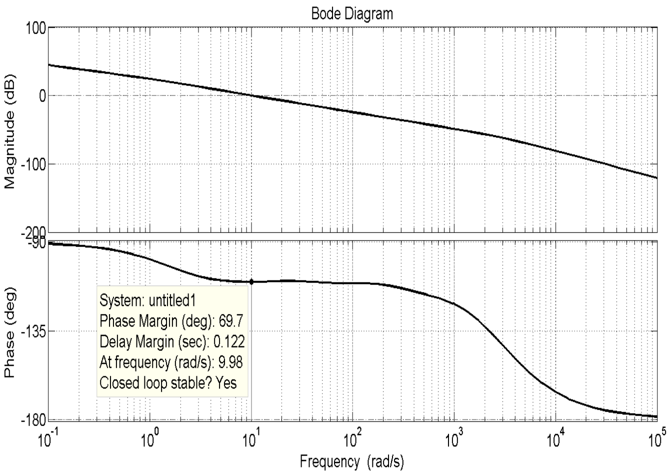

Fig. 13. Bode plot for $[PD]^\beta$ compensated plant (DC motor position servo system)

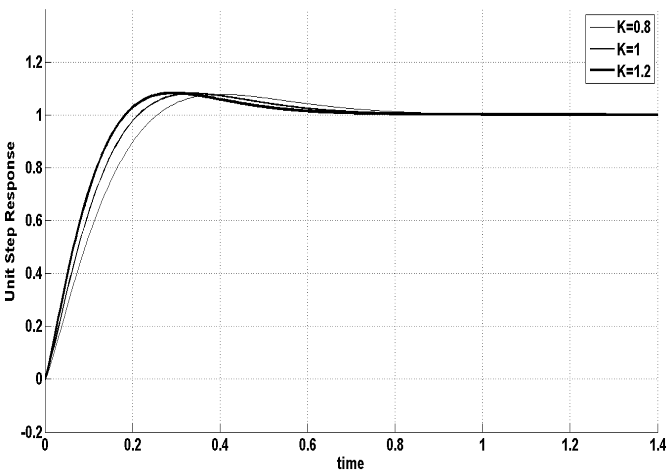

Fig. 14. Closed loop unit step response with $[PD]^\beta$ controller (DC motor position servo system)

Thus, (16), (17), and (18) are used to design $[PD]^\beta$ controller for different class of plants.

Thus, the illustration of our derivations for $[PI]^\alpha$ and $[PD]^\beta$ controllers is made by considering plants such as fractional thermal process, DC motor position servo system, precision modular servo system, fractional horsepower dynamometer, DC motor velocity servo system. The results are summarized in Table 2.

5. CONCLUSION

In this paper, the readily usable tuning expressions for three-parameter-fractional-controllers such as $[PD]^\beta$ and $[PI]^\alpha$ have been presented. These novel expressions are applicable to any class of integer or fractional order LTI plant. The designed controllers satisfy required phase margin, gain crossover frequency and isodamping property. Usefulness of such expressions has been demonstrated with several integer and fractional order plants which include:

- Fractional Thermal Process
- DC Motor Position Servo System
- Real Time Precision Modular Servo Set-up
- Fractional Horsepower Dynamometer
- DC Motor Velocity Servo System

The results with these cases confirm the correctness of the proposed derivations.

REFERENCES

Chen Y. Q. and Moore, K.L.. *Relay feedback tuning of robust PID controllers with iso-damping property.* Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on , vol. 35, no. 1, pp. 23-31, 2005.

Chen Y. Q., Petras I., and Xue D.u.. *Fractional order control- A tutorial.* in Proc. of American Control Conference, WeC02 Tutorial Session, Grand Ballroom B (east), 2009.

Kesarkar A. A., Selvaganesan N., *Tuning of Robust PI^α/PD^β Controller for Generalized Plant Structure*, Proceedings of the IEEE Conference on Recent Advancements in Electrical, Electronics and Control Engineering (ICONRAEeCE), pp. 104-108, 2011.

Kesarkar A. A., Selvaganesan N., *Design of Fractional Order Robust Controller for Universal Plant Structure*, Proceedings of the IEEE Conference on Current Trends in Technology (NUiCONE), Ahmedabad, India, pp. 1-4, 2011.

Li H. S. and Chen Y. Q. *A fractional order proportional and derivative (FOPD) controller tuning algorithm.* Chinese Control and Decision Conference (CCDC), pp. 4059-4063, 2008.

Li H. S., Luo Y., and Chen Y. Q. *A Fractional Order Proportional and Derivative (FOPD) Motion Controller: Tuning Rule and Experiments.* IEEE Transactions on Control Systems Technology, vol. 18, no. 2, pp. 516-520, 2010.

Luo Y. and Chen Y. Q.. *Fractional-order [proportional derivative] controller for robust motion control: Tuning procedure and validation.* Proceedings of the American Control Conference, St. Louis, Missouri, pp. 1412-1417, 2009.

Luo Y., Chen Y. Q., Wang C. Y., and Pi Y. G.. *Tuning fractional order proportional integral controllers for fractional order systems.* Journal of Process Control, vol. 20, no. 7, pp. 823-831, 2010.

Table 2. Results for $[PI]^{\alpha}$ and $[PD]^{\beta}$ Controllers

	Plant and Specifications	**Generalized Plant Structure Parameters**	**Designed Controller**
Example 1	$\frac{1}{0.4s^{0.5}+1}$ (Fractional Horsepower Dynamometer) $\omega_c = 10rad/s$ $\phi_m = 70°$	K=1, $L = 0$, a_0=1, α_0=0 b_0=0.4, β_0=0.5, b_1=1, β_1=0,	$0.2097\left(1+\frac{97.8062}{s}\right)^{1.007}$
Example 2	$\frac{1}{0.4s+1}$ (DC Motor Velocity Servo System) $\omega_c = 10rad/s$ $\phi_m = 70°$	K=1, $L = 0$, a_0=1, α_0=0 b_0=0.4, β_0=1 b_1=1, β_1=0,	$2.7482\left(1+\frac{18.1507}{s}\right)^{0.5567}$
Example 3	$\frac{1.4263\times10^7}{s^3+1000s^2+8.476\times10^4 s}$ (Precision Modular Servo) $\omega_c = 10rad/s$ $\phi_m = 70°$	K=1, a_0=1.4263×10^7, α_0=0, b_0=1, β_0=3 b_1=1000, β_1=2, b_2=8.476×10^4, β_2=1, L=0	$0.0524\left(1+\frac{13.7567}{s}\right)^{0.2459}$
Example 4	$\frac{1}{39.69s^{1.26}+0.598}$ (Fractional Thermal Process) $\omega_c = 0.5rad/s$ $\phi_m = 70°$	K=1, a_0=1, α_0=0 b_0=39.69, β_0=1.26 b_1=0.598, β_1=0, L=0	$16.2769\,(1+0.6484s)^{0.0824}$
Example 5	$\frac{1}{s(0.4s+1)}$ (DC Motor (Position Servo System) $\omega_c = 10rad/s$ $\phi_m = 70°$	K=1, a_0=1, α_0=0 b_0=0.4, β_0=2 b_1=1, β_1=1, L=0	$16.7780\,(1+0.2992s)^{0.7826}$

Manual: 33-927S, 'Precision Modular Servo Control Experiments', *Feedback Instruments Ltd.*, UK.

MATLAB version 7.10.0 (R2010a), Natick, Massachusetts, The MathWorks Inc., 2010.

Monje C. A., Vinagre B. M., Chen Y. Q., Feliu V., Lanusse P., and Sabatier J.. *Proposals for fractional $PI^{\lambda}D^{\mu}$ tuning.* The First IFAC Symposium on Fractional Differentiation and its Applications, Bordeaux, France, July 19-20, 2004.

Oldham K. B. and Spanier J.. *The Fractional Calculus.* Academic Press, New York, 1974.

Oustaloup A., Levron F., Mathieu B., and Nanot F. M.. *Frequency-Band Complex Noninteger Differentiator: Characterization and Synthesis.* IEEE Transactions On Circuits And Systems I: Fundamental Theory And Applications, vol. 47, no. 1, pp. 25-40, 2000.

Padula F. and Visioli A.. *Tuning rules for optimal PID and fractional-order PID controllers* Journal of Process Control, vol. 21, no. 1, pp. 69-81, 2011.

Petras I., Vinagre B. M., Dorcak L., and Feliu V.. *Fractional digital control of a heat solid: Experimental results.* In International Carpathian control conference, Malenovice, Czech Republic, pp. 365-370, 2002.

Podlubny I. *Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers.* IEEE Transactions on Automatic Control, vol. 44, no. 1, pp. 208-214, 1999.

Vinagre B. M., Podlubny I., Herndandez A. and Feliu V.. *Some Approximations Of Fractional Order Operators Used In Control Theory And Applications.* Fractional calculus and applied analysis, vol. 3, no. 3, pp. 231-248, 2000.

Wang C. Y., Luo Y., and Chen Y. Q.. *An Analytical Design of Fractional Order Proportional Integral and [Proportional Integral] Controllers for Robust Velocity Servo* Proceedings of the 4th IEEE Conference on Industry Electronics and Applications (ICIEA), Xian, China, pp. 3448-3453, 2009.

Wang C. Y., Luo Y., and Chen Y. Q.. *Fractional Order Proportional Integral (FOPI) and [Proportional Integral] (FO[PI]) Controller Designs for First Order Plus Time Delay (FOPTD) Systems.* Chinese Control and Decision Conference, pp. 329-334, 2009.

Xue D., Chen Y. Q. and Atherton D.. *Linear Feedback Control-Analysis and Design with Matlab.* Chapter-8: Fractional-order Controller - An Introduction, SIAM Press, 2007.

APPENDIX 1

Derivation for $[PI]^{\alpha}$ Parameters

The generalized plant structure as given in (5) is:

$$P(s) = K\frac{\sum_{i=0}^{m}(a_i s^{\alpha_i})}{\sum_{k=0}^{n}(b_k s^{\beta_k})}e^{-Ls}$$

The $[PI]^{\alpha}$ controller has following structure as given in (2):

$$C(s) = K_p\left(1+\frac{K_i}{s}\right)^{\alpha}$$

Therefore, $[P(s)C(s)]_{s=j\omega}$

$$= K\frac{\sum_{i=0}^{m}(a_i (j\omega)^{\alpha_i})}{\sum_{k=0}^{n}(b_k (j\omega)^{\beta_k})}e^{-L(j\omega)}K_p\left(1+\frac{K_i}{j\omega}\right)^{\alpha}$$

$$= K\frac{\sum_{i=0}^{m}(a_i \omega^{\alpha_i} j^{\alpha_i})}{\sum_{k=0}^{n}(b_k \omega^{\beta_k} j^{\beta_k})}e^{-L(j\omega)}K_p\left(1-j\frac{K_i}{\omega}\right)^{\alpha}$$

$$= K\frac{\sum_{i=0}^{m}(a_i \omega^{\alpha_i} e^{j\frac{\pi}{2}\alpha_i})}{\sum_{k=0}^{n}(b_k \omega^{\beta_k} e^{j\frac{\pi}{2}\beta_k})}e^{-L(j\omega)}K_p\left(1-j\frac{K_i}{\omega}\right)^{\alpha}$$

$$= K\frac{\sum_{i=0}^{m}\left(a_i \omega^{\alpha_i}\left(cos(\frac{\pi}{2}\alpha_i)+jsin(\frac{\pi}{2}\alpha_i)\right)\right)}{\sum_{k=0}^{n}\left(b_k \omega^{\beta_k}\left(cos(\frac{\pi}{2}\beta_k)+jsin(\frac{\pi}{2}\beta_k)\right)\right)}e^{-L(j\omega)}K_p\left(1-j\frac{K_i}{\omega}\right)^{\alpha}$$

Therefore, $[P(s)C(s)]_{s=j\omega_c}$

$$= K\frac{p_1+jq_1}{p_2+jq_2}e^{-L(j\omega_c)}K_p\left(1-j\frac{K_i}{\omega_c}\right)^{\alpha}$$

where,

$$p_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)$$

$$q_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)$$

$$p_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\cos\left(\frac{\pi}{2}\beta_k\right)\right)$$

$$q_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\sin\left(\frac{\pi}{2}\beta_k\right)\right)$$

Therefore, $P(j\omega_c)C(j\omega_c)$

$$= K\frac{p_1+jq_1}{p_2+jq_2}e^{-L(j\omega_c)}K_p\left(\sqrt{1+\left(\frac{K_i}{\omega_c}\right)^2}\,e^{jtan^{-1}\left(-\frac{K_i}{\omega_c}\right)}\right)^{\alpha}$$

Gain crossover frequency specification has been given in (7) as:

$$|C(j\omega_c)P(j\omega_c)| = 1$$

$$\therefore K\frac{\sqrt{p_1^2+q_1^2}}{\sqrt{p_2^2+q_2^2}}K_p\left(\sqrt{1+\left(\frac{K_i}{\omega_c}\right)^2}\right)^{\alpha} = 1$$

$$\therefore K_p = \frac{1}{K}\sqrt{\frac{(p_2^2+q_2^2)}{(p_1^2+q_1^2)\left(1+\left(\frac{K_i}{\omega_c}\right)^2\right)^{\alpha}}}$$

Phase margin specification has been given in (6) as:

$$\angle[C(j\omega_c)P(j\omega_c)] = -\pi+\phi_m$$

Therefore,

$$tan^{-1}\left(\frac{q_1}{p_1}\right)-tan^{-1}\left(\frac{q_2}{p_2}\right)-L\omega_c+\alpha tan^{-1}\left(-\frac{K_i}{\omega_c}\right) = -\pi+\phi_m$$

Therefore,

$$K_i = -\omega_c\tan\left(\frac{-tan^{-1}\left(\frac{q_1}{p_1}\right)+tan^{-1}\left(\frac{q_2}{p_2}\right)+L\omega_c-\pi+\phi_m}{\alpha}\right)$$

Isodamping condition has been expressed in (8) as:

$$\left(\frac{d(\angle[C(j\omega)P(j\omega)])}{d\omega}\right)_{\omega=\omega_c} = 0$$

Therefore,

$$\frac{p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right)-q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2+q_1^2}$$

$$-\frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right)-q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2+q_2^2}$$

$$-L+\alpha\frac{\frac{K_i}{\omega_c^2}}{1+\left(-\frac{K_i}{\omega_c}\right)^2} = 0$$

Let,

$$N = \frac{-p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right)+q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2+q_1^2}$$

$$+L+\frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right)-q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2+q_2^2}$$

Therefore,

$$\therefore -N+\frac{\alpha K_i}{\omega_c^2+K_i^2} = 0$$

Therefore,

$$K_i = \frac{\alpha\pm\sqrt{\alpha^2-4N^2\omega_c^2}}{2N}$$

Thus, the tuning expressions for $[PI]^{\alpha}$ controller have been derived.

APPENDIX 2

Derivation for $[PD]^{\beta}$ Parameters

The generalized plant structure as given in (5) is:

$$P(s) = K\frac{\sum_{i=0}^{m}(a_i s^{\alpha_i})}{\sum_{k=0}^{n}(b_k s^{\beta_k})}e^{-Ls}$$

The $[PD]^{\beta}$ controller has following structure as given in (4):

$$C(s) = K_p\left(1 + K_d s\right)^{\beta}$$

Therefore, $[P(s)C(s)]_{s=j\omega}$

$$= K\frac{\sum_{i=0}^{m}(a_i(j\omega)^{\alpha_i})}{\sum_{k=0}^{n}(b_k(j\omega)^{\beta_k})}e^{-L(j\omega)}K_p\left(1 + K_d(j\omega)\right)^{\beta}$$

$$= K\frac{\sum_{i=0}^{m}(a_i\omega^{\alpha_i}j^{\alpha_i})}{\sum_{k=0}^{n}(b_k\omega^{\beta_k}j^{\beta_k})}e^{-L(j\omega)}K_p\left(1 + K_d(j\omega)\right)^{\beta}$$

$$= K\frac{\sum_{i=0}^{m}(a_i\omega^{\alpha_i}e^{j\frac{\pi}{2}\alpha_i})}{\sum_{k=0}^{n}(b_k\omega^{\beta_k}e^{j\frac{\pi}{2}\beta_k})}e^{-L(j\omega)}K_p\left(1 + K_d(j\omega)\right)^{\beta}$$

$$= K\frac{\sum_{i=0}^{m}\left(a_i\omega^{\alpha_i}\left(cos(\frac{\pi}{2}\alpha_i) + jsin(\frac{\pi}{2}\alpha_i)\right)\right)}{\sum_{k=0}^{n}\left(b_k\omega^{\beta_k}\left(cos(\frac{\pi}{2}\beta_k) + jsin(\frac{\pi}{2}\beta_k)\right)\right)}e^{-L(j\omega)}K_p\left(1 + K_d(j\omega)\right)^{\beta}$$

Therefore, $[P(s)C(s)]_{s=j\omega_c}$

$$= K\frac{p_1 + jq_1}{p_2 + jq_2}e^{-L(j\omega_c)}K_p\left(1 + K_d(j\omega_c)\right)^{\beta}$$

where,

$$p_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)$$

$$q_1 = \sum_{i=0}^{m}\left(a_i\omega_c^{\alpha_i}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)$$

$$p_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\cos\left(\frac{\pi}{2}\beta_k\right)\right)$$

$$q_2 = \sum_{k=0}^{n}\left(b_k\omega_c^{\beta_k}\sin\left(\frac{\pi}{2}\beta_k\right)\right)$$

Therefore, $P(j\omega_c)C(j\omega_c)$

$$= K\frac{p_1 + jq_1}{p_2 + jq_2}e^{-L(j\omega_c)}K_p(\sqrt{1 + (K_d\omega_c)^2}e^{jtan^{-1}(K_d\omega_c)})^{\beta}$$

Gain crossover frequency specification has been given in (7) as:

$$|C(j\omega_c)P(j\omega_c)| = 1$$

Therefore,

$$K\frac{\sqrt{p_1^2 + q_1^2}}{\sqrt{p_2^2 + q_2^2}}K_p\left(\sqrt{1 + (K_d\omega_c)^2}\right)^{\beta} = 1$$

Therefore,

$$K_p = \frac{1}{K}\sqrt{\frac{(p_2^2 + q_2^2)}{(p_1^2 + q_1^2)\left(1 + (K_d\omega_c)^2\right)^{\beta}}}$$

Phase margin specification has been given in (6) as:

$$\angle[C(j\omega_c)P(j\omega_c)] = -\pi + \phi_m$$

Therefore,

$$tan^{-1}\left(\frac{q_1}{p_1}\right) - tan^{-1}\left(\frac{q_2}{p_2}\right) - L\omega_c + \beta tan^{-1}(K_d\omega_c) = -\pi + \phi_m$$

Therefore,

$$K_d = \frac{\tan\left(\frac{-tan^{-1}\left(\frac{q_1}{p_1}\right) + tan^{-1}\left(\frac{q_2}{p_2}\right) + L\omega_c - \pi + \phi_m}{\beta}\right)}{\omega_c}$$

Isodamping condition has been expressed in (8) as:

$$\left(\frac{d(\angle[C(j\omega)P(j\omega)])}{d\omega}\right)_{\omega=\omega_c} = 0$$

Therefore,

$$\frac{p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right) - q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2 + q_1^2}$$

$$-\frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right) - q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2 + q_2^2}$$

$$-L + \frac{\beta K_d}{1 + (\omega_c K_d)^2} = 0$$

Let,

$$N = \frac{-p_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\sin\left(\frac{\pi}{2}\alpha_i\right)\right)\right) + q_1\left(\sum_{i=0}^{m}\left(a_i\alpha_i\omega_c^{\alpha_i-1}\cos\left(\frac{\pi}{2}\alpha_i\right)\right)\right)}{p_1^2 + q_1^2}$$

$$+L + \frac{p_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\sin\left(\frac{\pi}{2}\beta_k\right)\right)\right) - q_2\left(\sum_{k=0}^{n}\left(b_k\beta_k\omega_c^{\beta_k-1}\cos\left(\frac{\pi}{2}\beta_k\right)\right)\right)}{p_2^2 + q_2^2}$$

Therefore,

$$-N + \frac{\beta K_d}{1 + (\omega_c K_d)^2} = 0$$

Therefore,

$$K_d = \frac{\beta \pm \sqrt{\beta^2 - 4N^2\omega_c^2}}{2N\omega_c^2}$$

Thus, the tuning expressions for $[PD]^{\beta}$ controller have been derived.