FOUR LECTURES ON STABILITY[1]

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Abstract: *Starting with the basic notions about Liapunov and input/output stability there are presented those elements that represent the minimal knowledge of Stability Theory and its methods.A particular attention is paid to absolute stability of feedback systems and connected framework (Liapunov functions, frequency domain inequalities, hyperstability and dissipativity).*
Acknowledgement*: In 1990 the Romanian Society for Automation and Industry Applied Information Processing emerged as an initiative of some outstanding control engineers active both in Education and Research. Systems and Control had been a long date field of interest in Romania however a dedicated academic and application oriented society was missing for, mildly said, non-academic reasons. Neither 1990 nor the following 1991 did not show very stimulating for normal daily research. For this reason the idea of Professor Ioan Dumitrache, first (and actual) President of the Society to organize a cycle of lectures on some basic (but freely chosen) topics of Systems and Control was highly stimulating for a new start. The cycle of lectures took place during the Academic Year 1991/1992 at "Politehnica" Technical University of Bucharest, Department of Automatic Process Control. These lecture notes represent author's contribution to that worth remembering event.*

Keywords*: Stability, Absolute stability, Hyperstability.*

1. LECTURE ONE. MODELS OF SYSTEMS AND ASSOCIATED STABILITY NOTIONS

The two systems representations are too well known to the specialists of the field to require a revisited presentation. We shall deal here only with those properties which are relevant to stability.

1.1. States, state equations and Liapunov stability

The modeling of the physical systems by state equations is based on the differential

[1] Presented at the Department of Automatic Process Control of "Politehnica" Technical University of Bucharest at the Lecture cycle organized by Romanian Society for Automation and Industry Applied Information Processing November 1991

equations which represent the mathematical expression of the physical laws ruling over the phenomena which take place within the system.

The only systemic aspect is here the "orientation" input/output in the sense of stimulus-response dependence between the terminal variables of the system. The choice of some terminal variables is essential in order to make a difference among inputs, disturbances and outputs - measurements that are performed upon the system.

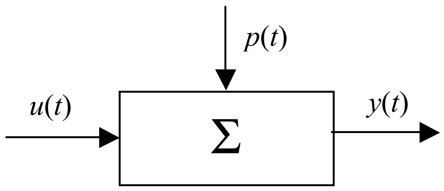

Fig. 1. The system

Writing the differential equations simultaneously with the orientation input/output leads, after giving to the equations the normal Cauchy form, to the standard input-state-output equations of the systems with lumped parameters:

$$\begin{aligned} \dot{x} &= f(x, u(t), p(t)) \\ y &= g(x, u, p) \end{aligned} \tag{1}$$

where $x \in \mathbf{P}^n$ is the state vector, $u \in \mathbf{P}^m$ is the input (control) vector, $p \in \mathbf{P}^l$ is the disturbance (also an input in the sense of stimulus but, unlike the control, inapplicable in generating controlled evolutions of the system) and $y \in \mathbf{P}^p$ is the output of the system.

Representation (1) is the expression of the *Newtonian physical determinism*: if the exogeneous signals $u(t)$ and $p(t)$ are known for $t \geq t_0$, the determination of the state trajectory $x(t)$ for $t \geq t_0$ is possible if it is solved the Cauchy problem associated to the system of differential equations; this requires the knowledge of the initial state $x(t_0) = x_0$. If we take into account the arbitrariness of the initial moment then it is obvious that the state at any moment can be an initial one in the sense that it generates an evolution of the system; by this way we arrive to the idea that the state is the *system information which is completely defined by its previous values*.

Such a Newtonian representation is "shadowed" by the exogeneous signals - control and disturbances - which have no correspondence in classical physics. It is nevertheless possible to introduce a notion of A.M.Liapunov [1] - *the basic motion*.

Definition 1 *Any state trajectory which is interesting for system analysis is called basic motion.*

Remark that in *Control Theory* any steady state motion could be considered as a basic motion. In a system of automatic stabilization of some operating point, the associated steady state can be considered as basic motion. In an electrical circuit with a.c. sources, the periodic currents through the branches represent the basic motion. The existence of the basic motion, especially for nonlinear systems, is itself a problem but it will not be tackled here. Our interest will be centered around the stability property.

According to a by now folk definition [2] stability is concerned with the influence of the disturbances on the motion of the physical systems (obviously motion is not only the mechanical motion but the state evolution in time).

Definition 2 *Any motion which is different from the basic one is called disturbed motion.*

In order to formulate a stability problem it is necessary to examine the types of disturbances that are met in systems. Obviously their complete knowledge is not possible: in such a case they will cease to be disturbances in the strict sense. But we may define classes of disturbances having in common one or several properties.

We shall call *short-period disturbances* those disturbances that appear at a certain moment and disappear at another moment. Mathematically a short-period disturbance is defined by

$$\begin{aligned} p(t) &\neq 0 \ , \quad t_{-1} \leq t \leq t_0 \\ p(t) &\equiv 0 \ , \quad \textit{elsewhere} \end{aligned} \tag{2}$$

There is a simple mechanism of describing the effect of the short-period disturbances which is illustrated in Fig.2: the system evoluates according to the basic motion (deduced from system's equations) when the disturbances occur at $t = t_{-1}$. As long as the disturbances are present, the dynamic laws of the motion are unknown hence the state trajectory cannot be determined. At $t = t_0$ the disturbances disappear hence the dynamic laws are valid again, according to the Newtonian determinism; the system has reached now some state $x_0 \neq \hat{x}(t_0)$, $\hat{x}(t_0)$ being the allowable state of the system if the disturbances were absent on $[t_{-1}, t_0]$ and the basic motion would have continued.

The new state x_0 which in fact incorporates the entire prehistory of the system is the source of a disturbed motion for $t > t_0$.

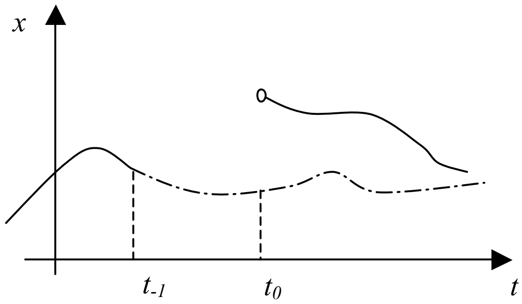

Fig. 2. The effect of short-period disturbances

basic motion $\hat{x}(t)$
real motion $x(t)$

Another way of generating perturbed motions, which is specific to Automatic Control, is represented by the maneuvers performed on physical systems endowed with automatic controllers (Fig. 3).

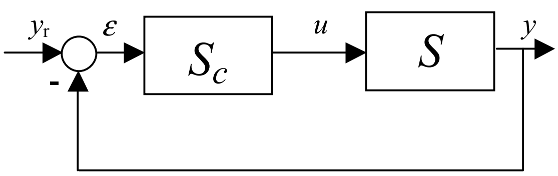

Fig. 3. Automatic control system

For $y_r = c_1$ a steady-state (operating point) is set up in the system; this operating point is described by some constant values of the parameters defining the state of the system. If there are no disturbances the system remains in that steady state. But system operation requires often some modifications of the steady state which are performed by modifying the reference signal y_r. Due to physical inertia (i.e. to system's dynamics) the system does not reach the new set point instantaneously but after a motion generated by the "old" steady state transformed in an initial condition. Indeed, consider the mathematical description of the system of Fig.3.

$$\begin{aligned} &\dot{x} = f(x, u(t)) \\ &y = g(x) \qquad (\boldsymbol{S}) \\ \\ &\dot{x}_c = f_c(x_c, \varepsilon(t)) \\ &u = g_c(x_c, \varepsilon(t)) \qquad (\boldsymbol{S}_c) \\ &\varepsilon = y_r - y \end{aligned} \tag{3}$$

From here the state equations of the feedback system are obtained

$$\begin{aligned} \dot{x} &= f(x, g_c(x_c, y_r - g(x))) \\ \dot{x}_c &= f_c(x_c, y_r - g(x)) \end{aligned} \tag{4}$$

To these equations we can add those outputs that might be of interest. According to the classical theory of linear control systems [3] such output signals might be the control error ε, system's output y, the control signal u.

For $t < t_0$ the following steady-state equations are valid

$$\begin{aligned} &f(x, g_c(x_c, c_1 - g(x))) = 0 \\ &f_c(x_c, c_1 - g(x)) = 0 \end{aligned} \tag{5}$$

At $t = t_0$ we have $y_r = c_2$ what asks for the following steady-state equations

$$\begin{aligned} &f(x, g_c(x_c, c_2 - g(x))) = 0 \\ &f_c(x_c, c_2 - g(x)) = 0 \end{aligned} \tag{6}$$

the new steady-state being, generally speaking, different of the previous one. If (x^i, x_c^i), $i = 1, 2$ are the two steady states then the perturbed motion will be generated by

$$\begin{aligned} &\dot{x} = f(x, g_c(x_c, c_2 - g(x))) \\ &\dot{x}_c = f_c(x_c, c_2 - g(x)) \\ &x(t_0) = x^1 \ , \quad x_c(t_0) = x_c^1 \end{aligned} \tag{7}$$

Now the way of introducing the stability concept becomes almost obvious. The basic motion is stable if the perturbed motions remain in its neighborhood. Even in the case of the maneuver we may say that the new steady state is stable provided the evolution of the system takes place in the neighborhood of that steady state. A stronger property is obtained when the disturbed motion tends asymptotically to the basic one or when the steady state imposed by the maneuver is reached asymptotically. These facts obtain a rigorous expression in the notion of stability in the sense of Liapunov.

Consider a general system described by

$$\dot{z} = F(z,t) \tag{8}$$

and let $\hat{z}(t)$ be a basic motion of it.

Definition 3 *The basic motion $\hat{z}(t)$ is called stable (in the sense of* ***Liapunov****) if for any $\varepsilon > 0$ and any $t_0 \in P$ there exists some $\delta(\varepsilon, t_0) > 0$ such that if $|\hat{z}(t_0) - z_0| < \delta$ then $|\hat{z}(t) - z(t;t_0,z_0)| < \varepsilon$, $\forall t \geq t_0$.*

Here $z(t;\ t_0,\ z_0)$ means the solution of (8) with the initial condition z_0 at $t = t_0$: $z(t_0;\ t_0,\ z_0) = z_0$.

Note also that in the above definition t_0 is the initial moment of the perturbed motion – when the short-period disturbances have just disappeared. Obviously in practice it is not possible to "catch" this moment and for this reason (as well as for many other) it is more useful to have a property which is independent of t_0.

Definition 4 *The basic motion is called uniformly stable if δ of Definition 3 is independent of t_0.*

A stronger property is that of asymptotic stability.

Definition 5 *The basic motion is called uniformly asymptotically stable if: i) it is uniformly stable; ii) it is equally attractive i.e. there exists $\delta_0 > 0$ such that for any z_0 with the property that $|\hat{z}(t_0) - z_0| < \delta_0$ and for any $\varepsilon > 0$ there exists $T(\delta_0, \varepsilon) > 0$ such that $|\hat{z}(t) - z(t;t_0,z_0)| < \varepsilon$, $\forall t > t_0 + T$.*

One can see that uniform asymptotic stability is exactly the property of asymptotic vanishing of the effect of short-period disturbances. The following remarks are also useful.

First, the property of stability concerns the basic motion and not the system itself which may have stable and unstable motions. The expression *stable system* corresponds to the reality only in the *linear case*.

Second, in all the definitions above there are involved only the differences between the basic and the perturbed motions - the so-called *deviations*. In order to simplify the writing there is introduced the system in deviations (with respect to the basic motion). Define the deviation

$$x(t) = z(t) - \hat{z}(t)$$

which will verify the system

$$\dot{x} = f(t,x) \tag{9}$$

where

$$f(t,x) := F\big(t, x(t) + \hat{z}(t)\big) - F\big(t, \hat{z}(t)\big)$$

Obviously

$$f(t,0) = F\big(t, \hat{z}(t)\big) - F\big(t, \hat{z}(t)\big) \equiv 0,$$

hence as one could expect the system in deviations has the trivial (identically zero) solution. The above definitions will be reformulated as follows

Definition 6 *a) The zero solution of system (9) is called stable if for any $\varepsilon > 0$ and any $t_0 \in P$ there exists $\delta(\varepsilon, t_0) > 0$ such that if $|x_0| < \delta$ then $|x(t;t_0,x_0)| < \varepsilon$, $t > t_0$.*

b) If δ can be chosen independently of t_0 the zero solution is called uniformly stable.

c) The zero solution is called uniformly asymptotically stable if it is uniformly stable and equally attractive i.e. there exists $\delta_0 > 0$ such *that* $\forall |x_0| < \delta_0$ *and any* $\varepsilon > 0$ *there exists* $T(\delta_0, \varepsilon) > 0$ *such that* $|x(t;t_0,x_0)| < \varepsilon$, $t > t_0 + T$.

To end this section remark that the introduction of the system in deviations and of the stability for the zero solution gives to the approach some degree of "universality" and hides (in a way) the basic motion; nevertheless this is only apparent since, according to its definition, the system in deviations depends on the basic solution. Moreover, if the initial system is time invariant but the basic motion is not constant, the system in deviations will result time-varying.

$$\dot{z} = F(z)\,, \quad f(t,x) = F\left(x + \hat{z}(t)\right) - F\left(\hat{z}(t)\right)$$

1.2. Signals, input/output equations and input/output stability

We return to the general model of the system as in Fig.1. If we are interested only in the behavior viewed at the terminal variables, the output $y(t)$ appears as a "result" of the control signal $u(t)$ and of the disturbance $p(t)$. For a self-contained theory of such systems there is no need of the notion of state but only of the notion of *ground state(zero-state)*.

Definition 7 *A system has the ground state if from* $u(t) \equiv 0, p(t) \equiv 0$ *it follows that* $y(t) \equiv 0$.

In this way the system is defined as *signal producer* based on other signals. Worth mentioning that the effect of the initial state may be incorporated in a perturbation; consider the linear system:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u(t) \\ y &= C(t)x\,, \quad x(t_0) = x_0 \end{aligned} \tag{10}$$

and let $X_A(t,\tau)$ be the state transition matrix defined by

$$\begin{aligned} &\frac{\partial}{\partial t} X_A(t,\tau) = A(t) X_A(t,\tau), \\ &X_A(\tau,\tau) = I \end{aligned} \tag{11}$$

The variations of parameters formula will give

$$x(t) = X_A(t,t_0)x_0 + \int_{t_0}^{t} X_A(t,\tau)B(\tau)u(\tau)d\tau$$

$$\begin{aligned} y(t) = {} & C(t)X_A(t,t_0)x_0 + \\ & + \int_{t_0}^{t} C(t)X_A(t,\tau)B(\tau)u(\tau)d\tau \end{aligned}$$

The last equality shows that the term which is dependent on the initial condition x_0 can be considered as an additional disturbance. The equality has in fact the form of an input (disturbance)/output relation:

$$y(t) = p(t) + \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau \tag{12}$$

where $H(t,\tau)$ is the matrix of the weighting patterns, defining an input/output operator. Starting from (12) the system appears to be a mapping between two sets of signals and, therefore, stability will result also as an input/output property; its natural definition is that of the *boundedness* of the above defined *input/output mapping* [4]. It is quite clear that the input/output stability is somehow arising from stability with respect to *persistent perturbations* [4]. The most important problem is here the suitable choice of the signal spaces with respect to which input/output stability is considered.

To fix the ideas consider the case of the systems described by nonlinear integral equations of *Hammerstein* type

$$y(t) = p(t) - \int_{0}^{t} h(t,\tau)\varphi\left(u(\tau)\right)d\tau \tag{13}$$

where $\varphi : \mathbf{P} \to \mathbf{P}$ is a nonlinear function, $\varphi(0) = 0$. Let U be a space of the input signals u and p and Y be a space of the output signals y which remain unspecified for a while.

Definition 8 *The ground state of system (13) is stable with respect to the pair* (U, Y) *or the pair* (U, Y) *is admissible with respect to (13) if* $p, u \in U \Rightarrow y \in Y$.

A way of specifying the spaces U, Y is to make use of the signal spaces with physical significance, well known from Signal Theory.

We give below some of them which are all Banach spaces:
a) the space of *finite energy* signals, defined on the time interval (*a, b*) with the norm

$$\int_a^b |x(t)|^2 dt < \infty$$

hence the space $\mathbf{L}^2(a, b)$.
b) the space of fi*nite action* signals, defined on the interval (*a, b*) with the norm

$$\int_a^b |x(t)| dt < \infty$$

hence the space $\mathbf{L}^1(a, b)$.
c) the space of *essentially bounded* on [*a, b*] signals with the norm

$$\operatorname*{ess\,sup}_{t \in [a,b]} |x(t)| < \infty$$

hence the space $\mathbf{L}^\infty(a, b)$; an important subspace of $\mathbf{L}^\infty(a, b)$ is $\boldsymbol{C}(a, b)$, the space of the continuous on [*a, b*] functions, with the uniform convergence norm (compatible with the *esssup* norm).

Using these spaces one may define the notions of stability that are widely used in Control and in Electronic engineering:

- $\mathbf{L}^2$ stability or energetic stability;
- BIBO (bounded input/bounded output) stability which is known since *Nyquist* or even earlier.

We have to mention here that stability with respect to persistent perturbations [2] is also defined within the framework of the space L^∞ i.e. of the bounded signals and perturbations.

1.3. Concluding remarks

We introduced above the two main concepts of stability which are associated to the main models of systems. Which of the two concepts is to be chosen depends on the application, more precisely on the interest paid to short-period or to persistent perturbations.

The state space is not relevant and is not involved too much in the input/output stability analysis. The results are in fact dependent of the system and of the signal spaces. The input/output stability is particularly suitable for systems with distributed parameters i.e. described by partial differential equations; for these systems the state space and the state transition might be complicated but they are somehow avoided in the developments.

On the contrary, the stability in the sense of Liapunov is an internal property of the system, the inputs and the outputs being of secondary interest. Nevertheless this concept is also important for control when short-period perturbations are considered. Also a connection between the two concepts as well as between the corresponding stability criteria may be established (at least in the linear case).

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