Robust Dynamic Integral Sliding Mode for MIMO Nonlinear Systems Operating Under Matched and Unmatched Uncertainties

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Abstract: In this research, a class of output feedback linearizable MIMO nonlinear systems is considered to be affected by both matched and unmatched uncertainties. The design of output feedback control law relies on an integral manifold, which permits subdivision of the control design architecture into two steps. In the first step, pole placement based continuous control component is designed, which regulates the system output when sliding mode is established. In the next step, a discontinuous control component is designed to cope with the uncertainties. In the proposed approach, the control input is applied to the actual system after passing through a chain of integrators. Consequently, the well-known chattering phenomenon, being caused by high frequency oscillations against the sliding manifold, is reduced and thus a continuous control input is fed into the system. This is a clear benefit in many applications, such as those of mechanical nature where a discontinuous control action could be inappropriate. The proposed control law is theoretically analysed and its performance in term of output regulation to zero is witnessed by the simulation results of two illustrative examples.

Keywords: Uncertain nonlinear systems, Integral manifold, Robust performance

1. INTRODUCTION

Sliding Mode Control (SMC) is a robust control technique capable of stabilizing nonlinear systems operating under matched uncertainties (Utkin (1992); Edwards and Spurgeon (1998)). However, there are many systems affected by uncertainties which do not satisfy the matching condition. These uncertainties are often called unmatched uncertainties. Different sliding mode approaches were proposed (Scarrat et al. (2000); Swaroop et al. (2000); Ferrara and Giacomini (2001); Capisani et al. (2010); Estrada and Fridman (2010); Guermouche et al. (2014))to cope with unmatched uncertainties. The main purpose of the above cited papers was to relax the matching conditions.

A wide class of dynamic system do not remain robust against uncertainties even of matched nature in the so-called reaching phase. Therefore, efforts were directed to eliminate the reaching phase so as to provide robustness against the uncertainties. In other words, sliding mode was established from the very start in the presence of the matched uncertainties Utkin et al. (1999) which is called Integral Sliding Mode Control (ISMC). ISMC has attracted many researchers to deal with the unmatched uncertain system (see for instance, Cao and Xin (2004); Castanos and Fridman (2006); Rubagotti et al. (2011); Bejarano et al. (2007, 2009); Basin et al. (2007); Basin and Ramirez (2013)). In these research works, it is assumed that all the states of the system are available since they are explicitly used to construct the control law.

In real applications, often we deal with systems where it is only the output that can be measured and its derivative can be estimated accurately. In such cases, output feedback SMC becomes a good candidate for the control of the systems. The control of systems with measurable output using SMC approach are studied in Edwards and Spurgeon (1998), Zak and Hui (1993) and Yallapragada et al. (1996). However, these researchers dealt with matched uncertainties and disturbances. An output feedback controller with a static output dependent integral manifold was designed for system operating under both matched and unmatched uncertainties in Choi (2002) using a Linear Matrix Inequalities (LMI) approach. This procedure was extended in Park et al. (2007) using dynamic output feedback variable structure control. Xiang et al. proposed an iterative LMI based approach Xiang et al. (2006) to solve the high gain problem associated with Choi (2002); Park et al. (2007). In this context, an LMI based output feedback SMC approach was proposed in Da Silva et al. (2009) to remove the limitations of the previously LMI based SMC approaches. Higher order SMC Bartolini et al. (1997, 1998); Levant (2003); Boiko et al. (2004); Levant (2005); Dinuzzo and Ferrara (2009) in combination with an integral manifold was studied in Levant and Alelishvili (2007) to improve the robustness and to alleviate chattering caused by high frequency oscillations. Dynamic output feedback ISMC was proposed in Chang (2009) for linear systems with both matched and unmatched uncertainties.

In contrast with most of the above cited research which either deal with matched uncertainties or consider linear systems, the present work deals with unmatched uncertainties in nonlinear MIMO systems. The goal of this paper is to extend the work presented in Khan et al. (2011a); Khan (2011b) to a MIMO nonlinear system operating under a class of matched and unmatched uncertainties. The sliding mode control, in the presence of uncertainties, is enforced along an integral manifold from the very beginning with enhanced robustness and the system outputs are asymptotically regulated to their respective equilibrium points.

In this work, we argue that the contribution is nontrivial in two major aspects. First, the system is considered under the effect of a class of matched and unmatched uncertainties which is different from Khan et al. (2011a); Khan (2011b) where either the system considered was SISO in nature operated under states dependent matched and unmatched uncertainties or the system was MIMO subjected to only matched uncertainties. Second, sliding mode is enforced, in finite time, along an integral manifold and a comprehensive stability analysis is presented in the presence of matched and unmatched uncertainties.

The rest of the paper is organized as follows: In Section 2, the problem formulation is presented while Section 3 outlines the design of the proposed control law. Section 4 discusses the stability analysis in the presence of matched and unmatched uncertainties. A couple of illustrative examples, one related to SISO and MIMO cases, affected by matched and unmatched uncertainties are reported in Section 5. Finally, Section 6 comments on conclusion and elaborates significance of this research.

2. PROBLEM FORMULATION

Consider a nonlinear MIMO system represented by the state space equation analogous to that considered in Cao and Xin (2004)

$$\dot{x} = f(x) + g(x)[(I + \delta_m)u + \Delta g_m(x,t)] + f_u(x,t) \quad (1)$$

$$y = h(x) \quad (2)$$

where $x \in R^n$ is a measurable states vector, $u \in R^q$ is controlled inputs vector, $f : R^n \to R^n$, $h : R^n \to R^q$ are sufficiently smooth vector fields, $g(x) \in R^{n\times q}$ is a full ranked state dependent matrix, δ_m, $\Delta g_m(x,t)$ represent matched uncertainties such that δ_m is a $q \times q$ diagonal matrix and $\Delta g_m(x,t)$ is a $q \times 1$ column vector and $f_u(x,t) = [f_{u_1}(x,t), f_{u_2}(x,t), ..., f_{u_n}(x,t)]$ points to unmatched uncertainties. The close loop system $\dot{x} = f^*(x,t)$ is Lipschitz in x which ensures the existence and uniqueness of the solution Capisani et al. (2010). In the present form, assume that the time is not appearing explicitly but it is just to show that the states are function of time. The following assumption is introduced:

Assumption 1. The uncertainties are assumed to be continuous, norm bounded with corresponding norm bounded derivatives for all $(x,t) \in R^n \times R^+$ i.e, $|\Delta g_{m_i}(x,t)| \le \rho_{m_i}$, $|\delta_{m_i}| \le (1 - \epsilon_{m_i})$ and $|f_{u_i}(x,t)| \le \rho_{u_i}$, where ρ_{m_i}, ϵ_{m_i} and ρ_{u_i} are some positive constants. In addition, each $0 \le \delta_{m_i} < 1$.

The problem we want to solve (Problem-1) is that of steering the vector of outputs to zero asymptotically i.e., an output regulation problem is considered here in the presence of a class of states dependent matched and unmatched uncertainties.

In order to design the control law, system 1 needs to be suitably transformed. Therefore, we consider an example of SISO system related to system 1 with a defined relative degree $r = 2$, and system order $n = 3$. Let

$$\dot{y} = \nabla h[f + gu + \delta_m gu + \Delta g_m g + f_u]$$
$$\dot{y} = L_f h + uL_g h + \delta_m uL_g h + \Delta g_m L_g h + L_{f_u} h$$

Assuming $L_g h = 0$, one has

$$\dot{y} = L_f h + L_{f_u} h$$
$$\ddot{y} = \nabla(L_f h)[f + gu + \delta_m gu + \Delta g_m g + f_u]$$
$$+\nabla(L_{f_u} h)[f + gu + \delta_m gu + \Delta g_m g + f_u]$$
$$\ddot{y} = L_f^2 h + uL_g L_f h + \delta_m uL_g L_f h + \Delta_{g_m} L_g L_f h + L_{f_u} L_f h$$
$$+L_f L_{f_u} h + uL_g L_{f_u} h + \delta_m uL_g L_{f_u} h + \Delta_{g_m} L_g L_{f_u} h + L_{f_u}^2 h$$

Now, once again it is needed that the unmatched uncertainty must appear in such a way that the definition of defined relative degree should not be disturbed. Therefore, by assuming $L_g L_{f_u} h = 0$ and $L_g L_f h \neq 0$, the above expression reduces to

$$\ddot{y} = L_f^2 h + uL_g L_f h + \delta_m uL_g L_f h + \Delta_{g_m} L_g L_f h$$
$$+L_{f_u} L_f h + L_f L_{f_u} h + L_{f_u}^2 h$$

Furthermore, assuming that $L_{f_u} L_g L_f h = 0$, $L_g L_{f_u} L_f h = 0$, $L_g L_{f_u}^2 h = 0$, and $L_g L_f L_{f_u} h = 0$ we have

$$y^3 = L_f^3 h + [u(1 + \delta_m) + \Delta_{g_m}]L_g L_f^2 h + L_{f_u} L_f^2 h$$
$$+\dot{u}(1 + \delta_m)L_g L_f h + (u(1 + \delta_m) + \Delta_{g_m})[L_f L_g L_f h$$
$$+[u(1 + \delta_m) + \Delta_{g_m}]L_g^2 L_f h] + L_f L_{f_u} L_f h$$
$$+L_{f_u}^2 L_f h + L_f^2 L_{f_u} h + L_{f_u} L_f L_{f_u} h + L_f L_{f_u}^2 h + L_{f_u}^3 h$$

Now, the system with $r = 2$ and $n = 3$ can be put in the following form by first defining $y^{(i-1)} = \xi_i$ while keeping in view for a moment, that the system is independent of uncertainties

$$\dot{\xi}_1 = \xi_2 + L_{f_u} h$$
$$\dot{\xi}_2 = \xi_3 + (u(1 + \delta_m) + \Delta_{g_m})L_g L_f h + L_{f_u} L_f h$$
$$+L_f L_{f_u} h + L_{f_u}^2 h$$
$$\dot{\xi}_3 = \Lambda(\xi_1, \xi_2, \xi_3, u) + \dot{u}(1 + \delta_n)L_g L_f h + \Lambda^*(\bullet, u)$$

where $\Lambda^*(\bullet, u)$ is an uncertainty composed of the matched and unmatched terms. It must be noted that the unmatched terms are independent of the control input as given by Assumption-1.

In the similar fashion, we assume that (1) has defined vector relative degree $\beta_1, \beta_2, \cdots, \beta_q$. So (1) can be transformed into the following alternate form:

$$\dot{\xi}_{i1} = \xi_{i2} + \zeta_{i1}(\hat{\xi}, t)$$
$$\dot{\xi}_{i2} = \xi_{i3} + \zeta_{i2}(\hat{\xi}, t)$$
$$\vdots \quad (3)$$
$$\dot{\xi}_{in_i} = \varphi_i(\hat{\xi}, \hat{u}) + \gamma_i(\hat{\xi})[(1 + \delta_{m_i})u_i^{(\beta_i)} + \Delta G_{m_i}(\hat{\xi}, \hat{u}, t)]$$

$$+F_{u_i}(\hat{\xi}, t)$$

where

$$\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n], \hat{\xi}_i = [\dot{\xi}_i, ..., \xi_i^{(n_i-1)}] = [\xi_{i1}, \xi_{i2}, ..., \xi_{in}],$$

$\hat{u} = [\hat{u}_1, \hat{u}_2, ..., \hat{u}_q]$, $\hat{u}_i = (\dot{u}_i, ..., u_i^{(\beta_i-1)})$, for $i = 1, 2, ..., q$. The subscript n_i represents the derivative of each output such that $\sum_i^q n_i = n$. The term $\varphi_i(\hat{\xi}, \hat{u})$ represents the nominal part of the system where as $\zeta_{ij}(\hat{\xi}, t)$ and $F_{u_i}(\hat{\xi}, t)$ refer to the unmatched uncertainties. The representation in (3) is analogous to the so-called Local Generalized Controllable Canonical (LGCC) form Fliess (1990) in the sense that it differs from the basic LGCC form since it is affected by matched and unmatched uncertainties. With reference to system (3), the forthcoming assumption (which is an alternative form of Assumption 1) is introduced:

Assumption 2. There exists β_{i-1} times continuous derivative of $\hat{u}$. So $\hat{u}$ can be taken as bounded term by some constants. Therefore, it can be assumed that $|\varphi_i(\hat{\xi}, \hat{u})| \leq C_i$, $|\gamma_i(\hat{\xi})| \leq K_{M_i}$, $|\Delta G_{m_i}(\hat{\xi}, \hat{u}, t)| \leq B_i$, $|F_{u_i}(\hat{\xi}, t)| \leq \lambda_i$, $|\zeta_{ij}(\hat{\xi}, t)| \leq \mu_i$ for $j = 1, 2, ..., n_{i-1}$, where C_i, K_{M_i}, B_i, λ_i and μ_i are positive constants. Furthermore, it is assumed that $\sum_{j=1}^{(n_i-1)} \zeta_{ij}(\hat{\xi}, t) + F_{u_i}(\hat{\xi}, t) \equiv \Theta_i(\hat{\xi}, t)$ and is bounded by the positive constants τ_i i.e., $|\Theta_i(\hat{\xi}, t)| \leq \tau_i$.

The following nominal system corresponding to (3) can be obtained when $\zeta_{ij}(\hat{\xi}, t) = 0$ and $\Theta_i(\hat{\xi}, t) = 0$.

$$\begin{aligned} \dot{\xi}_{i1} &= \xi_{i2} \\ \dot{\xi}_{i2} &= \xi_{i3} \\ &\vdots \\ \dot{\xi}_{in_i} &= \varphi_i(\hat{\xi}, \hat{u}) + \gamma_i(\hat{\xi}) {u_i}^{(\beta_i)} = \Psi_i\left(\hat{\xi}, \hat{u}, u^{(\beta_i)}\right) \end{aligned} \tag{4}$$

Definition 1. The aforementioned system is termed as proper Lu and Spurgeon (1999), if

- The number of inputs equal number of outputs
- $\Psi_i\left(\hat{\xi}, \hat{u}, u^{(\beta_i)}\right) \in C^1$
- The regularity condition $\left(\frac{\partial \Psi_i}{\partial {u_i}^{(\beta_i)}}\right) \neq 0$, holds in the neighbourhood of the equilibrium points.

Definition 2. The zero dynamics of the system in (4) are defined in (5) Lu and Spurgeon (1999)

$$\Psi_i\left(0, \hat{u}, u_i^{(\beta_i)}\right) = 0; 1 \leq k \leq q \tag{5}$$

The system in (4) is called strongly minimum phase if the zero dynamics are asymptotically stable.

Assumption 3. The nominal system (4) is proper and minimum phase as per mentioned definitions.

Now, Problem-1 can be reformulated with reference to system (3) under Assumption 2, and to the nominal system in (4). Therefore, the new control problem (Problem-2) is to regulate the vector of outputs $[\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n]$ to zero asymptotically in the presence of matched and unmatched uncertainties. In other words, a regulation problem is considered here. The solution to Problem-2 is a clear solution to Problem-1.

Note that the architecture of problem formulation presented in this section can be easily understood via the illustrative examples in Section 5.

3. CONTROL LAW DESIGN

The control law design proposed in this research is analogous to the control law designed in Khan (2011b), the only difference is that the system under study in the present article is considered to operate under a class of matched and unmatched uncertainties. In addition, the sliding mode is established asymptotically using strong reachability condition Lu and Spurgeon (1999). However, the control law proposed here establishes sliding mode, in finite time, along the respective integral manifolds in the presence of uncertainties. The control law, dynamic in nature, is composed of the following two components

$$u_i^{(\beta_i)} = u_{0i}^{(\beta_i)} + u_{1i}^{(\beta_i)} \tag{6}$$

The first component $\left(u_{0i}^{(\beta_i)}\right)$ refers to continuous control and is designed by pole placement. On the other hand, the second component $\left(u_{1i}^{(\beta_i)}\right)$ is designed via sliding mode approach with an integral manifold. In the forthcoming subsection, the design of both the control components is presented.

3.1 Design of $u_{0i}^{(\beta_i)}$

The nominal system (4) can be written in alternate form as (7)

$$\begin{aligned} \dot{\xi}_{i1} &= \xi_{i2} \\ \dot{\xi}_{i2} &= \xi_{i3} \\ &\vdots \\ \dot{\xi}_{in_i} &= \chi_i(\hat{\xi}, \hat{u}, {u_i}^{(\beta_i)}) + {u_i}^{(\beta_i)} \end{aligned} \tag{7}$$

where $\chi_i(\hat{\xi}, \hat{u}, {u_i}^{(\beta_i)}) = \varphi_i(\hat{\xi}, \hat{u}) + (\gamma_i(\hat{\xi}) - 1){u_i}^{(\beta_i)}$. To peruse for the design of $u_{0i}^{(\beta_i)}$, consider that the system (7) is independent of nonlinearities i.e., $\chi_i(\hat{\xi}, \hat{u}, {u_i}^{(\beta_i)}) = 0$ and the system is supposed to be under the action of the $u_{0i}^{(\beta_i)}$ only. Consequently, (7) becomes

$$\dot{\hat{\xi}}_i = A_i \hat{\xi}_i + B_i {u_{0i}}^{(\beta_i)} \tag{8}$$

where

$$A_i = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 \dots & 0 \\ \vdots & \dots & \vdots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \text{ and } B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This system is governed by the linear control law $u_{0i}^{(\beta_i)}$ which may be designed by simple pole placement with the following expression.

$$u_{0i}^{(\beta_i)} = -K_i^T \hat{\xi}_i \tag{9}$$

3.2 Design of $u_{1i}^{(\beta_i)}$

Now in order to achieve the desired performance, robust compensation of the uncertainties is needed. To this end we select the following sliding manifold of integral type Utkin et al. (1999)

$$\sigma_i = \sum_{l=1}^{n_i} r_{il}\xi_{il} + z_i \tag{10}$$

The first term on the right hand side of (10) represent the conventional sliding surface with r_{il} as the design parameters and the second term z_i indicates the integral term. These parameters are chosen in such a way that σ_i remains minimum phase. The time derivatives of (10) along (3) yields

$$\dot{\sigma}_i = \sum_{l=1}^{n_i-1} r_{il}\xi_{il+1} + \varphi_i(\hat{\xi},\hat{u}) + \gamma_i(\hat{\xi})\Big((1+\delta_{m_i})u_i^{(\beta_i)} + \Delta G_{m_i}(\hat{\xi},\hat{u},t)\Big) + \Theta_i(\hat{\xi},t) + \dot{z}_i \tag{11}$$

Now, choosing $\dot{z}_i$ as (12)

$$\dot{z}_i = -\left(\sum_{l=1}^{n_i-1} r_{il}\xi_{il+1} + u_{0i}^{(\beta_i)}\right), \quad z(0) = -\sigma_0(\xi(0)) \tag{12}$$

Then, (11) becomes

$$\dot{\sigma}_i = \varphi_i(\hat{\xi},\hat{u}) + (\gamma_i(\hat{\xi})-1)u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi})u_{1i}^{(\beta_i)} + \gamma_i(\hat{\xi})\Big((1+\delta_{m_i})u_i^{(\beta_i)} + \Delta G_{m_i}(\hat{\xi},\hat{u},t)\Big) + \Theta_i(\hat{\xi},t) \tag{13}$$

The initial conditions of the integral term dynamics are adjusted in such a way to meet the requirement $\sigma_i(0) \equiv 0$. For the sake of simplicity, it is assumed that there is no uncertainties and disturbances. The expression in (13) thus reduces to

$$\dot{\sigma}_i = \varphi_i(\hat{\xi},\hat{u}) + (\gamma_i(\hat{\xi})-1)u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi})u_{1i}^{(\beta_i)} \tag{14}$$

Taking into account the famous reachability condition Utkin (1992)

$$\dot{\sigma}_i = -K_i sign(\sigma_i) \tag{15}$$

Comparing (14) with (15), the expression of the discontinuous control component $u_{1i}^{(\beta_i)}$ becomes

$$u_{1i}^{(\beta_i)} = -\frac{1}{\gamma_i(\hat{\xi})}\Big(\varphi_i(\hat{\xi},\hat{u}) + (\gamma_i(\hat{\xi})-1)u_{0i}^{(\beta_i)} + K_i sign(\sigma_i)\Big) \tag{16}$$

This control law enforces sliding mode along the sliding manifold defined in (10). The constants K_i can be selected according to the subsequent stability analysis. Thus, the final control law becomes

$$u_i^{(\beta_i)} = -K_i^T\hat{\xi}_i - \frac{1}{\gamma_i(\hat{\xi})}\Big(\varphi_i(\hat{\xi},\hat{u}) + (\gamma_i(\hat{\xi})-1)u_{0i}^{(\beta_i)} + K_i sign(\sigma_i)\Big) \tag{17}$$

Note that this control law can be implemented by integrating the derivative of the control $\left(u_i^{(\beta_i)}\right)$ β_i times (which is referred in the following system (18) so that the actual control input applied to the system is continuous. This can be a benefit for various class of systems such as those of mechanical type, for which a discontinuous control action could be disruptive.

The components of the control input u for system in (1) can be obtained by solving the following differential equation

$$\begin{aligned} \dot{u}_{i1} &= u_{i2} \\ \dot{u}_{i2} &= u_{i3} \\ &\vdots \\ \dot{u}_{i\beta_i} &= \Psi_i'\Big(\hat{\xi},\hat{u}\Big), 1 \le i \le q \end{aligned} \tag{18}$$

The function $\Psi_i'(\hat{\xi},\hat{u})$ is discontinuous in nature and refers to the final expression of the dynamic controller (17) for system referred in (3).

4. STABILITY ANALYSIS

The proposed control law when applied to the uncertain nonlinear system in question has been theoretically analyzed. The first case considers only matched uncertainties while the more general case of matched and unmatched uncertainties follows afterwards.

4.1 The System Operating Under Matched Uncertainties

For this case of matched uncertainties, system (3) becomes

$$\begin{aligned} \dot{\xi}_{i1} &= \xi_{i2} \\ \dot{\xi}_{i2} &= \xi_{i3} \\ &\vdots \\ \dot{\xi}_{in_i} &= \varphi_i(\hat{\xi},\hat{u}) + \gamma_i(\hat{\xi})\Big((1+\delta_{m_i})u_i^{(\beta_i)} + \Delta G_{m_i}(\hat{\xi},\hat{u},t)\Big) \end{aligned} \tag{19}$$

To show that this system is stabilized, in finite time, in the presence of matched uncertainties, the following theorem can be stated.

Theorem 1. Consider that Assumptions 2 and 3 are satisfied. The sliding surface is chosen as $\sigma_i(\hat{\xi}) = 0$, where σ_i is defined in (10), and the control law is selected according to (17). If the gain is chosen according to (20), then the finite time enforcement of a sliding mode on $\sigma_i(\hat{\xi}) = 0$ is guaranteed in the presence of matched uncertainties.

$$K_i \ge \left(\frac{1}{(2-\epsilon_{m_i})}\right)\Big((1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{1i}\Big) \tag{20}$$

where η_{1i} are positive constants.

Proof. To prove that the sliding mode can be enforced in finite time, differentiating (10) along the dynamics of (19), and then substituting (12) and (17), (21) is obtained.

$$\dot{\sigma}_i = -K_i sign(\sigma_i) + \delta_{m_i}\Big((u_{0i}^{(\beta_i)} - \varphi_i(\hat{\xi},\hat{u}) - K_i sign(\sigma_i))\Big)$$

$$+\gamma_i(\hat{\xi})\Delta G_{m_i}(\hat{\xi},\hat{u},t) \tag{21}$$

Considering a Lyapunov candidate function $v_i = (1/2)(\sigma_i)^2$, the time derivative of this function along (21) becomes

$$\dot{v}_i \leq |\sigma_i|\Big(-(1+|\delta_{m_i}|)K_i + |\delta_{m_i}||u_{0i}^{(\beta_i)}| + |\varphi_i(\hat{\xi},\hat{u})| + |\gamma_i(\hat{\xi})\Delta G_{m_i}(\hat{\xi},\hat{u},t)|\Big) \tag{22}$$

In view of Assumption 2, the above expression in (22) can be rewritten as

$$\dot{v}_i \leq |\sigma_i|\Big(-K_i(2-\epsilon_{m_i}) + (1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i\Big)$$

or

$$\dot{v}_i \leq -\eta_{1i}|\sigma_i| \tag{23}$$

Provided that

$$K_i \geq \Big(\frac{1}{(2-\epsilon_{m_i})}\Big)\Big((1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{1i}\Big) \tag{24}$$

as in (20). Note that (23) can also be written as

$$\dot{v}_i + \sqrt{2}\eta_{1i}\sqrt{v_i} < 0 \tag{25}$$

This implies that $\sigma_i(\hat{\xi}) = 0$ is reached in finite time t_{s_i} (see Edwards and Spurgeon (1998)), such that

$$t_{s_i} \leq \sqrt{2}\eta_{1i}^{-1}\sqrt{v_i(\sigma_i(0))} \tag{26}$$

which completes the proof

Corollary 2. The dynamics of the system (19) in the absence of unmatched uncertainties, with control law (17) and integral manifold (10), in sliding mode is governed by the linear control law (9).

Proof. Considering (13) with only matched uncertainties, we can write

$$\dot{\sigma}_i = \varphi_i(\hat{\xi},\hat{u}) + (\gamma_i(\hat{\xi})-1)u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi})u_{1i}^{(\beta_i)} + \gamma_i(\hat{\xi})\Big((1+\delta_{m_i})u_i^{(\beta_i)} + \Delta G_{m_i}(\hat{\xi},\hat{u},t)\Big) \tag{27}$$

or

$$\dot{\sigma}_i = \varphi_i(\hat{\xi},\hat{u}) + \gamma_i(\hat{\xi})(1+\delta_{m_i})u_i{}^{(\beta_i)} - u_{0i}{}^{(\beta_i)} + \gamma_i(\hat{\xi})\Delta G_{m_i}(\hat{\xi},\hat{u},t) \tag{28}$$

Now, posing $\dot{\sigma}_i = 0$, and solving with respect to the control variable $u_i^{(\beta_i)}$, the so-called equivalent control Edwards and Spurgeon (1998) can be given by (29)

$$u_{eq}^{(\beta_i)} = \Big(\frac{1}{\gamma_i(\hat{\xi})(1+\delta_{m_i})}\Big)\Big(-\varphi_i(\hat{\xi},\hat{u}) + u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi})\Delta G_{m_i}(\hat{\xi},\hat{u},t)\Big) \tag{29}$$

Now, using (29) in (19), one has

$$\dot{\hat{\xi}}_{i,s} = A_i\hat{\xi}_{i,s} + B_i u_{0i}{}^{(\beta_i)} \tag{30}$$

where A_i and B_i have the form discussed in Section 3 and $\hat{\xi}_{i,s}$ is the state vector of the system (19). Thus, it is proved that the system in sliding mode operates under the continuous control law and the eigenvalues of the controlled transformed system in sliding mode are those of $A_i - B_iK_i^T$.

4.2 The System Operating Under Both Matched and Unmatched Uncertainties

Now, the control objective is to regulate the output of the system in the presence of both of these uncertainties. To prove that the proposed control law is capable of compensating for these uncertain terms, the following theorem can be stated.

Theorem 3. Consider that Assumptions 2 and 3 are satisfied. The sliding surface is chosen as $\sigma_i(\hat{\xi}) = 0$, where σ_i is defined in (10) and the control law is selected according to (17). If the gain is chosen according to the condition specified in (31), then the finite time enforcement of a sliding mode on $\sigma_i(\hat{\xi}) = 0$ is guaranteed in the presence of both matched and unmatched uncertainties.

$$K_i \geq \Big(\frac{1}{(2-\epsilon_{m_i})}\Big)\Big((1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{2i} + \tau_i\Big) \tag{31}$$

where η_{2i} are positive constants.

Proof. To prove that the sliding mode can be enforced in finite time, the time derivative of the Lyapunov candidate function $v_i = (1/2)(\sigma_i)^2$, along (13) becomes as

$$v_i \leq |\sigma_i|\Big(-(1+|\delta_{mi}|)K_i + |\delta_{m_i}||u_{0i}^{(\beta_i)}| + |\varphi_i(\hat{\xi},\hat{u})| + |\gamma_i(\hat{\xi})\Delta G_{m_i}(\hat{\xi},\hat{u},t)| + |\Theta_i(\hat{\xi},t)|\Big) \tag{32}$$

In view of Assumption 2, the above expression can be rewritten as

$$\dot{v}_i \leq |\sigma_i|\Big(-K_i(2-\epsilon_{m_i}) + (1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \tau_i\Big)$$

or

$$\dot{v}_i \leq -\eta_{2i}|\sigma_i| \tag{33}$$

Provided that

$$K_i \geq \Big(\frac{1}{(2-\epsilon_{m_i})}\Big)\Big((1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{2i} + \tau_i\Big) \tag{34}$$

The expression in (34) can be placed in the same format like that of (25). Note that the finite time t_{s_i} in the present case is given by (26) with η_{2i} instead of η_{1i}. Thus it is confirmed that, when the gain of the discontinuous component of the control law (17) is selected according to (31), the finite time enforcement of the sliding mode is guaranteed in the presence of matched and unmatched uncertainties, which proves the theorem.

Corollary 4. The dynamics of the system (3), with control law (17) and integral sliding manifold $\sigma_i = 0$, with $\sigma_i(\hat{\xi})$

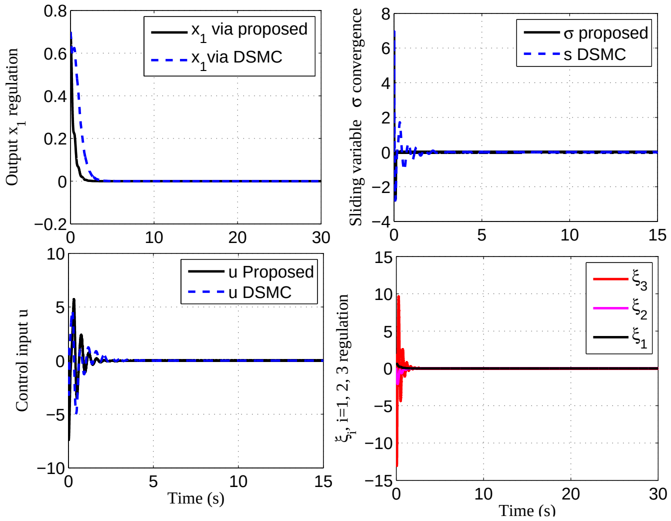

Fig. 1. **Solid line:** Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2, \xi_3]^T$ regulation in the presence of matched uncertainty via the proposed control law, **Doted line:** Output regulation, control effort, sliding variable convergence adopted from Lu and Spurgeon (1998).

defined in (10), in sliding mode is governed by the linear control law (9).

Proof. The proof can be performed in a similar fashion as Corollary 1 proof. The only difference is that in this case the equivalent control carry the form:

$$u_{eq}^{(\beta_i)} = \Big(\frac{1}{\gamma_i(\hat{\xi})(1+\delta_{m_i})}\Big)\Big(-\varphi_i(\hat{\xi}, \hat{u}) + u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi})\Delta G_{im}(x,t) + \Theta_i(\hat{\xi}, t)\Big) \quad (35)$$

5. APPLICATIONS

To verify the aforementioned claims and to validate the proposed control approach, two illustrative examples have been considered. In the first case, a SISO uncertain nonlinear system is studied while in the next case an application example of a three link robotic manipulator(MIMO) system is considered.

5.1 Academic Example

Consider the following SISO nonlinear system, whose nominal case is similar to that of Lu and Spurgeon (1998), operating under matched and unmatched uncertainties

$$\begin{aligned} \dot{x_1} &= x_2 + f_1(x,t) \\ \dot{x_2} &= x_1^2 + (x_2^2+1)((1+\delta_m)u + \Delta g_m(x,t)) \\ &+x_3 + f_2(x,t) \\ \dot{x_3} &= -x_3 + x_2 x_3^2 + f_3(x,t) \end{aligned} \quad (36)$$

The terms δ_m and Δg_m are matched uncertainties and $f_i(x,t)$, for $i = 1, 2, 3$ are components of the unmatched uncertainty which satisfy Assumptions 1 and 2 and contribute to the system uncertainty with the following mathematical expressions.

$$f_1(x,t) = -x_3 + x_2 x_3^2 + (-x_3 + x_2 x_3^2)^2$$

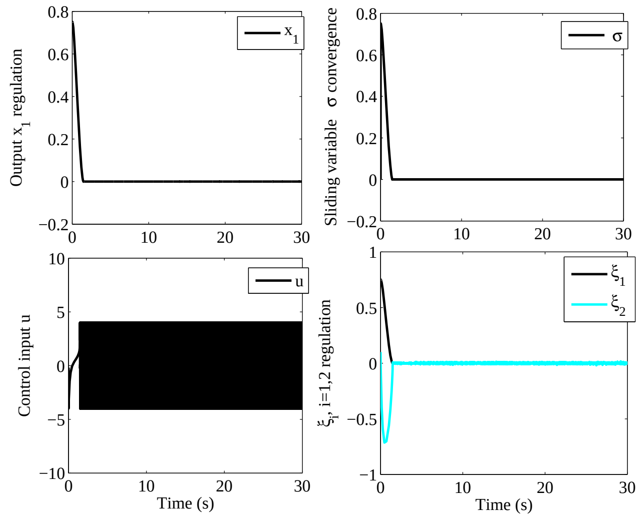

Fig. 2. Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2]^T$ regulation in the presence of matched uncertainty via 2-QCSMC.

$$\begin{aligned} &+0.25sin(t)cos(3x_2) + 0.26 \\ f_2(x,t) &= 0.25sin(t)cos(3x_2) + 0.1 \\ f_3(x,t) &= -x_3 + x_2x_3^2 + 3(-x_3 + x_2 x_3^2)^2 \\ &+0.25sin(t)cos(3x_2) + 0.1 \\ \Delta g_m(x,t) &= 3(-x_3 + x_2 x_3^2) \\ \delta_m &= 0.3cos(\pi t x_2) \end{aligned}$$

Furthermore, $[x_1, x_2, x_3]$ is the state vector and $y = x_1$ is the measurable output of the plant. The LGCCF of this system (for the nominal case) can be obtained by three time differentiation of the output along the dynamics of (36) for this nonlinear plant. The third derivative of the output takes the form of (37)

$$y^{(3)} = 2x_1x_2 + \dot{u}(x_2^2+1) + 2x_2u(x_1^2 + (x_2^2+1)u + x_3) - x_3 + x_2x_3^2 \quad (37)$$

The Definition 1 is satisfied and the zero dynamic mentioned in Definition 2, for the aforementioned system becomes Lu and Spurgeon (1998)

$$\dot{u} + 2u = 0$$

These exponentially stable dynamics shows that the system is minimum phase. The system in (37) can be written in the following LGCCF form

$$\begin{aligned} \dot{\xi_i} &= \xi_{i+1}, i = 1, 2 \\ \dot{\xi_3} &= \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi})\dot{u} \end{aligned}$$

where $y = \xi_1$, $\gamma(\hat{\xi}) = (x_2^2 + 1)$ and

$$\varphi(\hat{\xi}, \hat{u}) = 2x_1x_2 + 2x_2u(x_1^2 + (x_2^2+1)u + x_3) - x_3 + x_2x_3^2$$

The transformation being used here are $\hat{\xi} = [\xi_1, \xi_2, \xi_3]^T = [y, \dot{y}, \ddot{y}]^T$.

The sliding surface can be defined by

$$\sigma = c_1\xi_1 + c_2\xi_2 + \xi_3 + z$$

The compensator dynamics are given by the following expression

$$\dot{z} = -\dot{u}_0 + \left(-c_1 x_2 - c_2\left(x_1^2 + x_2^2 u + x_3\right)\right); \quad (38)$$

The expression of the dynamic integral controller can then be written as

$$\dot{u} = -k_1\xi_1 - k_2\xi_2 - k_3\xi_3 - \frac{1}{\gamma(\hat{\xi})}\Big(\varphi(\hat{\xi},\hat{u}) + (\gamma(\hat{\xi})-1)u_0^{(k)} + K_1(\sigma + W sign(\sigma))\Big)$$

Note that the uncertain terms are omitted for the sake of simplicity here.

In this study, we compare the results of the proposed control law with that of quasi continuous high order sliding mode controller proposed by Levant in Levant (2005) and with the Dynamic Sliding Mode Controller (DSMC) proposed by Lu et al. in Lu and Spurgeon (1998). To apply the approach of Levant, we denote as

$$s = x_1$$
$$\dot{s} = x_2$$

so that the expression of the Quasi Continuous Sliding Mode Controller in case of relative degree (2-QCSMC) takes the following form

$$u = -\left(\frac{\alpha\dot{s} + |s|^{1/2} sign(s)}{|\dot{s}| + |s|^{1/2}}\right) \quad (39)$$

where α is the controller gain which can be selected according to Bartolini et al. (2003). As proved in Levant (2005), the control law (39) provides a finite time sliding mode of the system with a control law which is continuous everywhere except on the second order sliding manifold $s = \dot{s} = 0$.

System Operated with Matched Uncertainty In this study, the system with matched uncertainties (i.e., with $f_i(x,t) = 0$ for $i = 1,2,3$) is simulated to confirm the aforementioned claim of the compensation of uncertain terms. This test with matched uncertainty is also performed with 2-QCSMC while the standard results of DSMC are adopted from Lu and Spurgeon (1998). In Figure 1, the standard results of DSMC reported in Lu and Spurgeon (1998) are compared with the proposed control law (with $K_1 = 230$). It can be seen that the output of the system via the proposed controller is regulated to zero in small time as compared to the regulation via DSMC. On the other hand, the sliding manifold convergence for DSMC is oscillatory while the sliding manifold convergence in our case is sharp and indicates no oscillation. Therefore, it justified that the proposed controller is performing better than the reported DSMC. Note that the results of DSMC Lu and Spurgeon (1998) were only for a nominal case with no uncertainties appearing in the system. The state vector $[\xi_1, \xi_2, \xi_3]^T$ is regulated in the presence of uncertainties.

In comparison with the results of the 2-QCSMC (with gain $\alpha = 4$) as depicted in Figure 2, it is noticeable that the proposed methodology provides a satisfactory regulation of the system output via a continuous control law. The 2-QCSMC also provides excellent performance yet with a control law which becomes discontinuous when the output

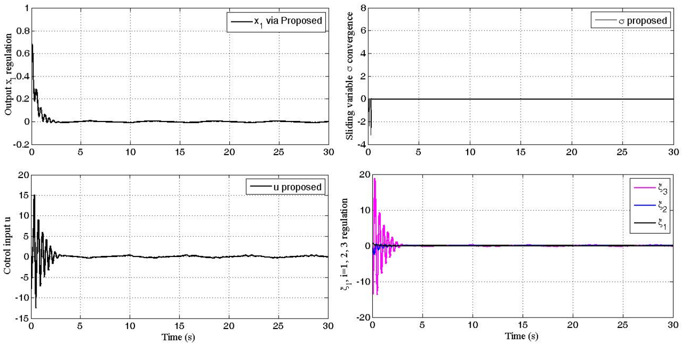

Fig. 3. Output regulation, control efforts, sliding variable convergence and $[\xi_1, \xi_2, \xi_3]^T$ regulation in the presence of matched and unmatched uncertainty via the proposed control law.

Table 1. Gains of the controllers in example 1

Constants	k_1	k_2	k_3	c_1	c_2
2-QCSMC	–	–	–	–	–
Proposed C. law $r = 2$	490.2	180.7	5.9	6	5

regulation objective is attained. Apart from that, both the controllers need to use a differentiator (Castanos and Fridman (2006) and Lu and Spurgeon (1998)) to construct the derivatives of the output variable necessary in the control laws.

The System Operated under Matched and Unmatched Uncertainties In this section, the test with both matched and unmatched uncertainty is performed. In view of the nature of the uncertainty here, we cannot compare our results with those of the 2-QCSMC algorithm and DSMC, since these algorithm were designed under the assumption of having only matched uncertainty Levant (2005) and Lu and Spurgeon (1998), respectively. The Output regulation, control efforts, sliding variable convergence and $[\xi_1, \xi_2, \xi_3]^T$ regulation in the presence of matched and unmatched uncertainty via the proposed control law is shown in Figure 3. It is clear from the results that the proposed control law is capable to regulate the system output to zero even in the presence of reported uncertainties. Note that, the gains of the proposed controller in this example are reported in Table-1.

5.2 Application to Serial Robotic Manipulator (MIMO System)

Consider the dynamic model of an n link robotic manipulator

$$u = B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) \quad (40)$$

where $q \in R^n$ is the measurable position vector such that $q^T = [q_1, q_2, \cdots, q_n]$, $u \in R^n$ are the controlled torques, $B(q)$ is the inertia matrix, $C(q,\dot{q})\dot{q}$ represents the Coriolis and centrifugal forces and $g(q)$ is the gravitational torques vector. The control objective, in this example, is tracking a predefined trajectory by the joints positions. To design a control law, the above system can be transformed to the following alternate form which is the LGCC form

$$\dot{\xi}_{i1} = \xi_{i2} \quad i = 1,2,3 \quad j = 1,2$$
$$\dot{\xi}_{i2} = f_i(\xi_{ij}, \bar{u}) + \bar{\gamma}_i u_i \quad (41)$$

where ξ_{ij} points to different joint positions and velocities, $\bar{u}$ represents the components of the control input other than u_i. The states ξ_{i1} refers to the joint i position angle while ξ_{i2} refers to the velocity. When $i = 1$, this means that the dynamics of the link-1 are considered and so on. The nonlinear components forces $f_i((\xi_{ij}, \bar{u}))$ are obtained by $[f_1, f_2, f_3]^T = B^{-1}(q)[-C(q, \dot{q})\dot{q} - g(q)]$ and are assumed to be known based on the following mathematical expressions.

$$f_1 = \frac{1}{b_{11}}\Big((L_1\xi_{12} + L_2\xi_{22} + L_3\xi_{32} + L_4 - L_5u_2 + L_6u_3)\Big)$$
$$f_2 = \frac{1}{b_{22}}\Big((P_1\xi_{12} + P_2\xi_{22} + P_3\xi_{32} + P_5 - P_4u_3)\Big)$$
$$f_3 = -\frac{1}{b_{33}}\Big(C_{31}\xi_{12} + C_{32}\xi_{22} + (C_{33} + \gamma_{11})\xi_{32} + g_3\Big)$$
$$\bar{\gamma}_1 = \frac{1}{b_{11}} \quad \bar{\gamma}_2 = \frac{1}{b_{22}} \quad \bar{\gamma}_3 = \frac{1}{b_{33}}$$

where

$$b_{11} = \gamma_3 + 2\gamma_5 l_1 cos(\xi_{21}) + 2\gamma_6 l_2 cos(\xi_{31}) + 2\gamma_6 l_1 cos(\xi_{31} + \xi_{31})$$
$$b_{12} = \gamma_2 + \gamma_5 l_1 cos(\xi 21) + 2\gamma_6 l_2 cos(\xi_{31}) + \gamma_6 l_1 cos(\xi_{31} + \xi 21)$$
$$b_{13} = \gamma_1 + \gamma_6 l_2 cos(\xi_{31}) + \gamma_6 l_1 cos(\xi_{31} + \xi_{21})$$
$$b_{22} = \gamma_2 + 2\gamma_6 l_2 cos(\xi_{31})$$
$$b_{23} = \gamma_1 + \gamma_6 l_2 cos(\xi_{31})$$
$$b_{31} = 0 \quad b_{32} = 0 \quad b_{33} = \gamma_1$$
$$c_{11} = -\xi_{22}\gamma_5 l_1 sin(\xi_{21}) - (\xi_{22} + \xi_{32})\gamma_6 l_1 sin(\xi_{21} + \xi_{31}) - \xi_{32}\gamma_6 l_2 sin(\xi_{31})$$
$$c_{12} = -(\xi_{12} + \xi_{22})\gamma_5 l_1 sin(\xi_3) - (\xi_{12} + \xi_{22} + \xi_{32})\gamma_6 l_1 sin(\xi_{21} + \xi_{31}) - \xi_{32}\gamma_6 l_2 sin(\xi_{31})$$
$$c_{13} = -(\xi_{12} + \xi_{22} + \xi_{32})\gamma_6(l_1 sin(\xi_{21} + \xi_{31}) + l_2 sin(\xi_{31}))$$
$$c_{21} = \xi 12\gamma_5 l_1 sin(\xi_{21}) + \xi_{12}\gamma_6 l_1 sin(\xi_{21} + \xi_{31}) - \xi_{32}\gamma_6 l_2 sin(\xi_{31})$$
$$c_{22} = -\xi_{32}\gamma_6 l_2 sin(\xi_{31})$$
$$c_{23} = -(\xi_{12} + \xi_{22} + \xi_{32})\gamma_6 l_2 sin(\xi_{31})$$
$$c_{31} = \gamma_6(\xi_{12} l_1 sin(\xi_{21} + \xi_{31}) + \xi_{12} l_2 sin(\xi_{31}) + \xi_{22} l_2 sin(\xi_{31}))$$
$$c_{32} = (\xi_{12} + \xi_{22})\gamma_6 l_2 sin(\xi_{31}) \quad c_{33} = 0$$
$$g_1 = \gamma_4 sin(\xi_{11}) + \gamma_5 g sin(\xi_{11} + \xi_{21}) + \gamma_6 g sin(\xi_{11} + \xi_{21} + \xi_{31})$$
$$g_2 = \gamma_5 g sin(\xi_{11} + \xi_{21}) + \gamma_6 g sin(\xi_{11} + \xi_{21} + \xi_{31})$$
$$g_3 = \gamma_6 g sin(\xi_{11} + \xi_{21} + \xi_{31})$$
$$p_1 = b_{23}\frac{1}{b_{33}}c_{31} - c_{21}, \quad p_2 = b_{23}\frac{1}{b_{33}}c_{32} - c_{22} - \gamma_9$$
$$p_3 = b_{23}\frac{1}{b_{33}}(c_{33} + \gamma_{11}) - c_{23}, \quad p_4 = b_{23}\frac{1}{b_{33}}$$
$$p_5 = b_{23}\frac{1}{b_{33}}g_3 - g_2$$
$$L_1 = b_{13}\frac{1}{b_{33}}c_{31} - b_{12}\frac{1}{b_{22}}p_1 - c_{11} - \gamma_7$$
$$L_2 = b_{13}\frac{1}{b_{33}}c_{32} - b_{12}\frac{1}{b_{22}}p_2 - c_{12}$$
$$L_3 = b_{13}\frac{1}{b_{33}}c_{32} - b_{12}\frac{1}{b_{22}}p_3 - c_{13}$$
$$L_4 = b_{13}\frac{1}{b_{33}}g_3 - g_1 - b_{12}\frac{1}{b_{22}}p_5$$
$$L_5 = b_{12}\frac{1}{b_{22}}, \quad L_6 = b_{12}\frac{1}{b_{22}}p_4 - b_{13}\frac{1}{b_{33}}$$

Note that, b_{ij}, c_{ij} and g_i are the components of the inertia matrix, Coriolis matrix and gravitational torque vector, respectively. In order to simplify the presentation of the dynamic model of the robotic manipulator, p_i and L_i are introduced.

Now, the over all system is divided into three subsystems with the following state vectors. $\hat{\xi}_1 = [\xi_{11}, \xi_{12}]^T$, $\hat{\xi}_2 = [\xi_{21}, \xi_{22}]^T$, $\hat{\xi}_3 = [\xi_{31}, \xi_{32}]^T$ and $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3]^T$.

Adapting the design procedure of this research, the corresponding linear systems becomes

$$\dot{\hat{\xi}}_i = A_i\hat{\xi}_i + B_i u_{0i}, i = 1, 2, 3 \tag{42}$$

where each $\hat{\xi}_i = [\xi_{i1}, \xi_{i2}]^T$ is the state vector of the outputs and its derivatives.

$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $i = 1, 2, 3$. The continuous components of the control law becomes

$$u_{0i} = k_{i1}\xi_{i1} + k_{i2}\xi_{i2}$$

Note that, the continuous components are designed via pole placement. The design of the discontinuous components is carried out by first designing the sliding surfaces as follows

$$\sigma_1 = r_{11}\xi_{11} + \xi_{12} + z_1$$
$$\sigma_2 = r_{21}\xi_{21} + \xi_{22} + z_2$$
$$\sigma_3 = r_{31}\xi_{31} + \xi_{32} + z_3$$

The dynamics of the integral terms become

$$\dot{z}_1 = -u_{01}$$
$$\dot{z}_2 = -u_{02}$$
$$\dot{z}_3 = -u_{03}$$

The discontinuous control components are calculated with the forthcoming mathematical expressions

$$u_{11} = -b_{11}\Big(r_{11}\xi_{12} + \frac{1}{b_{11}}\Big(L_1\xi_{12} + L_2\xi_{22} + L_3\xi_{32} + L_4 - L_5u_2 + L_6u_3\Big) + \Big(\frac{1}{b_{11}} - 1\Big)u_{01} + K_1 sign(\sigma_1)\Big)$$
$$u_{12} = -b_{22}\Big(r_{21}\xi_{22} + \frac{1}{b_{22}}\Big(p_1\xi_{12} + p_2\xi_{22} + p_3\xi_{32} + p_5 - p_4u_3\Big) + \Big(\frac{1}{b_{22}} - 1\Big)u_{02} + K_2 sign(\sigma_2)\Big)$$
$$u_{13} = -b_{33}\Big(r_{13}\xi_{32} - \frac{1}{b_{33}}\Big(c_{31}\xi_{12} + c_{32}\xi_{22} + (c_{33} + \gamma_{11})\xi_{32} + g_3\Big) + \Big(\frac{1}{b_{33}} - 1\Big)u_{03} + K_3 sign(\sigma_3)\Big)$$

Table 2. Controller's gains for 3DOF robotic manipulator

k_{11}	k_{12}	k_{21}	k_{22}	k_{21}	k_{22}	r_{11}	r_{21}	r_{31}
-6	-4	-2.5	-2.7	-4.2	-3.3	.2	.3	.2

Table 3. Measured parameters of the robotic manipulator Capisani et al. (2010)

γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
0.297	10.07	87.91	57.03	9.21	0.316
γ_7	γ_8	γ_9	l_1	l_2	l_3
190.5	66.343	21.0	0.65	0.6576	0.13

The final expression of the control input to link i can be calculated via the following formula

$$u_i = u_{0i} + u_{1i} \quad i = 1, 2, 3$$

In realistic sense, the uncertainties in case of this manipulator may be because of unmodeled dynamics, parametric variations and external loads. As far as the parametric variations and unmodeled dynamics are concerned, they are usually treated as unmatched uncertainties. Therefore, in this study the system is considered simultaneously under the parametric variations, unmodeled dynamics and time varying matched perturbations. All the parameters of the robotic system are kept under 200 percent change when the process starts. In addition, the last three terms in dynamics of link-1, the last two terms in the dynamics of link-2 and the last one term in the dynamics of link-3 were ignored. Furthermore, $0.5cos(t)$ was added as matched uncertainty in each control channel. The results are quite fruitful even in the presence of aforementioned perturbations.

The tracking performance of each link is displayed in Figure 4. The results confirm that the proposed control law is a good candidate for the fully actuated robotic manipulators. The ISMC manifolds shown in Figure 5 ensure that the sliding mode is enforced from the very beginning even in the presence of uncertainties. This once again guarantee the elimination of the reaching phase which resulting in the robustness enhancement. The control efforts applied to each link are displayed in Figure 6. This demonstrates that the proposed control technique is well suited for electro mechanical systems where inputs with reduced chattering(in case of SMC) is required. The two illustrative examples discussed here impressively justify the claim that the proposed methodology outshines the existing dynamics sliding mode and 2-QCSMC techniques. The gains of continuous components, coefficients of the integral manifolds are listed in Table 2 whereas the gains of the discontinuous components are set equal to seven i.e., $K_1 = K_2 = K_3 = 7$.

6. CONCLUSION

In this note, an output feedback SMC control law is presented for a class of MIMO nonlinear systems operating under a class of matched and unmatched uncertainties. The robustness analysis of the designed control control law is presented in term of two theorems. In addition, corollaries are presented which ensure that the system operates only under the action of a continuous control component in sliding mode. The effectiveness of the control law is

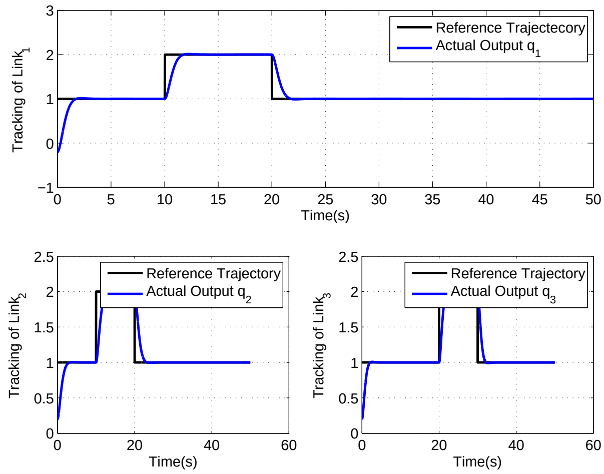

Fig. 4. Tracking performance of the three link of the robotic manipulator

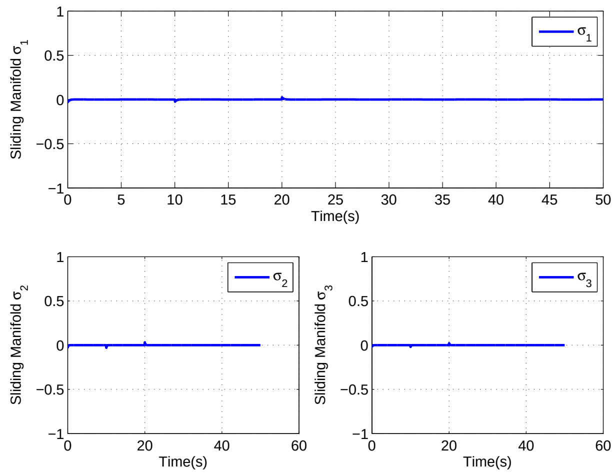

Fig. 5. The integral sliding manifold which ensures the sliding mode enforcement from the beginning of the process

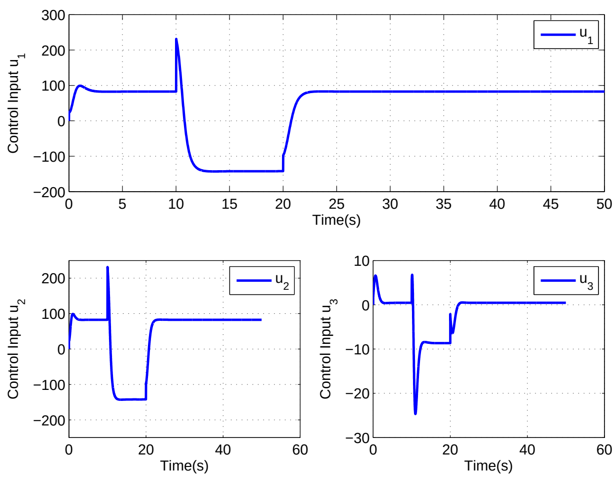

Fig. 6. The applied control inputs to each link of the robotic manipulators

test in two examples. The control input is applied to the actual system after passing thorough a chain of integrators. The integration performs like low pass filtering resulting in a continuous control input. This continuous nature is beneficial in term of chattering alleviation and therefore, offers as a good candidate for systems of mechanical type where a discontinuous control may cause damage to the system.

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