Pseudo Linear Systems: Stability Analysis and Limit Cycle Emergence

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Abstract: Nonlinear autonomous systems, often arise in different fields of science, are usually difficult to analyze. Pseudo linear representation of such systems recently has become very popular to deal with this difficulty. This paper presents a deep through analysis about the stability of nonlinear autonomous systems represented by pseudo linear forms. By discretization of the continuous-time dynamical systems, the stability of original nonlinear system is investigated via the discretized model and some new results are obtained for a class of pseudo linear systems. Due to the fact that the error between discrete model and that of the original continuous-time model vanishes as the sampling time goes to zero, in this paper we consider almost zero sampling time throughout of our analysis to make sure about the validity of results obtained for the continuous-time nonlinear systems. Based on the discretized model, some conclusive propositions are established; this apparently provides a framework to tackle the long struggle in the stability consideration of nonlinear systems via pseudo linear form by applying the crucial role of nonlinear eigenvectors neglected in the previous studies. In addition, based on these stability analysis results and the qualitative analysis tools provided through pseudo linear representation of nonlinear systems, the question of limit cycle emergence is also tackled. It is shown that, as an elegant application of the proposed qualitative analysis tool, the generation of limit cycles with desired shape and numbers can be easily performed. Some illustrative examples are finally given to highlight the validity of the proposed analysis technique.

Keywords: Pseudo Linear Systems, Stability Analysis, Limit Cycles, Nonlinear Eigenvalues, Nonlinear Eigenvectors.

1. INTRODUCTION

Knowing the fact that autonomous equations often arise in different field of science and almost all dynamical systems in nature are governed by weakly or strongly nonlinear equations of this kind, in recent decades analysis of nonlinear autonomous system has been a popular field of study. Since the 1970's, (Khalil, 1996), the formulation and analysis of procedures for systematic design of nonlinear controllers has attracted significant research interest. Applying the ease of linear system analysis leads to introduce pseudo linear form representation of nonlinear systems by (Banks and Mhana, 1992) which is also known as extended linearization, (Friedland, 1996), or state dependent coefficient (SDC) parameterization (Cloutier *et al.,* 1996; Mracek and Cloutier, 1998; Cloutier, 1997). The concern here is also to make a good and yet systematic trade-off between state error and input effort via a linear state dependent Riccati equation (SDRE).

The main problem with this approach is the lack of complete and yet reliable stability analysis. A review of the recently published papers on this field makes it evident that the stability analysis of nonlinear systems via pseudo linear representation is still a challenging issue. Originally, (Banks and Mhana, 1992) proposed that if for every $\mathbf{x} \in \mathbb{R}^n$, all eigenvalues of the pseudo linear form of a nonlinear system are located in the left half region of the complex plane, the global asymptotic stability of the nonlinear system is guaranteed. Later in (Tsiotras *et al.,* 1996), by a counter example it has been shown that the Banks' conclusion is not true in general. Despite meeting the condition of the proposed theorem in (Banks and Mhana, 1992), their counterexample has unbounded solutions for some initial states $\mathbf{x}(0)$. Then, (Banks and Mhana, 1996) proposed two other stability theorems based on the original one by 1) imposing some further assumptions on $\left|\frac{\partial A(\mathbf{x})}{\partial \mathbf{x}}\right|$ and 2) replacing $A(\mathbf{x})$ with $A(\mathbf{x})+A(\mathbf{x})^T$. These results were not completely right as fully discussed in (Langson and Alleyne, 2002), by presenting a counterexample and showing that the theorem in (Banks and Mhana, 1996) only leads to local stability. They also introduced some further assumptions to assure global stability. In (Muhammad and Van Der Woude, 2009), via some other counterexamples it has been shown that even the results proposed by (Langson and Alleyne 2002), may not be always true. In short, a reliable method for the stability analysis of nonlinear autonomous systems via pseudo linear form has not yet been found in the literature. In this paper, we try to tackle this issue based on the discretization of differential equation describing the underlying nonlinear system in the form of pseudo linear representation.

One of the most difficult problems connected with the study of nonlinear system is the question of limit cycles emergence. Multi-scale limit cycles also exist in many living systems as complex adaptive systems; hence it is no surprise that limit cycles are a topic of much interest due to the fact that complex adaptive systems happen to get benefit of limit cycles in order to survive on the edge of chaos (see Weisbuch, 1999 ; Pave, 2012). There are many researches related to the limit cycle behaviour including limit cycle generation, limit cycle detection, control of limit cycle (see Attabeigi *et al*., 2009 ; Kim and Robinson, 2008).

One of the principle motivations for the work reported here is the application of the stability analysis fully proposed and developed in the paper as well as our previous qualitative analysis tools (Ghane and Menhaj, 2013) to tackle the challenging issue of limit cycle emergence in some class of nonlinear systems. Through the proposed qualitative analysis tool, a systematic procedure is provided to generate limit cycles with desired shapes and numbers.

An outline of this paper is as follows. In section 2, the pseudo linear form of a nonlinear system is presented and the concept of NEValues and NEVectors are then introduced. This section also presents non-uniqueness of the pseudo linear forms of a nonlinear system along with a systematic method for obtaining infinite many pseudo linear forms by introducing a new concept of basis set for the space of pseudo linear forms. The main results about the stability of the nonlinear system via the aforementioned infinite pseudo linear forms representation is fully discussed in section 3. The qualitative analysis of limit cycles is thoroughly proposed in section 4. Section 5 is devoted to present some illustrative examples, in two parts: one for stability analysis including two counterexamples of Tsiotras *et al.* (1996) and Muhammad and Van Der Woude (2009) and two for limit cycle emergence examples along with the phase plane simulations. Finally, concluding remarks and future work are presented in section 6.

2. PSEUDO LINEAR SYSTEMS

An autonomous nonlinear system is a system of nonlinear ordinary differential equations which does not explicitly depend on the independent variable. It is of the form

$$\dot{\mathbf{x}}(t)=\mathbf{f}\left(\mathbf{x}(t)\right) \tag{1}$$

where $\mathbf{x}$ takes values in n - dimensional Euclidian space and the independent variable t is usually time. Inspiring from the linear system theory, assuming that $\mathbf{f}(\mathbf{0})=\mathbf{0}$, it is possible to transform an autonomous system of the form (1) to a new form as:

$$\frac{d}{dt}\mathbf{x}(t)=A\left(\mathbf{x}(t)\right)\mathbf{x}(t) \tag{2}$$

where $A:\mathbb{R}^n\rightarrow\mathbb{R}^{n\times n}$. This form is called pseudo linear (PL) and it was originally introduced in Banks and Mhana (1992) to cope with the difficulty of designing nonlinear optimal control laws.

2.1 Nonlinear Eigenvalues (NEValues) and Nonlinear Eigenvectors (NEVectors)

After obtaining the PL form of (2), it would be possible to extend the eigenstructure concept to these systems. In other words, by defining $\lambda(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ as nonlinear eigenvalue (NEValue) and its corresponding nonlinear eigenvector (NEVector), one can write:

$$A(\mathbf{x})\mathbf{v}(\mathbf{x})=\lambda(\mathbf{x})\mathbf{v}(\mathbf{x}) \tag{3}$$

Similar to the linear case, NEValues are achieved as the solution of the following equation:

$$\left|A(\mathbf{x})-\lambda(\mathbf{x})I_n\right|=0 \tag{4}$$

These NEValues and NEVectors are often functions of states and in every point in phase space, they take different values. It should be noted that there are some basic differences between this approach and the linearization method in which $A(\mathbf{x})$ may be viewed as the Jacobean matrix. In the linearization method, a nonlinear system is linearized near at the equilibrium point and the coefficient of $\mathbf{x}$ is indeed the Jacobean matrix quantified at the equilibrium point. In fact, a locally equivalent of the system obtained through linearization, can only represents the original system around every equilibrium point while the PL form represents exactly the original system over the whole underlying region with different form of representation.

2.2 Non-Uniqueness of Pseudo Linear Forms

For a scalar system, the PL form is unique for all x ≠ 0, given by:

$$\frac{d}{dt}x(t)=f\left(x(t)\right)\ \Rightarrow\ \frac{d}{dt}x(t)=a\left(x(t)\right)x(t)$$

where $a\left(x(t)\right)=\dfrac{f\left(x(t)\right)}{x(t)}$. However, for a general nonlinear system with order of $n>1$, the PL form is not necessarily unique. Indeed, in the case of multivariable nonlinear systems, $\mathbf{x}$ has n components $x_1,x_2\ ,\ldots x_n$. Assume $f_i(\mathbf{x})$, $i=1,2,..,n$ is the i th component of $\mathbf{f}(\mathbf{x})=\left[f_1(\mathbf{x})\ f_2(\mathbf{x})\ \cdots\ f_n(\mathbf{x})\right]^T$. Forming the PL form requires that each component of $\mathbf{f}(\mathbf{x})$ should be represented as a linear combination of state variables, most probably with state dependent coefficients in the form of:

$$\mathbf{f}(\mathbf{x})=\begin{bmatrix}f_1(\mathbf{x})\\ \vdots\\ f_i(\mathbf{x})\\ \vdots\\ f_n(\mathbf{x})\end{bmatrix}=\begin{bmatrix}\sum_{j=1}^{n}a_{1j}(\mathbf{x})x_j\\ \vdots\\ \sum_{j=1}^{n}a_{ij}(\mathbf{x})x_j\\ \vdots\\ \sum_{j=1}^{n}a_{nj}(\mathbf{x})x_j\end{bmatrix}=\left[a_{ij}(\mathbf{x})\right]_{n\times n}\mathbf{x} \tag{5}$$

It can be shown that the maximum possible number of distinct linear combination for $f_i(\mathbf{x})$ becomes:

$$m_i=\left(C(n,1)\right)^{k_i}=n^{k_i} \tag{6}$$

In this relation k_i is the number of distinct terms that constitute the $f_i(\mathbf{x})$. In addition, $C(n,i)$ stands for the combination of i states from the n states that can form a linear combination for representing $f_i(\mathbf{x})$. Indeed, the maximum possible number of PL forms, for a particular nonlinear system expressed by equation (1), can be given as:

$$m = \prod_{i=1}^{n} m_i = \prod_{i=1}^{n} \left(n^{k_i}\right) = n^{\sum_{i=1}^{n} k_i} \tag{7}$$

It should be emphasized that for a particular system, all of these m pseudo-linear forms may not be possible to be acquired.

However, by assigning $A_1(\mathbf{x})$, $A_2(\mathbf{x})$,…, $A_m(\mathbf{x})$ as system matrices for these m possible PL forms, infinite distinct PL forms can be obtained with system matrices of:

$$B_j(\mathbf{x}) = \sum_{i=1}^{m} \alpha_{ji} A_i(\mathbf{x}) \quad ; \quad \sum_{i=1}^{m} \alpha_{ji} = 1, \quad j = 1,2,... \tag{8}$$

Indeed, $\{A_1(\mathbf{x}), A_2(\mathbf{x}), \cdots, A_m(\mathbf{x})\}$ can be considered as a basis set for infinite PL forms system matrices.

Certainly, it is not possible to analyze all of these distinct infinite PL forms. On the other hand, each of these forms may give some information about the behaviour of the original nonlinear system. However, by an intuitive reasoning, it could be figured out that the information obtained from each PL form with system matrix of $B_j(\mathbf{x})$, is a subset of the information gained from PL forms with system matrices of $A_i(\mathbf{x})$, $i = 1,2,...,m$. This is inspired from the fact that each $B_j(\mathbf{x})$ is formed by a convex linear combination of $A_i(\mathbf{x})$'s as a basis set of PL forms system matrices. Therefore, among infinite PL forms obtained from a particular nonlinear system, only m PL forms named as basis set of PL space must be considered for analysis of the original nonlinear system. However, for a given system, it is not reasonable to analyze all of these m PL forms. On the other hand, further studies show that the information derived from all of these m PL forms is not necessarily correct. This issue has been specially considered in (Tsiotras *et al.,* 1996) as a counter example for a stability result about the PL systems. Indeed, we have previously shown that a PL representation of a nonlinear system, leads to certainly correct qualitative results about the behaviour of nonlinear system if the correspondent NEVectors are state independent (SI). This result has been fully proposed in the (Ghane and Menhaj, 2012).

With this introduction about the PL form representation of nonlinear systems, in the next section, we try to derive some useful results specially regarding to the stability analysis of PL systems with the help of discretization method. Indeed, it is shown that the aforementioned achievements obtained through the qualitative analysis can be backed up with a more reliable proof via the discretization of the nonlinear systems.

3. STABILITY ANALYSIS

In this section, it is tried to investigate the stability analysis of nonlinear systems via the PL representation. The approach used here is based on the analysis of discretized model of the nonlinear differential equations describing the nonlinear systems. Based on the stability of this discrete model, some useful and important results about the stability of the original system can be derived.

3.1 Closed Form Solution via Discretization

In this section, it is tried to obtain a closed form solution of the nonlinear continuous time system with the specified initial states. Since there is no general method to obtain such closed form solution for nonlinear systems, we tackle this issue by the help of discretization of the nonlinear differential equation of the underlying system. In this regard, we first need to discretize the continuous time system (1). To achieve this, there are several different methods like Euler method and Runge & Kutta method. The difference between these methods lies on the compromise between two important issues in real implementation and simulation of the original system: discretization error and computational time. The decrease of error oftenly leads to an increase in computational time. In general, the Runge & Kutta method is more accurate than the Euler method in the cost of more computational time and losing the simplicity. However in every selected method, the error goes to zero as the sampling time gets smaller and smaller. Very small sampling time consequently may lead to severe difficulties in practical implementations. Therefore, for the purpose of implementation, it is necessary to take into account this limitation leading to a lower bounded sampling time for the computational power we have. However, the main goal of this paper is to focus merely on the theoretical issues of the stability analysis and hence, the above practical limitation is not considered. On the other hand, it is known that as the sampling time goes to zero, the discrete model will approximate the original continuous model more accurately. Therefore, in the sequel, we use the Euler method for its simplicity to obtain the discretized model of the system and in the analysis, we let the sampling time goes to zero.

Now consider the original nonlinear system (1). By considering T as the sampling time and using the Euler method, we obtain the following discrete model of this system.

$$\mathbf{x}\left[(k+1)T\right] = \mathbf{x}\left[kT\right] + T\mathbf{f}\left(\mathbf{x}\left[kT\right]\right) \quad , k = 0,1,2,... \tag{9}$$

The solution of this difference equation strongly depends on the structure of $\mathbf{f}\left(\mathbf{x}\left[kT\right]\right)$. However, here we concentrate on the case that the structure of this function in some extent is known; i.e. it can be represented in the PL form. Therefore, we substitute the PL form of $\mathbf{f}\left(\mathbf{x}\left[kT\right]\right)$ in equation (9) to have:

$$\mathbf{x}\left[(k+1)T\right] = (I + T.A\left(\mathbf{x}\left[kT\right]\right))\mathbf{x}\left[kT\right] \tag{10}$$

The important point is that when a continuous system is represented in the PL form with the system matrix of $A(\mathbf{x})$, its discretized model obtained via the Euler method has still a PL form representation with the system matrix of

$$\tilde{A}\left(\mathbf{x}\left[kT\right]\right) = I + T.A\left(\mathbf{x}\left[kT\right]\right) \tag{11}$$

Then, the closed form solution for the difference equation of (10) is obtained as:

$$\mathbf{x}[(m+1)T] = \tilde{A}\left(\mathbf{x}[mT]\right)\mathbf{x}[mT] = \left(\prod_{i=0}^{i=m}\tilde{A}\left(\mathbf{x}[iT]\right)\right)\mathbf{x}[0] \tag{12}$$

This equation illustrates the crucial role of $\tilde{A}\left(\mathbf{x}[iT]\right)$, $i = 1,2,...$ on the stability feature of the solution. In the next subsection, stability analysis of the original nonlinear system based on its discretized model is fully discussed.

3.2 Main Stability Results

We first begin the stability analysis for the scalar case and then extend the results to the general n -dimensional case.

Consider a linear scalar discrete system as:

$$x[(k+1)T] = ax[kT]$$

Fact: The solution for an initial condition of $\mathbf{x}[0]$ is asymptotically stable if and only if $|a| < 1$.

This fact is apparently verified via the solution derived in equation (12) as:

$$\forall x(0) \in \mathbb{R} \qquad \lim_{m\to\infty} x[(m+1)T] = \lim_{m\to\infty} a^m x[0]\big|_{|a|<1} = 0$$

Therefore, the condition of $|a| < 1$ makes the origin of the system to be the globally asymptotic stable equilibrium point. However, for nonlinear discrete systems, there is not such a highlighting remark. However, representing the nonlinear discrete system in the PL form makes it possible to use this helpful remark of linear discrete systems. For a PL form representation of a scalar discrete dynamical system we have:

$$x[(k+1)T] = a\left(x[kT]\right)x[kT]$$

Using equation (12), the solution of this system with initial condition of $x[0]$ is obtained as:

$$x[(m+1)T] = \left(\prod_{i=0}^{i=m} a\left(x[iT]\right)\right)x[0]$$

Because the stability consideration of this solution is our main objective, it is desired to have:

$$\forall x(0) \in \mathbb{R} \qquad \lim_{m\to\infty} x[(m+1)T] = 0 \tag{13}$$

Here in the following proposition, we present the sufficient condition of asymptotic stability of a PL form representation of a scalar discrete dynamical system. This proposition can be considered as the sufficient condition for a nonlinear discrete system whose map is called contraction (Hunter and Nachtergaele, 2001).

PROPOSITION 3.1 Suppose a scalar autonomous discrete dynamical system with zero equilibrium point represented by the difference equation $x[(k+1)T] = f\left(x[kT]\right)$ can be transformed to a unique PL form as :

$$x[(k+1)T] = a\left(x[kT]\right)x[kT]$$

The origin of this system is globally asymptotically stable if

$$\forall x \in \mathbb{R}, \quad |a(x)| < 1.$$

PROOF: For the globally asymptotically stability of the origin it is required that for $\forall x(0) \in \mathbb{R}$:

$$\lim_{m\to\infty} x[(m+1)T] = \lim_{m\to\infty}\left(\prod_{i=0}^{i=m} a\left(\mathbf{x}[iT]\right)\right)x(0) = 0$$

If we have $\forall x \in \mathbb{R}$, $|a(x)| < 1$, the above conclusion is obvious based on the fact that the multiplication of m terms with the magnitude less than unity goes to zero when m goes to infinity.

In the next proposition, we extend the previous result to the multivariable system.

PROPOSITION 3.2 Suppose an n - dimensional autonomous discrete dynamical system with zero equilibrium point represented by the difference equation $\mathbf{x}[(k+1)T] = \mathbf{f}\left(\mathbf{x}[kT]\right)$ can be transformed to the following PL form: $\mathbf{x}[(k+1)T] = A\left(\mathbf{x}[kT]\right)\mathbf{x}[kT]$ where $\mathbf{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. The sufficient conditions for global asymptotic stability of the origin are:

1. $A\left(\mathbf{x}[kT]\right)$ is in diagonal form.

2. Every element of $A\left(\mathbf{x}[kT]\right)$ have a magnitude less than unity i.e. $|a_{ii}(\mathbf{x}(kT))| < 1, \quad \forall i = 1,2,..,n$.

PROOF: By satisfying the first condition, $A\left(\mathbf{x}[kT]\right)$ is written as: $A(\mathbf{x}[iT]) = diag[a_{ii}]$, $i = 1,2,...,n$. Therefore, solution of the above dynamical system is obtained as:

$$\mathbf{x}[(m+1)T] = \begin{bmatrix} \prod_{i=0}^{i=m} a_{11}\left(\mathbf{x}[iT]\right)x_1[0] \\ \prod_{i=0}^{i=m} a_{22}\left(\mathbf{x}[iT]\right)x_2[0] \\ \vdots \\ \prod_{i=0}^{i=m} a_{nn}\left(\mathbf{x}[iT]\right)x_n[0] \end{bmatrix}.$$

Then, if the second condition is satisfied, based on the proposition 3.1, we have:

$$\forall \mathbf{x}(0) \in \mathbb{R}^n \;|\; \lim_{m\to\infty} \mathbf{x}[(m+1)T] = 0.$$

The proposition 3.2 illustrates the only condition in which we can determine the stability property of nonlinear discrete dynamical systems based on its PL form. Based on this proposition, the stability of discrete dynamical system of equation (10) should be tackled.

For satisfying condition 1 of proposition 3.2, the system matrix $\tilde{A}(\mathbf{x}[kT])$ should be diagonal and knowing that,

$$\tilde{A}\left(\mathbf{x}[kT]\right) = I + T.A\left(\mathbf{x}[kT]\right)$$

the matrix $A\left(\mathbf{x}\left[kT\right]\right)$ should then be diagonal.

The concluding result about the stability analysis of nonlinear system based on the PL form representation is given in the following proposition.

PROPOSITION 3.3 Knowing the fact that any autonomous dynamical system with zero equilibrium point described by the differential equation $\dot{\mathbf{x}}(t)=\mathbf{f}\left(\mathbf{x}(t)\right)$ can be represented by a PL form of $\dot{\mathbf{x}}(t)=A\left(\mathbf{x}(t)\right)\mathbf{x}(t)$ in which $\mathbf{x}\in\mathbb{R}^n$ and $A\in\mathbb{R}^{n\times n}$, the following conditions are sufficient for globally asymptotically stability of the origin.

1. $A(\mathbf{x}(t))$ has diagonal form.

2. All NEValues of matrix $A(\mathbf{x})$ satisfy the following criterion:

$\forall\mathbf{x}\in\mathbb{R}^n \mid \lambda_i(\mathbf{x})<0\,;\quad i=1,2,...,n$.

PROOF: See appendix.

In the following proposition, we try to extend the previous result to cover a broader category of nonlinear autonomous systems. The following remarks are needed in advance.

REMARK 3.1 Suppose $A(\mathbf{x})$ is a matrix valued function as $A:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ and $\lambda_i(\mathbf{x})$ and $\mathbf{v}_i(\mathbf{x})$, $i=1,2,..,n$ are its eigenvlues and eigenvectors, respectively. The eigenvalues and eigenvectors of matrix $\tilde{A}(\mathbf{x})=I_n+TA(\mathbf{x})$ denoted by $\tilde{\lambda}_i(\mathbf{x})$, $\tilde{\mathbf{v}}_i(\mathbf{x})$, $i=1,2,..,n$, are then obtained as:

$\tilde{\lambda}_i(\mathbf{x})=T\,\lambda_i(\mathbf{x})+1$, $\tilde{\mathbf{v}}_i(\mathbf{x})=\mathbf{v}_i(\mathbf{x})$ $\forall i=1,2,...,n$

PROOF: See appendix.

PROPOSITION 3.4: Suppose an autonomous dynamical system with zero equilibrium point is presented by the differential equation $\dot{\mathbf{x}}(t)=\mathbf{f}\left(\mathbf{x}(t)\right)$ with the following PL form $\dot{\mathbf{x}}(t)=A\left(\mathbf{x}(t)\right)\mathbf{x}(t)$ where $\mathbf{x}\in\mathbb{R}^n$ and $A\in\mathbb{R}^{n\times n}$. The sufficient conditions for global asymptotic stability of the origin are:

1. The NEValues of matrix $A(\mathbf{x})$ satisfy the criterion: $\forall\mathbf{x}\in\mathbb{R}^n \mid \mathrm{Re}\{\lambda_i(\mathbf{x})\}<0\,;\quad i=1,2,...,n$.

2. The geometric multiplicity of every multiple NEValue should be equal to its corresponding algebraic multiplicity.

3. All NEVectors of matrix $A(\mathbf{x})$ are state independent (SI) or constant.

PROOF: See appendix.

COROLLARY 3.1 Consider the system defined in proposition 3.4. The sufficient conditions for instability of the origin are:

1. The NEValues of matrix $A(\mathbf{x})$ satisfy the criterion: $\forall\mathbf{x}\in\mathbb{R}^n \mid \mathrm{Re}\{\lambda_i(\mathbf{x})\}>0\,;\quad i=1,2,...,n$.

2. All NEVectors of matrix $A(\mathbf{x})$ are state independent (SI) or constant.

PROOF: The trend of the proof here is fundamentally the same as that of the proposition 3.4 except that every 1^- in the proof should be replaced by 1^+. This replacement consequently leads to:

$\lim_{\substack{T\to 0\\ m\to\infty}} \left|\mathbf{x}\left[(m+1)T\right]\right|\to\infty$

In addition, it can be easily seen that $\forall l\in\mathbb{N} \mid \left|\mathbf{x}\left[(l+1)T\right]\right|>\left|\mathbf{x}\left[lT\right]\right|$. This ends the proof.

It is worthwhile mentioning that based on proposition 3.4, for having an unstable origin equilibrium point, it is suffice to have:

$\exists i\ ;\ i=1,2,...,n \mid \forall\mathbf{x}\in\mathbb{R}^n\ :\ \mathrm{Re}\{\lambda_i(\mathbf{x})\}>0$. However, the corollary 3.1 provides more restrictive conditions in which all system trajectories move away from the origin in all direction in the state space. This condition can be useful in the rest of the paper. To proceed further, the following definition is required.

DEFINITION 3.1 Suppose there is an autonomous dynamical system with zero equilibrium point presented by the differential equation $\dot{\mathbf{x}}(t)=\mathbf{f}\left(\mathbf{x}(t)\right)$, in which $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$.

The region $D_S\subset\mathbb{R}^n$ is called 0-attracting, if $\forall\mathbf{x}(\mathbf{0})\in\mathbb{R}^n$, all solution trajectories of the system move toward the origin, exponentially or spirally, when they are in the region $D_S\subset\mathbb{R}^n$. Subsequently, the subspace $D_U\subset\mathbb{R}^n$ is called 0-repelling, if $\forall\mathbf{x}(\mathbf{0})\in\mathbb{R}^n$, all solution trajectories of this system move away from the origin, exponentially or spirally, when the trajectories are in $D_U\subset\mathbb{R}^n$.

Comment 1. The aforementioned 0-attracting and 0-repelling regions can be considered as a generalized nonlinear version of stable and unstable subspaces of linear dynamical systems, which are defined along with the eigenvectors. Indeed, the indices 'S' and 'U' in the D_S and D_U are due to this fact.

Comment 2. If a 0-attracting region contains the origin, it can be considered as the region of attraction for the origin equilibrium, which is indeed locally asymptotically stable.

In general, it is not so straightforward to obtain these 0-attracting and 0-repelling regions for nonlinear systems. However, for a class of nonlinear systems considered in the proposition 3.4 it is possible to determine these regions based on the sign of $\mathrm{Re}\{\lambda_i(\mathbf{x})\}$ with $i=1,2,...,n$.

Comment 3. For a nonlinear system of order n which can transformed to a PL form with SI NEVectors, if the geometric multiplicity of every multiple NEValue is equal to its corresponding algebraic multiplicity, the 0-attracting and 0-repelling regions can be respectively determined as: $D_S = \{\mathbf{x} | \mathrm{Re}\{\lambda_i(\mathbf{x})\} < 0, i = 1,2,\cdots,n\}$, $D_U = \{\mathbf{x} | \mathrm{Re}\{\lambda_i(\mathbf{x})\} > 0, i = 1,2,\cdots,n\}$.

COROLLARY 3.2 Suppose in proposition 3.4, the conditions 2 and 3 are satisfied while the condition 1 is partially satisfied as:

$$\forall \mathbf{x} \in D_S \mid \mathrm{Re}\{\lambda_i(\mathbf{x})\} < 0; \quad i = 1,2,...,n \quad , \quad D_S \subset \mathbb{R}^n .$$

where D_S is a closed region including the origin. Then, local asymptotic stability of origin can be guaranteed. In addition, the region of attraction for this system is indeed the D_S region.

PROOF: The trend of the proof here is fundamentally the same as that of the proposition 3.4, except that the solution is considered for $\forall \mathbf{x}(0) \in D_S$.

The propositions 3.3 and 3.4 along with the aforementioned corollaries provide us an analytical framework for the stability analysis of nonlinear systems through the PL form representation. As highlighted in proposition 3.4, the critical point ignored in the previous studies was to consider the impact of NEVectors on the stability property of these systems. Indeed, the SD NEVectors can completely change the results about the qualitative behaviour of nonlinear systems obtained thorough the NEValues analysis reported in the literature. This ignorance led to the investigation of some counterexamples about the ability of PL form in stability analysis of nonlinear systems (Tsiotras *et al.*, 1996). In the section 5, via some illustrative examples the validity of our findings is highlighted.

4. LIMIT CYCLE EMERGENCE

A limit cycle is an isolated closed trajectory. Isolated means that the neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say that the limit cycle is stable or attracting. Otherwise, the limit cycle is unstable or, in exceptional cases, half stable. Stable limit cycles are very important in science. They model systems that exhibit self-sustained oscillations. Limit cycles are an inherently nonlinear phenomenon; they cannot occur in linear systems. The qualitative features of a nonlinear system behavior which exhibit a limit cycle can be listed as:

a. Spiral type dynamics: by translating this feature to an eigenstructure terminology, we may equivalently say that there exists a region in the state space in which the NEValues are complex.

b. Limiting center type dynamics: similar to the previous feature, it means that in the same region that the nonlinear system has complex NEValues, there should exist a closed curve on which the NEValues are pure imaginary.

The above mentioned closed curve is the expected limit cycle. Indeed, we consider these two features as the qualitative translation of the limit cycle behavior.

Based on the results reported in (Ghane and Menhaj, 2013)

and in the previous section, it is possible to synthesize the desired qualitative behavior for a class of nonlinear system based on the NEValues analysis. Therefore, it is expected that by investigating these qualitative features via the NEValue analysis of a nonlinear system, the emerging of limit cycle behavior can be systematically tackled. This idea can be presented by the following criterion.

CRITERION 4.1 Consider a pseudo linearizable nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^{n\times n}$. The sufficient conditions for emerging limit cycle are:

a) There exists a region $D \subset \mathbb{R}^n$ in which the NEValus are complex.

b) In the region $D \subset \mathbb{R}^n$, there exists a closed orbit χ on which the NEValues are purely imaginary.

c) The imaginary parts of NEValues do not change sign.

In some cases, the closed curve χ is the exact boundary of limit cycle. However, since the eigenstructure analysis tool proposed here leads to qualitative results, in general, χ does indeed represent the approximate boundary of the predicted limit cycle. The third condition is added to assure the mono direction spin of the spiral dynamics. In the next section, via some illustrative examples, the validity of this criterion is approved.

5. ILLUSTRATIVE EXAMPLES

In this section, the capabilities of PL form representation of nonlinear systems for stability analysis and limit cycle generation are investigated via some illustrative examples.

5.1 Stability Analysis

Examples 1 through 4 given below are devoted the stability analysis results of this paper. To highlight better, it is tried to apply the proposed results to a wide range of systems including the aforementioned counterexample systems.

Example 1: Consider the following system

$$\dot{x} = \sin^2(x)x - 2x \quad .$$

The origin is the equilibrium point of this system and hence its PL form becomes: $\dot{x} = a(\mathbf{x})x$ with $\mathbf{a(x)} = \sin^2(x) - 2$ .This PL form is unique since the system is of order 1. The single NEValue equals $\lambda(x) = a(x) = \sin^2(x) - 2$. According to proposition 3.1, since $\sin^2(x)x - 2 < 0, \forall x \in \mathbb{R}$, this system is globally asymptotically stable. For this system, the results obtained from NEVlue analysis are the same as those obtained from Lyaponuv analysis and they can be easily validated through simulation results shown in figure 1.

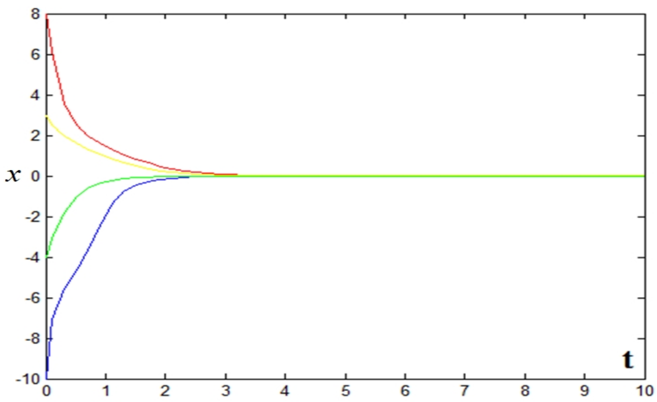

Fig. 1. Solution of example 1 for some different initial conditions.

Example 2: In this example, a second order system is considered with the following state equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_1^2 x_2 \\ -x_2 \end{bmatrix}$$

This is indeed the counterexample presented in (Tsiotras *et al.,* 1996) and can be precisely analyzed through proposition 3.4. Obviously, the origin is an equilibrium point. Therefore, the PL form representation of this system is possible. However, the only PL form representation of this system with SI NEVectors has the following system matrix:

$$A_1(\mathbf{x}) = \begin{bmatrix} x_1 x_2 - 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The SI NEVectors of this PL form are: $\mathbf{v}_1(\mathbf{x}) = [1 \quad 0]^T$ and $\mathbf{v}_2(\mathbf{x}) = [0 \quad 1]^T$. Consequently, merely the PL form with this system matrix, which satisfies the condition of SI NEVectors of the proposition 3.4, can be used in the stability analysis. Since the first NEValue of this matrix, $\lambda_1(\mathbf{x}) = x_1 x_2 - 1$, does not satisfy the condition of $\lambda_1(\mathbf{x}) < 0$, $\forall \mathbf{x} \in \mathbb{R}^n$, the globally asymptotic stability of the origin fails.

However, in (Tsiotras *et al.,* 1996), asymptotic stability of the origin was concluded through NEValues of PL form with system matrix $A_2(\mathbf{x}) = \begin{bmatrix} -1 & x_1^2 \\ 0 & -1 \end{bmatrix}$, while the claim was not supported with the simulation; this reflects the vital importance of considering the SI NEVectors to correctly determine the stability of nonlinear systems. Indeed, this wrong result is originated from the state dependent NEVecors of matrix $A_2(\mathbf{x})$. This state dependency is the important ignorant point which has led to the wrong result.

Example 3: In this example the counterexample proposed by (Muhammad and Van Der Woude, 2009) is investigated. The state equations of this system are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 + c(\mathbf{x})x_2 \\ -bx_1 + ax_2 - c(\mathbf{x})x_1 \end{bmatrix}$$

where $a, b \in \mathbb{R}$ and $c : \mathbb{R}^2 \to \mathbb{R}$ is a scalar smooth function. For this system, four distinct PL forms with the following system matrices constitute the basis set of PL forms.

$$A_1(\mathbf{x}) = \begin{bmatrix} a & b + c(\mathbf{x}) \\ -b - c(\mathbf{x}) & a \end{bmatrix}, A_2(\mathbf{x}) = \begin{bmatrix} a + \frac{c(\mathbf{x})}{x_1} x_2 & b \\ -b - c(\mathbf{x}) & a \end{bmatrix}$$

$$A_3(\mathbf{x}) = \begin{bmatrix} a + \frac{c(\mathbf{x})}{x_1} x_2 & b \\ -b & a - \frac{c(\mathbf{x})}{x_2} x_1 \end{bmatrix}$$

and

$$A_4(\mathbf{x}) = \begin{bmatrix} a & b + c(\mathbf{x}) \\ -b & a - \frac{c(\mathbf{x})}{x_2} x_1 \end{bmatrix}$$

Of course, in the above derivation, it is supposed that $c(\mathbf{x})$ is analytical with respect to each of its arguments. However, none of these PL forms has SI NEVectors. Therefore, these PL forms are of no use for the stability of the nonlinear system based on proposition 3.4.

Example 4: In this example, we apply proposition 3.4 in the control design for a 3rd order nonlinear system as given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1 \sin(x_2) - 2x_1 \\ x_2 x_3^2 \\ x_3 \cos^2(x_1) \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The objective is to design a stabilizer controller for this system. Here, we suppose that all states are accessible.

We first obtain the PL form of this system with SI NEVectors. It can then be shown that the following is indeed the only PL form with SI NEVectors:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = A(\mathbf{x}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Where $A(\mathbf{x}) = diag\{\sin(x_2) - 2, x_3^2, \cos^2(x_2)\}$

The NEValues and NEVectors of this PL form are:

$$\begin{cases} \lambda_1(\mathbf{x}) = \sin(x_2) - 2 \\ \lambda_2(\mathbf{x}) = x_3^2 \\ \lambda_3(\mathbf{x}) = \cos^2(x_2) \end{cases}, \quad \begin{cases} \mathbf{v}_1(\mathbf{x}) = [1 \quad 0 \quad 0]^T \\ \mathbf{v}_2(\mathbf{x}) = [0 \quad 1 \quad 0]^T \\ \mathbf{v}_3(\mathbf{x}) = [0 \quad 0 \quad 1]^T \end{cases}$$

This system is not asymptotically stable due to $\lambda_2(\mathbf{x}) \geq 0$ and $\lambda_3(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$. Here, we are not really interested in designing a proper state feedback controller; however, it is very straightforward to stabilize the origin of this system using proposition 3.4 by selecting properly the control signals

u_1, u_2 and u_3. Indeed, due to $\lambda_1(\mathbf{x})<0$, $\forall\,\mathbf{x}\in\mathbb{R}^n$, we only need to determine u_2 and u_3 such that both $(\lambda_2(\mathbf{x}))_{CL}<0$ and $(\lambda_3(\mathbf{x}))_{CL}<0$, $\forall\,\mathbf{x}\in\mathbb{R}^n$ are satisfied. The sub-index CL is referring to the closed loop system. This object is satisfied with the following controls:

$\begin{cases} u_1(\mathbf{x})=0 \\ u_2(\mathbf{x})=-2x_2x_3^{\,2} \\ u_3(\mathbf{x})=-3x_3 \end{cases}$. With these controls, the closed loop PL form system can be achieved as $\dot{\mathbf{x}}=A_{CL}(\mathbf{x})\mathbf{x}$ with:

$$A_{CL}(\mathbf{x})=diag\left\{\sin(x_2)-2\,,-2x_3^{\,2}\,,\cos^2(x_2)-3\right\}.$$

This closed loop system is obviously globally asymptotically stable. The closed loop solution of the system depicted in figure 2 for some initial states better highlights the effectiveness of the stabilizing control design.

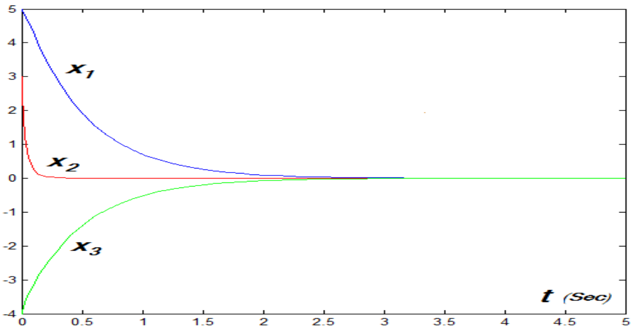

Fig. 2. Solution for closed loop system of example 4; $x_1(0)=5$, $x_2(0)=3$ and $x_3(0)=-4$.

5.1 Limit Cycle Emergence

The examples of this subsection are focused on the limit cycle emergence. It is tried to show the capability of the proposed approach to generate limit cycles with desired shapes and numbers. In addition, in the first example of this subsection, the detection problem of limit cycle is also slightly considered.

Example 5: This example is devoted to a 2nd order nonlinear system with a more complex dynamic behaviour. The system is of the following form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\alpha(\mathbf{x})x_1-cx_2\\ cx_1+\beta(\mathbf{x})x_2\end{bmatrix}$$

where $c\in\mathbb{R}$ and $\begin{cases}\alpha(\mathbf{x})=a^2-x_1^{\,2}-x_2^{\,2}\\ \beta(\mathbf{x})=b^2-x_1^{\,2}-x_2^{\,2}\end{cases}$, $a\in\mathbb{R}$ and $b\in\mathbb{R}$

. The origin is the equilibrium point of this system. Therefore, the PL form representation of this system can be achieved. The following PL form in the basis set is the only one which meets the SI NEVectors condition of proposition 3.4.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\alpha(\mathbf{x}) & -c\\ c & \beta(\mathbf{x})\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The NEValues of this PL form can be obtained as:

Case1: for $\left|a^2-b^2\right|\geq 2c$,

$$\lambda_{1,2}(\mathbf{x})=(\frac{a^2+b^2}{2}-x_1^{\,2}-x_2^{\,2})\pm\frac{1}{2}\sqrt{(a^2-b^2)^2-4c^2}$$

Case 2: for $\left|a^2-b^2\right|<2c$,

$$\lambda_{1,2}(\mathbf{x})=(\frac{a^2+b^2}{2}-x_1^{\,2}-x_2^{\,2})\pm j\,\frac{1}{2}\sqrt{4c^2-(a^2-b^2)^2}$$

First, the real NEValues are considered by assigning the parameters as $a=3$, $b=5$ and $c=5$, which meets the condition of case 1. By this assignment, the NEValues and NEVectors are obtained as:

$$\lambda(\mathbf{x})_{1,2}=(17-x_1^{\,2}-x_2^{\,2})\pm 6.25=\begin{cases}10.75-x_1^{\,2}-x_2^{\,2}\\ 23.25-x_1^{\,2}-x_2^{\,2}\end{cases}$$

$\mathbf{v}_1(\mathbf{x})=\begin{bmatrix}1 & -2.85\end{bmatrix}^T$ and $\mathbf{v}_2(\mathbf{x})=\begin{bmatrix}-2.85 & 1\end{bmatrix}^T$.

Since the condition of SI NEVectors is satisfied, we can use both proposition 3.4 and corollary 3.1 for stability analysis of this system, In the region $D_U: x_1^{\,2}+x_2^{\,2}<10.75$, both NEValues are real and positive, and therefore, this is a 0-repelling for this system. This means that all solution trajectories initiating from this region, $\forall\mathbf{x}(0)\in D_U$, move away from the origin. On the other hand, the 0-attracting region of this system is obtained as $D_S: x_1^{\,2}+x_2^{\,2}>23.25$, in which both NEValues are real and negative. This in turn means that every solution trajectories with initial state $\forall\mathbf{x}(0)\in D_S$, moves toward the origin. These two statements say that, a solution trajectory, as long as the trajectory is in the region D_U, goes away from the origin and as long as is in region D_S move toward the origin. The validity of this stability analysis results, can be illustratively highlighted in the figure 3. In this figure, the NEVectors are shown with the yellow lines. Based on the criterion 4.1, in the case 2 it is expected to have a limit cycle in the phase plane.

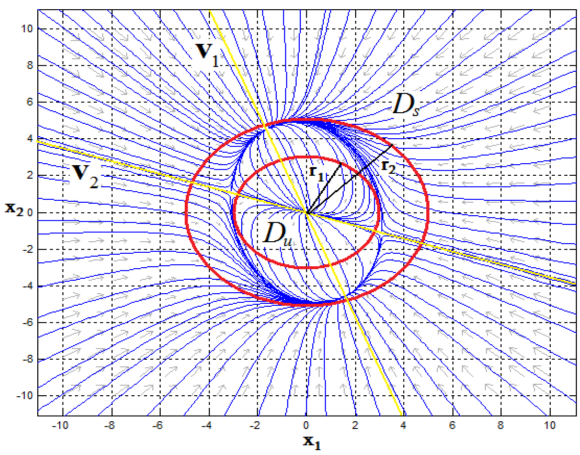

Fig. 3. Phase plane simulation of example 5; Case 1: $r_1=\sqrt{10.75}$ and $r_2=\sqrt{23.25}$

Therefore, by assigning the parameters of this example as $a=5$, $b=5$ and $c=10$, the condition of case 1 is satisfied. By this assignment, NEValues and NEVectors are obtained as: $\lambda(\mathbf{x})_{1,2}=(25-x_1^2-x_2^2)\pm j10$ and $\mathbf{v}(\mathbf{x})_{1,2}=\begin{bmatrix}1\\ \pm j\end{bmatrix}$.

In this case, the condition of SI NEVectors is satisfied and therefore like in case 1, based on the proposition 3.4 and comment 3, the 0-attracting and 0-repelling regions of the nonlinear system can be obtained respectively as $D_S: x_1^2+x_2^2>25$ and $D_U: x_1^2+x_2^2<25$. The phase plane simulation of the system with parameters satisfying case 2, is shown in figure 4, which easily verifies this stability analysis.

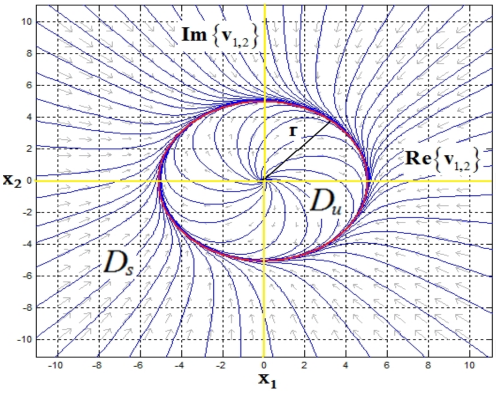

Fig. 4. Phase plane simulation of example 5; Case 2: $r=5$

In this figure, the yellow lines are the real part and imaginary parts of NEVectors. Since in the phase plane simulation, the states are to be real and hence only vectors with real component should be shown, the imaginary and real parts of NEVectors are taken as independent eigenvectors.

As is illustrated in figure 4, the system possesses a limit cycle in this case. We expect that the limit cycle is exactly the same as the boundary between 0-attracting and 0-repelling regions, i.e. $x_1^2+x_2^2=25$, and this is the case in the figure 4.

Example 6: In this example, by utilizing the criterion 4.1, we show the possiblity to generate multiple limit cycle with desired boundaries. For example if it is desired to have 3 limit cycles with boundaries of $\alpha_1(\mathbf{x})=0$, $\alpha_2(\mathbf{x})=0$ and $\alpha_3(\mathbf{x})=0$, the synthesized system can be something like:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\alpha_1(\mathbf{x})\alpha_2(\mathbf{x})\alpha_3(\mathbf{x})x_1-\omega x_2\\ \omega x_1+\alpha_1(\mathbf{x})\alpha_2(\mathbf{x})\alpha_3(\mathbf{x})x_2\end{bmatrix}$$

whose unique PL form with SI NEVectors is obtained as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\alpha_1(\mathbf{x})\alpha_2(\mathbf{x})\alpha_3(\mathbf{x})-\omega\\ \omega+\alpha_1(\mathbf{x})\alpha_2(\mathbf{x})\alpha_3(\mathbf{x})x_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The NEValues of these PL form become:

$$\lambda(\mathbf{x})_{1,2}=\alpha_1(\mathbf{x})\alpha_2(\mathbf{x})\alpha_3(\mathbf{x})\pm j\omega$$

By the assumptions that:

1. There is no intersection between $\alpha_i(\mathbf{x})=0 \quad ; i=1,2,3$.

2. For $i<j$, the curve $\alpha_i(\mathbf{x})=0$ is entirely in the region enclosed by the curve $\alpha_j(\mathbf{x})=0$ in the phase plane, or equivalently $\alpha_i(\mathbf{x})<\alpha_j(\mathbf{x}) \quad \forall \mathbf{x}\in\mathbb{R}^2$. Without loss of generality, a proper choice for these boundaries is:

$\alpha_1(\mathbf{x})=x_1^2+x_2^2-a_1^2$, $\alpha_2(\mathbf{x})=x_1^2+x_2^2-a_2^2$ and

$\alpha_3(\mathbf{x})=|x_1|+|x_2|-a_3$ with $a_1<a_2<a_3$.

Therefore, the qualitative analysis of the synthesized nonlinear system based on NEValues analysis is illustrated in table 1.

Based on the qualitative analysis proposed in this paper and criterion 4.1, it can be concluded that this system has three limit cycles, two unstable and one stable. The phase plane simulation in the figure 5, highlights the validity of these qualitative results.

In the above examples, we tried to cover a wide range of nonlinear system behaviours to emphasize the ability and validity of our findings.

Table 1. NEValue analysis of example 6.

Reign	**Complex, Real or Imaginary**	**Sign of $\mathrm{Re}\{\lambda(\mathbf{x})_1\}$**	**Sign of $\mathrm{Re}\{\lambda(\mathbf{x})_2\}$**	**Behavior type**
$x_1^2+x_2^2<a_1$	Complex	+	+	Unstable Spiral
$x_1^2+x_2^2=a_1$	Imaginary	ND	ND	Center
$a_1<x_1^2+x_2^2<a_2$	Complex	-	-	Stable Spiral
$x_1^2+x_2^2=a_2$	Imaginary	ND	ND	Center
$\{x_1^2+x_2^2>a_2\}\cap\{\lvert x_1\rvert+\lvert x_2\rvert<a_3\}$	Complex	+	+	Unstable Spiral
$\lvert x_1\rvert+\lvert x_2\rvert=a_3$	Imaginary	ND	ND	Center
$\lvert x_1\rvert+\lvert x_2\rvert>a_3$	Complex	-	-	Stable Spiral

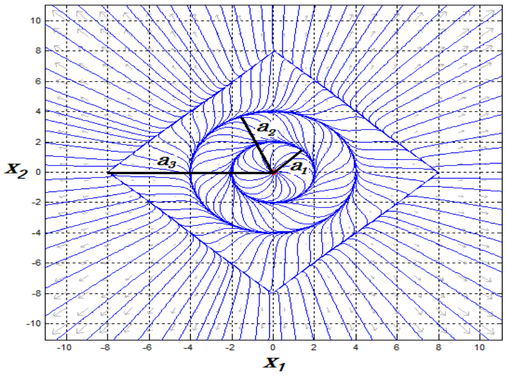

Fig. 5. Phase plane simulation of example 6.
$a_1 = 2$, $a_2 = 4$, $a_3 = 8$ **and** $\omega = 100$

6. CONCLUSIONS AND REMARKS

In this paper, the stability analysis of nonlinear autonomous systems through the PL form representation was thoroughly investigated. Based on the discretized model of the system, the closed form solution of the nonlinear system for every set of initial state was obtained. In the analysis level, to have zero error between discretized model and the original nonlinear system, in the solution, we let the sampling time go to zero. Then, the stability of the nonlinear system trajectories was analyzed based on the stability of closed form solution of the discretized model. Through this approach, the crucial effect of SI NEVectors in the stability analysis based on the NEValues was highlighted as the important fact ignored in the previous studies which had been led to some incorrect results about the stability of nonlinear systems.

Through four propositions, the globally asymptotically stability conditions of nonlinear autonomous systems, which can be represented in PL form with SI NEVectors, were fully derived and in addition, via some comments and corollaries, the conditions for local asymptotically stability of these nonlinear systems were presented. The illustrative examples have shown that it is possible to have a global insight about the stability property of the nonlinear system, by transforming it to a PL form with SI NEVectors. By applying the results of the proposed stability analysis and our previous qualitative analysis method for PL systems, some sufficient conditions for limit cycle emergence were proposed. Although the stability consideration of nonlinear systems is a significantly important, however, by utilizing the results of this paper, our qualitative analysis of nonlinear systems reported in our previous paper was completed. Then, the limit cycle emergence as one of the challenging issues in nonlinear system theory was tackled. In this regard, first, the qualitative conditions for limit cycle emergence was presented through one criterion and then by satisfying these conditions via the proposed qualitative analysis tool, limit cycles with desired shape and numbers were generated. Simulation results at the end of the paper easily verify the correctness of these theoretical results.

As future work, some challenging issues in nonlinear control theory like controllability, observability, non-minimum phase system determination can be systematically approached via the proposed analysis tool. In addition, the case in which no PL form with SI NEvector can be obtained still remaines as an open problem and may be the subject of future studies. We are currently working on these issues and the results will be reported soon.

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APPENDIX A

PROOF of PROPOSITION 3.3:

From the equation (11), the discretized model of this nonlinear system can be obtained as:

$$\mathbf{x}\left[(k+1)T\right]=\tilde{A}\left(\mathbf{x}\left[kT\right]\right)\mathbf{x}\left[kT\right]$$

Due to condition 1, $\tilde{A}(\mathbf{x}[kT])$ has diagonal form with the following elements:

$\tilde{A}(\mathbf{x}[kT])=\left[\tilde{a}_{ii}(\mathbf{x}[kT])\right]_{n\times n}$ and $A(\mathbf{x}[kT])=\left[a_{ii}(\mathbf{x}[kT])\right]_{n\times n}$, in which $\tilde{a}_{ii}=a_{ii}+1$. Furthermore, the NEValues can be determined as:

$\lambda_i(\mathbf{x}(t))=a_{ii}(\mathbf{x}(t))$, $i=1,2,..,n$.

Therefore, the solution of the discretized system can be rewritten as:

$$\mathbf{x}\left[(m+1)T\right]=\begin{bmatrix}\prod_{i=0}^{i=m}\left(\tilde{a}_{11}\left(\mathbf{x}\left[iT\right]\right)\right)x_1[0]\\ \prod_{i=0}^{i=m}\left(\tilde{a}_{22}\left(\mathbf{x}\left[iT\right]\right)\right)x_2[0]\\ \vdots\\ \prod_{i=0}^{i=m}\left(\tilde{a}_{nn}\left(\mathbf{x}\left[iT\right]\right)\right)x_n[0]\end{bmatrix}=\begin{bmatrix}\prod_{i=0}^{i=m}\left(1+T\lambda_1\left(\mathbf{x}\left[iT\right]\right)\right)x_1[0]\\ \prod_{i=0}^{i=m}\left(1+T\lambda_2\left(\mathbf{x}\left[iT\right]\right)\right)x_2[0]\\ \vdots\\ \prod_{i=0}^{i=m}\left(1+T\lambda_n\left(\mathbf{x}\left[iT\right]\right)\right)x_n[0]\end{bmatrix}$$

As stated, if the sampling time T goes to zero, the approximate derivative used in Euler discritazation method goes to the actual derivative and the error between the discrete model and the original continuous system vanishes. To be more precise, we compute the solution when T goes to zero. Based on the proposition 3.2 it is known that for the resulting discretized system, the globally asymptotically stability of the origin is guaranteed if all elements of $\tilde{A}(\mathbf{x}[kT])=\left[\tilde{a}_{ii}(\mathbf{x}[kT])\right]_{n\times n}$ have magnitude less than unity. This is so as shown below due to condition 2.

$$\lim_{\substack{T\to 0\\ m\to\infty}}\mathbf{x}\left[(m+1)T\right]=\begin{bmatrix}\lim_{\substack{T\to 0\\ m\to\infty}}\prod_{i=0}^{i=m}\left(1+T\lambda_1\left(\mathbf{x}\left[iT\right]\right)\right)x_1[0]\\ \lim_{\substack{T\to 0\\ m\to\infty}}\prod_{i=0}^{i=m}\left(1+T\lambda_2\left(\mathbf{x}\left[iT\right]\right)\right)x_2[0]\\ \vdots\\ \lim_{\substack{T\to 0\\ m\to\infty}}\prod_{i=0}^{i=m}\left(1+T\lambda_n\left(\mathbf{x}\left[iT\right]\right)\right)x_n[0]\end{bmatrix}\qquad(14.A)$$

in which $T>0$ and $\lambda_i(\mathbf{x}[iT])<0$; $i=1,2,...,n$. In addition, since $\lambda_i(\mathbf{x}[iT])s$ are analytical functions of $\mathbf{x}[kT]$, they have in worst case an exponential growth rate, say α_i. By making the decay rate γ of the sampling time $T>0$ exponentially fast with $\gamma>\max\{\alpha_i\}$, we can have

$\lim_{T\to 0}\left|1+T\lambda_i(\mathbf{x}[kT])\right|=1^- \quad | \quad \forall i=1,2,..,n$

The above condition is satisfied for every $k\in\mathbb{N}$.

Substitution of the above limit in equation (14) yields:

$$\lim_{m\to\infty}\left(\lim_{T\to 0}\mathbf{x}\left[(m+1)T\right]\right)=\lim_{m\to\infty}\begin{bmatrix}\prod_{i=0}^{i=m}\left(1^-\right)x_1[0]\\ \prod_{i=0}^{i=m}\left(1^-\right)x_2[0]\\ \vdots\\ \prod_{i=0}^{i=m}\left(1^-\right)x_n[0]\end{bmatrix}=\mathbf{0}$$

On the other hand,

$$\lim_{T\to 0}\left(\lim_{m\to\infty}\mathbf{x}\left[(m+1)T\right]\right)=\lim_{T\to 0}\begin{bmatrix}\prod_{i=0}^{\infty}\left(1+T\lambda_1\left(\mathbf{x}\left[iT\right]\right)\right)x_1[0]\\ \prod_{i=0}^{\infty}\left(1+T\lambda_2\left(\mathbf{x}\left[iT\right]\right)\right)x_2[0]\\ \vdots\\ \prod_{i=0}^{\infty}\left(1+T\lambda_n\left(\mathbf{x}\left[iT\right]\right)\right)x_n[0]\end{bmatrix}$$

$$=\begin{bmatrix}\prod_{i=0}^{\infty}\left(1^-\right)x_1[0]\\ \prod_{i=0}^{\infty}\left(1^-\right)x_2[0]\\ \vdots\\ \prod_{i=0}^{\infty}\left(1^-\right)x_n[0]\end{bmatrix}=\mathbf{0}$$

This completes the proof.

PROOF of Remark 3.1:

The eigenvectors of $A(\mathbf{x})$ and $\tilde{A}(\mathbf{x})$ are determined as:

$$\begin{cases}\left|A(\mathbf{x})-\lambda(\mathbf{x})I_n\right|=0\\ A(\mathbf{x})\mathbf{v}(\mathbf{x})=\lambda(\mathbf{x})\mathbf{v}(\mathbf{x})\end{cases}\text{ and }\begin{cases}\left|\tilde{A}(\mathbf{x})-\tilde{\lambda}(\mathbf{x})I_n\right|=0\\ \tilde{A}(\mathbf{x})\tilde{\mathbf{v}}(\mathbf{x})=\tilde{\lambda}(\mathbf{x})\tilde{\mathbf{v}}(\mathbf{x})\end{cases}.$$

Substitute (11) in the above equation to obtain:

$$\left|TA(\mathbf{x})+I_n-\tilde{\lambda}(\mathbf{x})I_n\right|=0 \;\Rightarrow\; \left|TA(\mathbf{x})-I_n(\tilde{\lambda}(\mathbf{x})-1)\right|=0$$

$$\Rightarrow\;\left|A(\mathbf{x})-I_n\left(\frac{\tilde{\lambda}(\mathbf{x})-1}{T}\right)\right|=0\;\Rightarrow\;\frac{\tilde{\lambda}(\mathbf{x})-1}{T}=\lambda(\mathbf{x})$$

$$\Rightarrow\;\tilde{\lambda}(\mathbf{x})=T\lambda(\mathbf{x})+1$$

and,

$$\tilde{A}(\mathbf{x})\tilde{\mathbf{v}}(\mathbf{x})=\tilde{\lambda}(\mathbf{x})\tilde{\mathbf{v}}(\mathbf{x})\;\Rightarrow\;(TA(\mathbf{x})+I_n)\tilde{\mathbf{v}}(\mathbf{x})=(T\lambda(\mathbf{x})+1)\tilde{\mathbf{v}}(\mathbf{x})$$
$$\Rightarrow\;TA(\mathbf{x})\tilde{\mathbf{v}}(\mathbf{x})=T\lambda(\mathbf{x})\tilde{\mathbf{v}}(\mathbf{x})\;\Rightarrow\;\tilde{\mathbf{v}}(\mathbf{x})=\mathbf{v}(\mathbf{x})$$

PROOF of PROPOSITION 3.4:

Consider equation (10), which is the discretized model of the original nonlinear system:

$$\mathbf{x}\left[(k+1)T\right]=\tilde{A}\left(\mathbf{x}\left[kT\right]\right)\mathbf{x}\left[kT\right]$$

Based on the remark 3.1 we can write:

$\tilde{\lambda}_i\left(\mathbf{x}\left[kT\right]\right)=T\lambda_i\left(\mathbf{x}\left[kT\right]\right)+1$ and

$\tilde{\mathbf{v}}_i\left(\mathbf{x}\left[kT\right]\right)=\mathbf{v}_i\left(\mathbf{x}\left[kT\right]\right)\quad\forall i=1,2,...,n$.

For distinct NEValues the Jordan form of $\tilde{A}\left(\mathbf{x}\left[kT\right]\right)$ can be obtained as:

$$\tilde{\Lambda}\left(\mathbf{x}\left[kT\right]\right)=\left(Q\left(\mathbf{x}\left[kT\right]\right)\right)^{-1}\tilde{A}\left(\mathbf{x}\left[kT\right]\right)\left(Q\left(\mathbf{x}\left[kT\right]\right)\right)\qquad(15.A)$$

in which

$$Q\left(\mathbf{x}[kT]\right)=\left[\mathbf{v}_1\left(\mathbf{x}[kT]\right)\quad \mathbf{v}_2\left(\mathbf{x}[kT]\right)\quad \cdots \quad \mathbf{v}_n\left(\mathbf{x}[kT]\right)\right]$$

is the so-called modal matrix. Then, the Jordan form becomes:

$$\tilde{\Lambda}\left(\mathbf{x}[kT]\right)=diag\left\{\tilde{\lambda}_1\left(\mathbf{x}[kT]\right),\tilde{\lambda}_2\left(\mathbf{x}[kT]\right),\cdots,\tilde{\lambda}_n\left(\mathbf{x}[kT]\right)\right\}$$

If the condition 2 of the proposition is held, the above Jordan form is also obtained even in the case of existing non distinct NEValues. From (15.A) it is clear that

$$\tilde{A}\left(\mathbf{x}[kT]\right)=\left(Q\left(\mathbf{x}[kT]\right)\right)\tilde{\Lambda}\left(\mathbf{x}[kT]\right)\left(Q\left(\mathbf{x}[kT]\right)\right)^{-1} \tag{16.A}$$

By substituting (16.A) into equation (12) which is the solution of discrete system of equation (10), we have:

$$\begin{aligned}\mathbf{x}[(m+1)T]&=\left(\prod_{i=0}^{i=m}\tilde{A}\left(\mathbf{x}[iT]\right)\right)\mathbf{x}[0]\\&=\left(\prod_{i=0}^{i=m}\left(Q\left(\mathbf{x}[iT]\right)\right)\tilde{\Lambda}\left(\mathbf{x}[iT]\right)\left(Q\left(\mathbf{x}[iT]\right)\right)^{-1}\right)\mathbf{x}[0]\end{aligned} \tag{17.A}$$

Proposition 3.3 cannot be applied to check the stability of (17.A) because the system is not in diagonal form. However, to elaborate more, we rewrite (17.A) as:

$$\begin{aligned}&\mathbf{x}[(m+1)T]=\\&\left(Q\left(\mathbf{x}[0]\right)\right)\tilde{\Lambda}\left(\mathbf{x}[0]\right)\left(Q\left(\mathbf{x}[0]\right)\right)^{-1}\left(Q\left(\mathbf{x}[T]\right)\right)\tilde{\Lambda}\left(\mathbf{x}[T]\right)\left(Q\left(\mathbf{x}[T]\right)\right)^{-1}\cdots\\&\cdots\left(Q\left(\mathbf{x}[mT]\right)\right)\tilde{\Lambda}\left(\mathbf{x}[mT]\right)\left(Q\left(\mathbf{x}[mT]\right)\right)^{-1}\mathbf{x}[0]\end{aligned} \tag{18.A}$$

Easily observed that if the condition 3 of the proposition is satisfied, i.e., $Q\left(\mathbf{x}[0]\right)=Q\left(\mathbf{x}[T]\right)=\cdots=Q\left(\mathbf{x}[kT]\right)=Q$, we have:

$$\begin{aligned}\mathbf{x}[(m+1)T]&=\left(Q\left(\mathbf{x}[0]\right)\right)\prod_{i=0}^{i=m}\tilde{\Lambda}\left(\mathbf{x}[iT]\right)\left(Q\left(\mathbf{x}[iT]\right)\right)^{-1}\mathbf{x}[0]\\&=Q\left(\prod_{i=0}^{i=m}\tilde{\Lambda}\left(\mathbf{x}[iT]\right)\right)Q^{-1}\mathbf{x}[0]\end{aligned} \tag{19.A}$$

Use remark 3.1 and write:

$$\begin{aligned}&\mathbf{x}[(m+1)T]=\\&Q\left(diag\left\{\prod_{i=0}^{i=m}\tilde{\lambda}_1\left(\mathbf{x}[iT]\right),\prod_{i=0}^{i=m}\tilde{\lambda}_2\left(\mathbf{x}[iT]\right),\cdots,\prod_{i=0}^{i=m}\tilde{\lambda}_n\left(\mathbf{x}[iT]\right)\right\}\right)Q^{-1}\mathbf{x}[0]\\&=Q\begin{bmatrix}\prod_{i=0}^{i=m}\left(1+T\lambda_1\left(\mathbf{x}[iT]\right)\right) & 0 & \cdots & 0\\ 0 & \prod_{i=0}^{i=m}\left(1+T\lambda_2\left(\mathbf{x}[iT]\right)\right)\cdots & & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \prod_{i=0}^{i=m}\left(1+T\lambda_n\left(\mathbf{x}[iT]\right)\right)\end{bmatrix}Q^{-1}\mathbf{x}[0]\end{aligned} \tag{20.A}$$

Now, we first consider the case in which the NEVlaues of original continuous systems are real. This case is qualitatively similar to the case of proposition 3.3. The constant matrix Q and its inverse do not impact on the qualitative behaviour of the system especially its stability properties as stated below.

$$\begin{aligned}&\lim_{\substack{T\to 0\\ m\to\infty}}\mathbf{x}[(m+1)T]=\\&\lim_{m\to\infty}Q\left(diag\left\{\prod_{i=0}^{i=m}\left(1^-\right),\prod_{i=0}^{i=m}\left(1^-\right),\ldots,\prod_{i=0}^{i=m}\left(1^-\right)\right\}\right)Q^{-1}\mathbf{x}[0]\\&=Q\,[\mathbf{0}]_{n\times n}\,Q^{-1}\mathbf{x}[0]=\mathbf{0}\end{aligned}$$

In the second case we consider the complex NEValues.

For this case, we need to show that

$$\lim_{\substack{T\to 0\\ m\to\infty}}\prod_{i=0}^{i=m}\left(1+T\lambda\left(\mathbf{x}[iT]\right)\right)=0,\quad \forall\mathbf{x}\in\mathbb{R}^n \tag{21.A}$$

where

$$\lambda\left(\mathbf{x}[kT]\right)=-a\left(\mathbf{x}[kT]\right)\pm jb\left(\mathbf{x}[kT]\right) \tag{22.A}$$

and $a\left(\mathbf{x}[kT]\right)>0$.

To prove this, we use (22.A) in (21.A) and write the left hand side of (21.A) as

$$\lim_{\substack{T\to 0\\ m\to\infty}}\prod_{i=0}^{i=m}\left(1-Ta\left(\mathbf{x}[iT]\right)\pm jTb\left(\mathbf{x}[iT]\right)\right) \tag{23.A}$$

Knowing that

$$\begin{aligned}&\lim_{T\to 0}\left|\left(1-Ta\left(\mathbf{x}[iT]\right)\pm jTb\left(\mathbf{x}[iT]\right)\right)\right|\\&=\lim_{T\to 0}\left(\left(1-Ta\left(\mathbf{x}[iT]\right)\right)^2+T^2\left(b\left(\mathbf{x}[iT]\right)\right)^2\right)^{1/2}\end{aligned}$$

and $a\left(\mathbf{x}[iT]\right)$ and $b\left(\mathbf{x}[iT]\right)$ are finite valued functions for every $\mathbf{x}\in\mathbb{R}^n$ and $T>0$, the two terms $\lim_{T\to 0}Ta\left(\mathbf{x}[iT]\right)$ and $\lim_{T\to 0}Tb\left(\mathbf{x}[iT]\right)$ are in the same order and without loss of generality we can have:

$$\begin{aligned}&\lim_{T\to 0}Ta\left(\mathbf{x}[iT]\right)\sim\Delta\\&\lim_{T\to 0}Tb\left(\mathbf{x}[iT]\right)\sim\alpha\Delta\quad ;\quad 0<\Delta\ll 1\end{aligned}$$

where $\alpha\in\mathbb{R}$ is the constant proportional factor and

$$\lim_{T\to 0}\left(\left(1-Ta\left(\mathbf{x}[iT]\right)\right)^2+T^2\left(b\left(\mathbf{x}[iT]\right)\right)^2\right)^{1/2}$$

$$\sim\left((1-\Delta)^2+\alpha^2\Delta^2\right)^{1/2}=\left(1-2\Delta(1-\left(\frac{1+\alpha^2}{2}\right)\Delta)\right)^{1/2}$$

Since

$\forall\alpha\ ,\ \exists\,\Delta<\dfrac{2}{1+\alpha^2}$ such that $\left(\dfrac{1+\alpha^2}{2}\right)\Delta<1$, the limit becomes:

$$\lim_{T\to 0}\left(\left(1-Ta\left(\mathbf{x}[iT]\right)\right)^2+T^2\left(b\left(\mathbf{x}[iT]\right)\right)^2\right)^{1/2}\sim\left(1-0^+\right)^{1/2}=1^-$$

Therefore, in (21.A), each diagonal element, which contains complex NEValues, will reduce to multiplication of m 1^- s; consequently, it goes to zero as m goes to infinity. This ends the proof.