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FOUR LECTURES ON STABILITY[1]

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Abstract: *Starting with the basic notions about Liapunov and input/output stability there are presented those elements that represent the minimal knowledge of Stability Theory and its methods.A particular attention is paid to absolute stability of feedback systems and connected framework (Liapunov functions, frequency domain inequalities, hyperstability and dissipativity).*

Keywords: *Stability, Absolute stability, Hyperstability.*

4. LECTURE FOUR. HYPERSTABILITY OF DYNAMICAL SYSTEMS

Hyperstability is a general theory issued from absolute stability theory or, to be closer to the basic idea [1], from the study of the stability considered as a property *not only of isolated systems but also of families of systems*.

We have already seen that absolute stability of the trivial solution for the system

$$\dot{x} = Ax - b\varphi(c^* x) \qquad (1)$$

from Fig.1 is in fact global asymptotic stability of that solution for the family of systems defined by the family of nonlinear functions satisfying the sector condition

$$\underline{\varphi} \le \frac{\varphi(\sigma)}{\sigma} \le \overline{\varphi} \,. \qquad (2)$$

It has been also seen that if the control function $u(t) = -\varphi(\sigma(t))$ with $\sigma = c^* x$ is introduced, then (2) can be replaced by the following quadratic type restrictions on u and x

[1] *Presented at the Department of Automatic Process Control of "Politehnica" Technical University of Bucharest at the Lecture cycle organized by Romanian Society for Automation and Industry Applied Information Processing November 1991*

$$(u+\underline{\varphi}c^*x)(u+\overline{\varphi}c^*x)\le 0$$

$$\int_0^t (u(\tau)+\underline{\varphi}c^*x(\tau))c^*(Ax(\tau)+bu(\tau))d\tau$$

$$=\underline{\Psi}(c^*x(0))-\underline{\Psi}(c^*x(t))$$

$$-\int_0^t (u(\tau)+\overline{\varphi}c^*x(\tau))c^*(Ax(\tau)+bu(\tau))d\tau$$

$$=\overline{\Psi}(c^*x(0))-\overline{\Psi}(c^*x(t))$$

where

$$\underline{\Psi}(\sigma)=\int_0^\sigma (\varphi(\theta)-\underline{\varphi}\theta)d\theta\,,$$

$$\overline{\Psi}(\sigma)=\int_0^\sigma (\overline{\varphi}\theta-\varphi(\theta))d\theta\,.$$

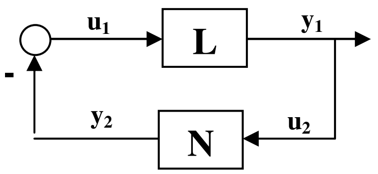


Fig.1 Feedback structure associated to the problem of absolute stability

Moreover, the application of the frequency domain method of Popov lead to the construction of the integral index

$$\eta(0,t)=\alpha_0\int_0^t \left(u(\tau)+\underline{\varphi}c^*x(\tau)\right)\left(u(\tau)+\overline{\varphi}c^*x(\tau)\right)d\tau$$

$$+\alpha_1\int_0^t \left(u(\tau)+\underline{\varphi}c^*x(\tau)\right)c^*\left(Ax(\tau)+bu(\tau)\right)d\tau$$

$$-\alpha_2\int_0^t \left(u(\tau)+\overline{\varphi}c^*x(\tau)\right)c^*\left(Ax(\tau)+bu(\tau)\right)d\tau$$

which, after an integration by parts, leads to the form

$$\eta(0,t)=\frac{1}{2}\left(\alpha_1\underline{\varphi}-\alpha_2\overline{\varphi}\right)x^*cc^*x\Big|_0^t$$

$$+\alpha_0\int_0^t \left(u(\tau)+\underline{\varphi}c^*x(\tau)\right)\left(u(\tau)+\overline{\varphi}c^*x(\tau)\right)d\tau \qquad (3)$$

$$+(\alpha_1-\alpha_2)\int_0^t u(\tau)c^*\left(Ax(\tau)+bu(\tau)\right)d\tau.$$

This integral index was used together with the state differential equation

$$\dot{x}=Ax+bu(t)\,. \qquad (4)$$

The first natural generalization of the above index structure was to consider a *general quadratic* form under the integral

$$\eta(0,t)=x^*Jx\Big|_0^t+\int_0^t W\left(x(\tau),u(\tau)\right)d\tau\,. \qquad (5)$$

Along the solutions of system (1) the integral (3) satisfies the inequality

$$\eta(0,t)\le\alpha_1\underline{\Psi}(c^*x_0)+\alpha_2\overline{\Psi}(c^*x_0)$$

$$-\alpha_1\underline{\Psi}(c^*x(t))-\alpha_2\overline{\Psi}(c^*x(t)).$$

Even for the case of integral blocks we still have

$$(u+\underline{\varphi}\sigma)(u+\overline{\varphi}\sigma)\le 0$$

$$\int_0^t (u(\tau)+\underline{\varphi}\sigma(\tau))\dot{\sigma}(\tau)d\tau=\underline{\Psi}(\sigma(0))-\underline{\Psi}(\sigma(t))$$

$$-\int_0^t (u(\tau)+\overline{\varphi}\sigma(\tau))\dot{\sigma}(\tau)d\tau=\overline{\Psi}(\sigma(0))-\overline{\Psi}(\sigma(t))$$

hence

$$\eta(0,t)\le\alpha_1\underline{\Psi}(\sigma(0))+\alpha_2\overline{\Psi}(\sigma(0))$$

$$-\alpha_1\underline{\Psi}(\sigma(t))-\alpha_2\overline{\Psi}(\sigma(t)).$$

Such inequalities are considered to define a *generalized feedback*. The generalized feedback thus introduced, as well as the way of considering the frequency domain method of Popov justifies the introduction of *a new mathematical object*: equation (differential, difference, integral) plus a “cummulative” (integral, sum) index. It is for *such* systems (sometimes called Popov-type systems) that the notion of hyperstability is introduced.

4.1 Hyperstability - definition and basic elements

Our main object will be in the following the Popov system defined by

$$\dot{x}=f(x,u,t) \qquad (6)$$

$$\eta(t_0,t_1)=\varphi\left(x(t),t\right)\Big|_{t_0}^{t_1}+\int_{t_0}^{t_1}\psi\left(x(t),u(t),t\right)dt \qquad (7)$$

where $u \in \mathrm{P}^m$ is the control vector and $x \in \mathrm{P}^n$ is the state vector. We assume that f is such that satisfies the conditions of existence of a solution on any interval $[t_0, t_1]$ and φ, ψ are continuous with respect to all variables. The integral index has to be considered as defined along the solutions of (6); by solution we understand any pair $(x(t), u(t))$ that satisfies (6).

As in the Liapunov theory we shall need in the following the so-called *Kamke-Massera functions*: a function $\alpha : \mathrm{P}_+ \mapsto \mathrm{P}_+$ is a Kamke-Massera function if $\alpha(0) = 0$, it is continuous, strictly increasing for all positive arguments and $\lim_{\rho \to \infty} \alpha(\rho) = \infty$.

Definition 4.1 *By a solution of a system having the form (6)-(7), on the interval* $[t_0, T_0]$, $T_0 > t_0$ *we mean any set consisting of: a) a pair* $u(t), x(t)$ *of functions defined on that interval and satisfying (6); b) a function* $t_1 \mapsto \eta(t_0, t_1)$ *defined on the interval* $[t_0, T_0]$ *connected with* $u(t), x(t)$ *by (7).*

We are now in position to state the main definition

Definition 4.2 *(definition of hyperstability) System (6)-(7) is said to be hyperstable (in the sense of* V. M. Popov*) if there exist four* Kamke-Massera *functions* α, β, γ, δ *such that the following two properties hold*

(H_s) For every interval $[t_0, T_0]$, $T_0 > t_0$ *and any solution on that interval, the fulfilment of the inequality*

$$\eta(t_0, t) \le \beta_0^2, \quad \forall t \in [t_0, T_0], \ \beta_0 \ge 0 \tag{8}$$

implies the fulfillment of

$$\alpha(|\, x(t)\,|) \le \beta_0 + \beta(|\, x(t_0)\,|), \quad \forall t \in [t_0, T_0] \tag{9}$$

(H_p) For every interval $[t_0, T_0]$, $T_0 > t_0$ *and every solution on that interval the following holds*

$$\begin{aligned} \eta(t_0, t) \ge & -[\gamma(|\, x(t_0)\,|)]^2 \\ & - \delta(|\, x(t_0)\,|) \sup_{t_0 \le \tau \le t} \alpha(|\, x(\tau)\,|) \end{aligned} \tag{10}$$

The property seems somehow aside the mainstream of the stability theory but the connection with stability will appear later. Consider first the following result - a direct consequence of the hyperstability

Proposition 4.1 *If (H_s) holds then for every pair of constants* $\tilde{\beta}_1 \ge 0, \tilde{\beta}_2 \ge 0$ *satisfying*

$$\eta(t_0, t) \le \tilde{\beta}_1^2 + \tilde{\beta}_2 \sup_{t_0 \le \tau \le t} \alpha(|\, x(\tau)\,|) \tag{11}$$

one has also

$$\alpha(|x(t)|) \le 2\left(\tilde{\beta}_1 + \tilde{\beta}_2 + \beta(|x(t_0)|)\right), \ t \in [t_0, T_0] \tag{12}$$

Remark that if $\tilde{\beta}_1 = \tilde{\beta}_2 = 0$ then (12) means such a boundedness of $x(t)$ that can give *at once Liapunov stability* (because of the invertibility of the Kamke-Massera functions). Another remark is that if (H_p) holds for $\gamma = \delta = 0$ we have $\eta(0, t) \ge 0$. This inequality, called by Popov *hyperstability "in the large"*, is nothing more but *passivity* as it is defined in Circuit Theory.

Sometimes it appears useful to replace properties (H_s) and (H_p) by a single one that may be viewed as a *sufficient condition of hyperstability*.

Proposition 4.2 *Assume the existence of Kamke-Massera functions* $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ *such that for any solution of system (6)-(7) the following inequality holds*

$$\begin{aligned} [\tilde{\alpha}(|\, x(t)\,|)]^2 \le \eta(t_0, t) + [\tilde{\beta}(|\, x(t_0)\,|)]^2 \\ + \tilde{\gamma}(|\, x(t_0)\,|) \sup_{t_0 \le \tau \le t} \tilde{\alpha}(|\, x(\tau)\,|) \end{aligned} \tag{13}$$

Then system (6)-(7) is hyperstable in the sense of Definition 4.2.

The proofs of **Propositions 4.1** and **4.2** which are straightforward may be found in [1].

4.2 The "sum" of two hyperstable systems

Consider two systems of the form (6)-(7)

$$\dot{x}_i = f_i(x_i, u_i, t) \tag{14}$$

$$\eta_i(t_0,t_1)=\varphi_i\big(x_i(t),t\big)\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\psi_i\big(x_i(t),u_i(t),t\big)dt\,,\quad i=\overline{1,2} \tag{15}$$

By definition the "sum" of these two systems is also a system (6)-(7) obtained as follows: equation (6) is obtained as the Cartesian product of the two equations (14) and the index (7) is defined as the sum of the two indices

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}f_1(x_1,u_1,t)\\ f_2(x_2,u_2,t)\end{pmatrix} \tag{16}$$

$$\eta(t_0,t_1)=\eta_1(t_0,t_1)+\eta_2(t_0,t_1)\,. \tag{17}$$

To the above relations one may add, possibly, some restrictions of the form

$$h(x_1,x_2,u_1,u_2,t)=0 \tag{18}$$

where h is a continuous vector-valued function. It is clear that the introduction of (18) means that one considers a subset of the sum system defined above.

The main property of the sum system is contained in the following result

Proposition 4.3 *The sum of two hyperstable systems is a hyperstable system.*

The proof may be found in [1] and we shall omit it because of its very special but enough straightforward character. All one has to do is to use the estimates (inequalities) for the two systems and, starting from the 8 Kamke-Massera functions, to associate the 4 Kamke-Massera functions for the sum.

4.3 Hyperstable blocks and their connections

A block is, as previously, a couple of input-state-output relations: a state differential equation

$$\dot{x}=f(x,u,t) \tag{19}$$

and a *readout mapping*

$$y=g(x,u,t) \tag{20}$$

In the following we shall assume that the block is "square" i.e. the number of the output variables equals the number of the input variables. For any block (19)-(20) one can associate a Popov system described by

$$\begin{aligned}\dot{x}&=f(x,u,t)\\ \eta(t_0,t_1)&=\int_{t_0}^{t_1}u^*(t)y(t)dt\\ &=\int_{t_0}^{t_1}u^*(t)g\big(x(t),u(t),t\big)dt.\end{aligned} \tag{21}$$

After associating the above Popov system to the block all the definitions and results of the previous sections are immediately extended to the blocks. For instance, we have

Definition 4.3 *We shall call block (19)-(20) a hyperstable block if the associated system (21) is hyperstable in the sense of Definition 4.2.*

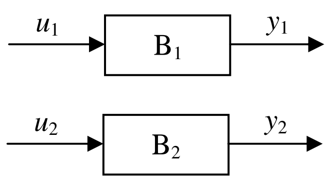
Remark that a block is defined whenever the set of all its solutions is known, what allows a considerable generalization of the concept of block. An important system-like feature of the blocks is that they may be combined together in various ways thus obtaining from two or more distinct blocks new and more complicated ones. Some of these combinations are well known but they are also of particular interest since they have the property that *if the component blocks are hyperstable then the resulting block is also hyperstable*.

Consider for instance two blocks B_i, i = 1, 2 described by relations of the form (19)-(20)

$$(\mathrm{B}_i):\quad \begin{aligned}\dot{x}_i&=f_i(x_i,u_i,t)\\ y_i&=g_i(x_i,u_i,t),\quad i=1,2\end{aligned} \tag{22}$$

and assume that all vectors u_i, y_i have the same number of components.

One may define a new block with the input u and the output y by introducing, according to the chosen interconnection rule (Fig. 2) these new vector valued functions.



a)

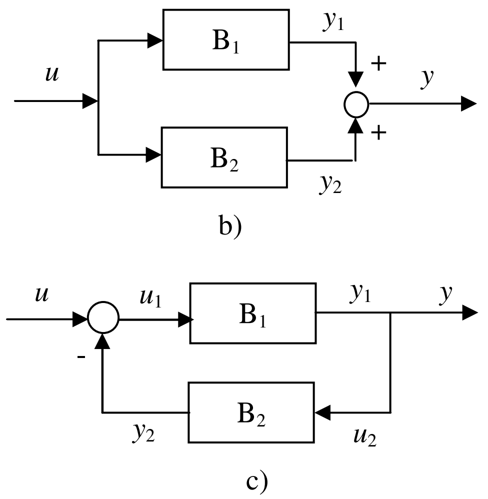


Fig. 2. Block interconnections

For instance, if

$$u_1 = u_2 = u, \quad y = y_1 + y_2 \tag{23}$$

this corresponds to the parallel connection (Fig. 2b) and yields the block

$$(\mathrm{B}_p): \begin{array}{l} \dot{x}_i = f_i(x_i, u_i, t), \quad i = 1, 2 \\ y = g_1(x_1, u, t) + g_2(x_2, u, t) \end{array} \tag{24}$$

We have

$$\begin{aligned} u^* y &= u_1^* g_1(x_1, u_1, t) + u_2^* g_2(x_2, u_2, t) \\ &= u_1^* y_1 + u_2^* y_2 \end{aligned}$$

and if the associated Popov systems of the blocks $(\mathrm{B}_1), (\mathrm{B}_2), (\mathrm{B}_p)$ are introduced we have

$$\eta(t_0, t_1) = \eta_1(t_0, t_1) + \eta_2(t_0, t_1)$$

It can be also seen that any solution of the system associated to (B_p) is also a solution of the sum associated to (B_1) and (B_2). To be more formal, the set of solutions defined by $u_1 = u_2 = u$ is a subset of the solutions of the Cartesian product of the systems associated to (B_i). Therefore we have

Proposition 4.4 *Any block resulting from the parallel connection of two hyperstable blocks is hyperstable.*

A similar property holds for the negative feedback connection (Fig. 2c). In this case we have

$$u_1 = u - y_2, \quad u_2 = y_1, \quad y = y_1. \tag{25}$$

Assume also that the nonlinear equations

$$\begin{aligned} y_1 &= g_1(x_1, u - y_2, t) \\ y_2 &= g_2(x_2, y_1, t) \end{aligned}$$

have a solution of the form

$$y_2 = g_0(x_1, x_2, u, t)$$

where g_0 is piecewise continuous for all the values of the variables involved. We define further the block (B_f) as follows

$$(\mathrm{B}_f): \begin{array}{l} \dot{x}_1 = f_1\big(x_1, u - g_0(x_1, x_2, u, t), t\big) \\ \dot{x}_2 = f_2\big(x_2, g_1\big(x_1, u - g_0(x_1, x_2, u, t), t\big), t\big) \\ y = g_1\big(x_1, u - g_0(x_1, x_2, u, t), t\big) \end{array} \tag{26}$$

Introducing again the associated systems $(\Sigma_1), (\Sigma_2), (\Sigma_f)$ we have

$$\eta_f(t_0, t_1) = \eta_1(t_0, t_1) + \eta_2(t_0, t_1)$$

since

$$u^* y = (u_1 + y_2)^* y_1 = u_1^* y_1 + y_2^* u_2$$

From (25) and (26) it follows once again that the solutions of (26) are a subset of the Cartesian product of (B_1) and (B_2). This subset is in fact determined by the feedback equations (25). We have thus proved

Proposition 4.5 *Any block resulting from the negative feedback connection of two hyperstable blocks is hyperstable.*

Remark that *negative* is crucial: it is not possible to change sign of the feedback including the sign in the equations of the block because this would imply that the block is no longer hyperstable (hyperstability is defined by input/output inequalities).

Remark also that the parallel and negative feedback connections are not the only ones with the property that the resulting block is hyperstable whenever the component blocks are hyperstable. This property holds whenever one replaces (23) or (24) by any other relation which leads to the sum of the integral indices.

4.4 Inherent stability of hyperstable blocks

Proposition 4.6 *Assume (19)-(20) be hyperstable. Then the equation*

$$\dot{x} = f(x, 0, t)$$

obtained from (19) by letting $u \equiv 0$ has the zero solution $x(t) \equiv 0$ and this solution is uniformly stable (in the sense of Liapunov). Moreover, all solutions of this equations are bounded.

Indeed if $u \equiv 0$ then $\eta(t_0, t_1) \equiv 0$ for $\eta(t_0, t_1)$ defined by (21). The block being hyperstable (H_s) holds for $\beta_0 = 0$ hence

$$| x(t) | \le \alpha^{-1}\big(\beta(| x(0) |)\big)$$

and all the solutions are bounded. If we consider the solution that corresponds to $x(t_0) = 0$ it follows from the above inequality that $x(t) \equiv 0$ hence the equation admits the zero solution. But in this case the above inequality combined with the properties of the functions α and β signifies uniform Liapunov stability with $\delta(\varepsilon) = \beta^{-1}\big(\alpha(\varepsilon)\big)$.

Combining **Proposition 4.6** with **Proposition 4.5** we see, for instance, that the block with negative feedback has a uniformly stable zero solution for the input $u \equiv 0$ if the blocks (B_1) and (B_2) are hyperstable.

4.5 Some hyperstable blocks

It becomes quite clear from the previous development that the problem is finally to recognize hyperstable blocks. The book of Popov [1] contains necessary and sufficient conditions for hyperstability and even a list of hyperstable blocks, both linear and nonlinear. Without giving details on theorems and proofs, we give below some hyperstable blocks.

1. *Linear time invariant blocks with lumped parameters* described by:

$$\begin{aligned} \dot{x} &= Ax + Bu(t) \\ y &= Cx + Du(t) \end{aligned} \tag{27}$$

where $\dim x = n$, $\dim u = \dim y = m$ and matrices A, B, C, D have appropriate dimensions. This block is hyperstable if and only if its matrix transfer function

$$H(s) = D + C(sI - A)^{-1}B \tag{28}$$

is real positive i.e. $H(s) \ge 0$ (in the sense of the quadratic forms) for all $s \in X$ with $\mathrm{Re}\,(s) \ge 0$.

2. *Nonlinear blocks containing sector restricted nonlinear functions*:

$$\beta\frac{d\xi}{dt} + \left(\xi - \frac{1}{\overline{\varphi}}\varphi(\xi)\right) = u(t), \quad \sigma = \varphi(\xi) \tag{29}$$

where $\beta > 0$ and $0 \le \varphi(\xi)\xi \le \overline{\varphi}\xi^2$. Indeed we have:

$$\eta(0,t) = \int_0^t \sigma(\tau)u(\tau)d\tau$$

$$= \int_0^t \varphi(\xi(\tau))\left[\left(\xi(\tau) - \frac{1}{\overline{\varphi}}\varphi(\xi(\tau))\right) + \beta\frac{d\xi}{dt}(\tau)\right]d\tau$$

and taking into account the sector inequalities we obtain

$$\eta(0,t) \ge \beta\int_{\xi(0)}^{\xi(t)} \varphi(\theta)d\theta$$

which reads as

$$\begin{aligned} &\beta\psi(\xi(t)) \le \eta(0,t) + \beta\psi(\xi(0)), \\ &\psi(\sigma) = \int_0^{\sigma} \varphi(\theta)d\theta \end{aligned} \tag{30}$$

and we have to apply **Proposition 4.2** with

$$\alpha(\rho) \equiv \beta(\rho) \equiv \beta\psi(\rho), \quad \gamma(\rho) \equiv 0.$$

3. *Nonlinear blocks containing monotone functions* $\varphi(0) = 0$ and $\varphi(v_2) - \varphi(v_1) > 0$ for all $v_2 > v_1$

$$\begin{aligned} &\frac{d\xi}{dt} = -\rho_2\xi + u(t), \quad \sigma = \varphi\big(\rho_1\xi + u(t)\big) \\ &\rho_i > 0, \quad i = 1,2 \end{aligned} \tag{31}$$

Indeed

$$\eta(0,t)=\int_0^t \sigma(\tau)u(\tau)d\tau=\int_0^t \varphi(\rho_1\xi(\tau)+u(\tau))u(\tau)d\tau$$

$$=\int_0^t u(\tau)(\varphi(\rho_1\xi(\tau)+u(\tau))-\varphi(\rho_1\xi(\tau)))d\tau+$$

$$+\int_0^t u(\tau)\varphi(\rho_1\xi(\tau))d\tau$$

$$=\int_0^t u(\tau)(\varphi(\rho_1\xi(\tau)+u(\tau))-\varphi(\rho_1\xi(\tau)))d\tau+$$

$$+\int_0^t \left(\frac{d\xi}{dt}(\tau)+\rho_2\xi(\tau)\right)\varphi(\rho_1\xi(\tau))d\tau$$

$$\geq \int_0^t \varphi(\rho_1\xi(\tau))\frac{d\xi}{dt}(\tau)d\tau$$

The last inequality is due to the property of monotonicity that $\varphi(\sigma)$ has. Next

$$\frac{1}{\rho_1}\psi(\rho_1\xi(t))\leq \eta(0,t)+\frac{1}{\rho_1}\psi(\rho_1\xi(0)), \qquad (31)$$

where $\psi(\sigma)$ is the same as previously and we have to apply again **Proposition 4.2** with the same choices of the Kamke-Massera functions as above.

4.6 Conclusions and perspectives (in 2007)

As mentioned from the beginning, these lectures are from 1991-1992 and they display somehow the state of the art and the scientific/engineering philosophy of the time (the reader may consult e.g. the author's monographs of 1987 or 1993 [2], [3]. With respect to this we are entitled to ask the evergreen philosophical question: “was bleibt?” (“what stands”, in German); this is, let us say, always urgent, but in Control Theory we have in mind a remark made by R.E. Kalman to A. Halanay: “Technology chose other ways than those prescribed by us” [4].

However the new dynamics models that arose from the technology of the last two-three decades e.g. hybrid systems, networked and embedded systems etc. still contain the requirement of inherent or feedback stability. This speaks for the fact that the “stability postulate” coined by one of the classics of the stability theory, namely N.G. Četaev [5] still remains. And this confers some actuality to these lectures.

ACKNOWLEDGEMENT

In 1990 the Romanian Society for Automation and Industry Applied Information Processing emerged as an initiative of some outstanding control engineers active both in Education and Research. Systems and Control had been a long date field of interest in Romania however a dedicated academic and application oriented society was missing for, mildly said, non-academic reasons. Neither 1990 nor the following 1991 did not show very stimulating for normal daily research. For this reason the idea of Professor Ioan Dumitrache, first (and actual) President of the Society to organize a cycle of lectures on some basic (but freely chosen) topics of Systems and Control was highly stimulating for a new start. The cycle of lectures took place during the Academic Year 1991/1992 at "Politehnica" Technical University of Bucharest, Department of Automatic Process Control. These lecture notes represent author's contribution to that worth remembering event.

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