FOUR LECTURES ON STABILITY[1]

Vladimir Rasvan

Department of Automatic Control,
University of Craiova,
A.I.Cuza, 13, Craiova, RO-1100, Romania
e-mail: vrasvan@automation.ucv.ro

Abstract: *Starting with the basic notions about Liapunov and input/output stability there are presented those elements that represent the minimal knowledge of Stability Theory and its methods.A particular attention is paid to absolute stability of feedback systems and connected framework (Liapunov functions, frequency domain inequalities, hyperstability and dissipativity).*

Acknowledgement: *In 1990 the Romanian Society for Automation and Industry Applied Information Processing emerged as an initiative of some outstanding control engineers active both in Education and Research. Systems and Control had been a long date field of interest in Romania however a dedicated academic and application oriented society was missing for, mildly said, non-academic reasons. Neither 1990 nor the following 1991 did not show very stimulating for normal daily research. For this reason the idea of Professor Ioan Dumitrache, first (and actual) President of the Society to organize a cycle of lectures on some basic (but freely chosen) topics of Systems and Control was highly stimulating for a new start. The cycle of lectures took place during the Academic Year 1991/1992 at "Politehnica" Technical University of Bucharest, Department of Automatic Process Control. These lecture notes represent author's contribution to that worth remembering event.*

Keywords: *Stability, Absolute stability, Hyperstability.*

2. LECTURE TWO. THE LIAPUNOV FUNCTION AND THE ABSOLUTE STABILITY

We shall not insist on the stability results for linear systems but enter directly in the subject by presenting the first important result concerning nonlinear system stability: stability by the first approximation.

2.1 Stability by the first approximation

We shall consider a system written in deviations with respect to a steady state

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$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n \tag{1}$$

and assume that f Î $\boldsymbol{C}^1$ i.e. it has continuous partial derivatives. Let

$$A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}(0) \tag{2}$$

be the Jacobian matrix of f computed at the equilibrium point $x = 0$. Defining

$$g(x) = f(x) - Ax \tag{3}$$

system (1) is written as

$$\dot{x} = Ax + g(x) \tag{4}$$

where the linear system

$$\dot{z} = Az \tag{5}$$

is called *the linearized system* of (1). Starting from the assumption that for small deviations the higher order terms represented by $g(x)$ may be neglected, all sciences that were based on linear models restricted themselves to the study of (5), including stability studies (see, for instance, the celebrated reference [1]). In fact (4) and (5) are two completely different systems and, according to a remark of Liapunov himself, the automatic extension on (4) of the properties of (5) is not rigorous.

Definition 9 *If the stability of (5) implies stability of the zero solution of (4) it is said that this solution is stable by the first approximation.*

The stability by the first approximation is a special division of Stability theory; a *State of the Art* paper in the field is [2]; we shall restrict ourselves to give here the best known theorem of stability by the first approximation, belonging to Liapunov himself.

Theorem 1 *Consider system (4) under the following assumptions: i) the linear system (5) of the first approximation is exponentially stable i.e.matrix A is a Hurwitz matrix* ($s(A)$) Ì $\mathbb{C}^-$)*; ii)* $g(x)$ *is sublinear in a neighbourhood of the origin i.e. for sufficiently small* $g > 0$ *there exists* $b(g) > 0$ *such that if* $|x| < b$ *then* $|g(x)| < g|x|$*. Then the zero solution of (4) is exponentially stable.*

The proof of this theorem may be found, for instance, in [3]; one might consult also [2] for other types of proofs. We shall not insist any more on this topic.

2.2 The theorems of the Liapunov function

The method of the Liapunov function (sometimes called also direct method or second method of Liapunov) originates in Rational Mechanics. It is a recognized truth of this science that those systems, whose *total energy (which is a nonnegative definite state function) is decreasing* for any state except a single one which is an equilibrium of the system and a minimum of the energy, have their evolutions ending (asymptotically) in that equilibrium state.

The Liapunov function has analogous properties but, generally speaking, *it is not an energy function*; this gives to the method of the Liapunov function a wider power in applications especially when dealing with systems whose energy is not easy to express.

We shall give below the main mathematical results concerning the function of Liapunov.

Theorem 2 *Consider system (1) and assume there exists a continuous and positive definite in a neighbourhood of the origin* $|x| < d_0$ *function V(x), such that* $V^*(t) = V(x(t))$ *is nondecreasing along any solution of (1) starting in that neighbourhood i.e. with* $|x(0)| < d_0$*. Then the zero solution of (1) is uniformly stable.*

Theorem 3 *Consider system (1) and assume there exists a* $\boldsymbol{C}^1$ *and positive definite in a neighbourhood of the origin* $|x| < d_0$ *function V(x), such that the function* $W(x) = \frac{\partial V}{\partial x} f(x)$ *should be negative definite. Then the zero solution of (1) is asymptotically stable.*

The classical example of a Liapunov function is the positive definite quadratic form

$$V(x)=x^{*}Px$$

where $P>0$ is the solution of the *Matrix Liapunov equation*

$$A^{*}P+PA=-Q$$

A being the matrix of the linear system (5). It is quite well known that for any positive definite matrix Q the above equation has a unique positive definite solution if and only if (5) is exponentially stable. Obviously $W(x)=-x^{*}Qx$.

A simple example from Rational Mechanics (but not only!) is the linear second order system

$$\ddot{x}+a\dot{x}+bx=0,\quad a>0,\quad b>0 \tag{6}$$

Multiplying both sides by $\dot{x}$ and integrating from 0 to t along a solution of the equation we find

$$\int_0^t \ddot{x}(t)\dot{x}(t)dt+a\int_0^t(\dot{x}(t))^2dt+b\int_0^t x(t)\dot{x}(t)dt=0$$

and integrating by parts

$$\frac{1}{2}(\dot{x}(t))^2+\frac{1}{2}b(x(t))^2=$$
$$=\frac{1}{2}(\dot{x}(0))^2+\frac{1}{2}b(x(0))^2-a\int_0^t(\dot{x}(t))^2dt$$

Denoting

$$V(x,y)=\frac{1}{2}\left(y^2+bx^2\right) \tag{7}$$

we have

$$V(x(t),\dot{x}(t))=V(x(0),\dot{x}(0))-a\int_0^t(\dot{x}(t))^2dt$$

For the mechanical system composed of an inertial mass, a spring and a viscous friction linear element - described by (5) - the function defined in (7) represents the total energy of the system. The system is obviously exponentially stable (it can be integrated) but the Liapunov function associated in a natural way - the energy - has its derivative only nonpositive definite. Such a situation occurs in many real cases, when the derivative of the most natural Liapunov function is only nonpositive definite while the system is nevertheless asymptotically stable. These cases led to a less restrictive theorem on asymptotic stability.

Theorem 4 (Barbashin-Krasovskii) *If* $V(x)$ *satisfies the assumptions of* **Theorem 3** *with* $W(x)\le 0$ *and the set* $G=\{x\,|\,W(x)=0\}$ *does not contain other trajectories than the zero solution, this zero solution is asymptotically stable.*

It can be easily seen that for equation (6) and the Liapunov function (7) the assumptions of **Theorem 4** hold. Further development, due to J.P. La Salle, allowed important extensions of **Theorem 4** (see, for instance, [3]).

Remark also the character of sufficient (not necessary) conditions for stability of the above theorems. Therefore, *the choice of the Liapunov function can have various issues in evaluating stability*. We shall illustrate this by an example. Consider the equation

$$\ddot{q}+a\dot{q}+j(q+b\dot{q})=0,\quad a>0 \tag{8}$$

which may describe a simplified model for ship stabilization. The function $j:\mathbb{R}\to\mathbb{R}$ of (8) verifies

$$s\,j(s)>0,\quad j(s)=0\Leftrightarrow s=0$$

In order to construct a Liapunov function we shall proceed as previously: we multiply both sides of (8) by $q+b\dot{q}$ and integrate, obtaining

$$\int_0^t\left[\ddot{q}(t)+a\dot{q}(t)\right]\left[q(t)+b\dot{q}(t)\right]dt+$$
$$+\int_0^t j(s(t))s(t)dt=0$$

After some integration by parts we obtain

$$\frac{1}{2}\left[\sqrt{a}\,q(t)+\frac{1}{\sqrt{a}}\dot{q}(t)\right]^2+\frac{1}{2}\left(b-\frac{1}{a}\right)(\dot{q}(t))^2=$$
$$=\frac{1}{2}\left[\sqrt{a}\,q(0)+\frac{1}{\sqrt{a}}\dot{q}(0)\right]^2+\frac{1}{2}\left(b-\frac{1}{a}\right)(\dot{q}(0))^2-$$
$$-(ab-1)\int_0^t(\dot{q}(t))^2dt-\int_0^t j(s(t))s(t)dt$$

We remark that the quadratic form

$$V(q,\Omega)=\frac{1}{2}\left(\sqrt{a}q+\frac{1}{\sqrt{a}}\Omega\right)^2+\frac{1}{2}\left(b-\frac{1}{a}\right)\Omega^2 \qquad (9)$$

satisfies the requirements of a Liapunov function provided $ab>1$; this inequality together with $a>0$ can be considered as a sufficient condition for asymptotic stability.

Multiply now (8) by $\dot{q}+b\ddot{q}$ and integrate from 0 to *t*. We shall have, in the same way as above

$$\frac{1}{2}(1+ab)\left(\dot{q}(t)\right)^2+\int_0^{s(t)} j(l)dl=\frac{1}{2}(1+ab)\left(\dot{q}(0)\right)^2+$$
$$+\int_0^{s(0)} j(l)dl-\int_0^t\left[a\left(\dot{q}(t)\right)^2+b\left(\ddot{q}(t)\right)^2\right]dt$$

where $s=q+b\dot{q}$. The function *quadratic form plus integral of the nonlinearity*

$$V(q,\Omega)=\frac{1}{2}(1+ab)\Omega^2+\int_0^{q+b\Omega} j(l)dl \qquad (10)$$

satisfies the requirements of a Liapunov function if $b>0$; together with $a>0$ this is a sufficient condition for stability.

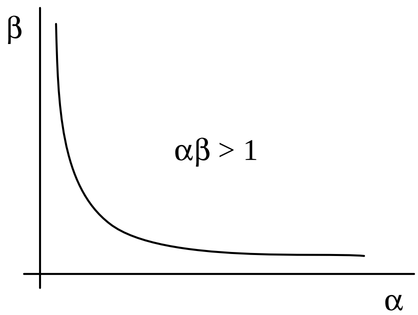

Fig. 1. Parameter plane stability region

The diagram of Fig. 1 shows that while the Liapunov function (9) prescribes as stability region the area above the hyperbola, the function defined by (10) prescribes the entire positive quadrant i.e. the stability region includes some area under the hyperbola too. This is a simple example of the way a suitable Liapunov function may improve the stability region.

2.3 The problem of the absolute stability

Consider the feedback control system of Fig. 2 with *constant reference signal* (stabilization system).

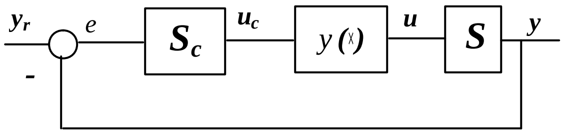

Fig. 2. Stabilization feedback system with a nonlinear element

The system includes two linear blocks, one of the controlled system (***S***) and one of the compensator (***Sc***) and also a static block described by a monotonic nonlinear (possibly globally Lipschitz) function. The two linear blocks, (***S***) being strict proper and (***Sc***) at most proper, may be specified either by their transfer functions or by state representations. In any case the following state equations can be written

$$(\boldsymbol{S}) \qquad \begin{cases}\dot{z}=Az+bu(t)\\ y=c^*z\end{cases};$$

$$(\boldsymbol{S_c}) \qquad \begin{cases}\dot{z}_c=A_cz_c+b_ce(t)\\ u_c=f_c^*z_c+g_ce(t);\\ e(t)=y_r-y\end{cases} \qquad (11)$$

$$(\boldsymbol{N}) \qquad u=y(u_c)$$

After eliminating some intermediate variables the following state equations are obtained

$$\begin{aligned}&\dot{z}=Az+by(u_c)\\ &\dot{z}_c=-b_cc^*z+A_cz_c+b_cy_r\\ &u_c=-g_cc^*z+f_c^*z_c+g_cy_r\end{aligned} \qquad (12)$$

From here we deduce the equations of the operating point (steady state) as prescribed by the reference y_r

$$\begin{aligned}&A\hat{z}+by(\hat{u}_c)=0\\ &-b_cc^*\hat{z}+A_cz_c+b_cy_r=0\\ &\hat{u}_c=-g_cc^*\hat{z}+f_c^*\hat{z}_c+g_cy_r\end{aligned} \qquad (13)$$

Supposing that matrices *A* and *Ac* are nonsingular we can eliminate $\hat{z}$ and $\hat{z}_c$ obtaining the basic steady state equation

$$\hat{u}_c+H_c(0)H(0)y(\hat{u}_c)=H_c(0)y_r \qquad (14)$$

with $H_c(s) = g_c + f_c^*(sI - A_c)^{-1}b_c$ and $H(s) = c^*(sI - A)^{-1}b$ being the tranfer functions of ($\boldsymbol{S_c}$) and ($\boldsymbol{S}$) respectively.

If $\hat{u}_c$, the solution of the above nonlinear equation, is known, then $\hat{z}$ and $\hat{z}_c$ can be determined from two linear systems. We then introduce the deviations

$$x = z - \hat{z}, \quad x_c = z_c - \hat{z}_c$$

which verify the system

$$\begin{aligned} \dot{x} &= Ax + b[y(u_c) - y(\hat{u}_c)] \\ \dot{x}_c &= -b_c c^* x + A_c x_c \\ u_c &= -g_c c^* x + f_c^* x_c + \hat{u}_c \end{aligned}$$

Introducing the new nonlinear function

$$j(s) = y(\hat{u}_c) - y(\hat{u}_c - s) \tag{15}$$

the following system is obtained

$$\begin{aligned} \dot{x} &= Ax - b\,j(s) \\ \dot{x}_c &= -b_c c^* x + A_c x_c \\ s &= g_c c^* x + f_c^* x_c \end{aligned} \tag{16}$$

From the monotonicity of y we obtain the condition $j(s)s > 0$; if, additionally, y is globally Lipschitz, then the sector condition holds

$$0 < \frac{j(s)}{s} < k \tag{17}$$

System (16) may be given a vector-matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A & 0 \\ -b_c c^* & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix} j\left(g_c c^* x - f_c^* x_c\right) \tag{18}$$

what corresponds to the general form

$$\dot{x} = Ax - b\,j(c^* x) \tag{19}$$

(with some abuse of notations).

The nonlinear system (19) can be considered as an autonomous (without exogeneous signals) feedback structure (Fig.3)

The linear block contains the system's dynamics

$$\begin{aligned} \dot{x} &= Ax + bu_1 \\ y_1 &= c^* x \end{aligned}$$

while the nonlinear block is static

$$y_2 = j(u_2)$$

To this we have to add the feedback equations

$$u_1 = -y_2, \quad u_2 = y_1$$

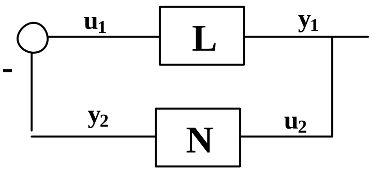

Fig.3 Feedback structure associated to the problem of absolute stability

For system (19) a usual Liapunov stability problem of the zero solution can be formulated. This was exactly the case of the pioneering paper [4]; in that paper the stability result occurs as valid for any nonlinear function satisfying $j(s)s > 0$. This fact led to the validity of the property for an entire class of nonlinear functions hence for an entire class of systems. Further, the advancement of the understanding of the basic aspects led to the statement of the problem as stability of those systems with poor information on the nonlinearity. Using the contemporary language this signifies a robust stability with respect to nonlinear function uncertainty. Indeed this may be seen from problem statement.

Absolute stability problem *Given system (19) where $j(s)$ may be any function satisfying (17), find conditions on (A, b, c, k) or, equivalently, on ($H(s)$, k), where $H(s) = c^*(sI - A)^{-1}b$ is the transfer function of the linear part, in order that the trivial solution should be globally asymptotically stable for all nonlinear functions satisfying (17).*

For this problem which is almost half-century old there exists a long list of references. We send to [3], [5], [6] and in the following we shall deal only with some basic aspects.

Remark first that (8) belongs to the class defined by (19); the Liapunov function, either

(9) or (10) is of the type *quadratic form of the state variables* or *quadratic form plus integral of the nonlinear function*. This type of Liapunov function occurs in most of the papers on absolute stability (see [5], [6]).
In the general case it is a function of the form

$$V(x) = x^* P x + \int_0^{c^* x} j(s) ds \tag{20}$$

where P and b have to be determined in order that (20) and its derivative along system's solutions should have the property required by the fundamental theorems. A quite simple computation will give the following derivative function

$$\begin{aligned} W(x) = {} & x^* P\left(Ax - bj(c^* x)\right) + \left(Ax - bj(c^* x)\right)^* Px \\ & + bj(c^* x)\left(c^* Ax - c^* bj(c^* x)\right) \end{aligned} \tag{21}$$

and it can be easily seen that W can be viewed as a quadratic form on x, j

$$W(x) = \\ = \begin{pmatrix} x \\ j \end{pmatrix}^* \begin{pmatrix} PA + A^* P & -Pb + \frac{1}{2} bA^* c \\ \left(-Pb + \frac{1}{2} bA^* c\right)^* & -\frac{1}{2} b\left(b^* c + c^* b\right) \end{pmatrix} \begin{pmatrix} x \\ j \end{pmatrix} \tag{22}$$

It is obvious that the variables x, j are not quite independent but $V > 0$, $W \le 0$ will be sufficient conditions for stability. The working experience in the field showed that in many cases an easier problem is to judge the sign of the modified quadratic form

$$S(x, j) = W(x, j) + j\left(kc^* x - j\right) \tag{23}$$

The explanation is the following: the variables x, j verify the cone-type restriction

$$j\left(kc^* x - j\right) \ge 0$$

a direct consequence of (17). The sign analysis for a quadratic form in a cone is a quite difficult problem; for this reason the "trick"

$$W(x, j) = S(x, j) - j\left(kc^* x - j\right)$$

reduces this problem to the *sign analysis of the form $S(x, j)$ on the entire space*; because the second term is negative in the cone the conditions on S are sufficient for $W \le 0$. Moreover, a result of V.A.Yakubovich [7] shows that these conditions are necessary and sufficient i.e. the above algorithm called *S-procedure* does not restrict the class of the absolutely stable systems that a direct procedure (without S-procedure) could point out.

By explicitely writing $S(x, j)$ we find

$$S(x) = \left(x^* \; j\right) \begin{pmatrix} S_1 & S_2 \\ S_2^* & S_4 \end{pmatrix} \begin{pmatrix} x \\ j \end{pmatrix} \tag{24}$$

where we denoted

$$\begin{aligned} S_1 &= PA + A^* P \\ S_2 &= -Pb + \frac{1}{2}\left(kc + bA^* c\right) \\ S_4 &= -\frac{1}{2} b\left(b^* c + c^* b\right) - 1 \end{aligned}$$

The fact that $S(x, j) \le 0$ means existence of a scalar g and of a n-dimensional vector w such that

$$S(x, j) = -\left|-gj + w^* x\right|^2 \tag{25}$$

Therefore we may write

$$W(x, j) = -\left|-gj + w^* x\right|^2 - j\left(kc^* x - j\right)$$

where $W(x, j)$ is that of (21). Taking $x = x(t)$, $j = j\left(c^* x(t)\right)$ where $x(t)$ is a solution of (19), and integrating from 0 to t we find

$$\begin{aligned} V\left(x(t)\right) = {} & V\left(x(0)\right) - \int_0^t \left|-gj\left(c^* x(t)\right) + w^* x(t)\right|^2 dt \\ & - \int_0^t j\left(c^* x(t)\right)\left(kc^* x(t) - j\left(c^* x(t)\right)\right) dt \end{aligned} \tag{26}$$

where $V(x)$ is that of (20). The equality (26) *represents a generalization of the equalities obtained for the linear and nonlinear second order systems considered previously*.

Summarizing, it is clear that the problem of finding a Liapunov function *quadratic form plus integral* for the absolute stability can be reduced finally to the finding a triple (g, w, P) such that the following equalities hold

$$
\begin{aligned}
PA + A^*P &= -ww^* \\
-Pb + \frac{1}{2}\left(kI + bA^*\right)c &= gw \\
1 + \frac{1}{2}b\left(b^*c + c^*b\right) &= |g|^2
\end{aligned}
\tag{27}
$$

These are called *the equations of A.I.Lurie*. The fulfillment of (27) ensures that the derivative of the Liapunov function is at least nonpositive definite. Additional conditions may ensure the positive sign of the Liapunov function and the asymptotic stability. In this way the problem of the absolute stability is solved by a Liapunov function of the type (20).

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