# Nonlinear observer-based optimal control using the state-dependent Riccati equation for a class of non-affine control systems

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**Abstract:** The present study uses a combination of a state-dependent Riccati equation (SDRE) controller and SDRE observer to control a class of nonlinear non-affine control systems. This type of nonlinear system features control nonlinearities that change the formulation of the SDRE and its solution techniques. Control law is obtained using the exact solution method for low-order and uncomplicated systems; numerical approaches are available for complex and high order systems. Both solution methods must use integral control formulation or solve the system equation using control law simultaneously to treat control nonlinearities. The closed loop feedback of the system is partially available, providing an observer for estimation of full-state feedback. The efficacy, advantages and compatibility of the controller with the observer guides the choices to the SDRE observer. The stability proof of the SDRE controller in the non-affine structure is presented and four case studies are simulated and discussed. Two were selected from previous studies to verify the proposed algorithm. The third, more complicated model, shows the ability of the controller to solve complex and high order systems. A brief discussion of the suboptimal cheap control problem and an example are also presented.

*Keywords:* SDRE controller; SDRE observer; non-affine system; nonlinear control; optimal control.

## 1. INTRODUCTION

Non-affine systems are used in two major areas. In the first category, nonlinear systems are considered with uncertainties in states or control inputs (Lee *et al.,* 2009; Bartolini & Punta, 2012; Huang & Wu, 2011; Huang *et al.,* 2012; Gang *et al.,* 2012). Different methods have been applied in this category, including (Lee *et al.,* 2009) introduced direct adaptive backstepping and a recurrent wavelet neural network method to control a class of non-affine systems. They simulated a double pendulum as a non-affine system and the results expressed the good performance of the algorithm. (Bartolini & Punta, 2012) presented a variable structure method for controlling a non-affine system with uncertainties. The state vector was not completely available, so an observer was combined in the structure of the controller. (Huang & Wu, 2011) used robust decentralized adaptive output feedback fuzzy control for a class of large-scale non-affine systems. The model was two inverted pendulums on carts connected to each other (Huang & Wu, 2011) and a string of vehicles (Huang *et al.,* 2012). Gang *et al.,* 2012) introduced a fuzzy approach for robust control of a class of stochastic non-affine systems. Non-affine problems for these approaches have been examined for mathematical models and experimental systems. Such models are more common in reality, but models with nonlinearities in control (non-affine in control) are less approachable.

In the second category, nonlinear systems were considered as control nonlinearities or as non-affine-in-control systems. This category has been less studied. Such a model is rare in the real world and most research has been based on theoretical concepts and mathematical models; however, some real models have been introduced in aerospace. (Lavretsky & Hovakimyan, 2005) proposed an adaptive dynamic inversion for non-affine systems using time scale separation. Details of the algorithm and stability analysis were explained and simulated for a mathematical model by (Lavretsky & Hovakimyan, 2005) and for a double inverted pendulum by (Young *et al.,* 2006). (Young *et al.,* 2006) also considered control nonlinearities due to actuator limits. (Tsai, 2013) reported an improved fuzzy modelling method for a class of multi-input non-affine nonlinear systems. (Arefi & Motlagh, 2011) presented an observer-based adaptive neural control for a class of nonlinear non-affine systems with unknown gain signs. The first example of this article is selected from (Arefi & Motlagh, 2011). (Tamimi, 2012) and Wang *et al.,* 2012) used optimal control to solve discrete non-affine-in-control systems.

In the present study, the state-dependent Riccati equation method is employed to control non-affine systems in the second category. An SDRE controller was proposed for non-affine-in-control systems by (Wernli & Cook, 1975). A power-series approximation method was also developed for complicated models as an approximate numerical solution to SDRE. (Cimen & Banks, 2004) introduced a general formulation for non-affine systems and applied it to modelling an aircraft with control nonlinearities. The algorithm provided an approximating sequence for Riccati equations to construct non-linear time-varying optimal state-feedback controllers. (Beeler, 2004) introduced an online control update (OCU) formulation for an SDRE with control nonlinearities. Although valuable research has been done on

SDRE in the non-affine area, this controller has not been sufficiently examined for these types of systems, especially with a connection to an observer.

(Mracek *et al.,* 1996) used the SDRE technique for nonlinear estimation and applied it to a simple pendulum problem. (Friedland, 1998) expressed the state-dependent differential Riccati equation method for estimation. (Beikzadeh & Taghirad, 2012; Iratni *et al.,* 2012) and other studies have examined other aspects of the SDRE estimators, such as robustness. (Hassan, 2012) proposed an observer-based controller for discrete time systems using the SDRE approach. A recursive regularized least-square state estimator was also used.

A stability proof of a SDRE controller for non-affine systems was presented by (Wernli & Cook, 1975) for power series approximation as a numerical solution technique. Solving the system and control laws simultaneously was termed a tedious task by (Wernli & Cook, 1975).

In the present study, the main contribution is a formulation for solving this type of system and its control law simultaneously with a stability proof. A combination of SDRE controller and SDRE observer is presented to control non-affine systems with partial feedback available.

*Notations:* Throughout this work, $\Re^n$ denotes the $n$-dimensional Euclidean space and $\Re^{m\times n}$ is a set of $m\times n$ real matrices. When a matrix or vector is $C^k$, that matrix or vector is $k$ times continuously differentiable. $\mathbf{\Omega}$ is a bounded Euclidean space that contains the origin. Bold letters in formulas are dedicated to matrices and vectors.

The structure of the SDRE controller and its stability analysis are developed in Section 2. The formulation of the SDRE observer is expressed in Section 3. The tuning of the observer based controller is presented in Section 4. Section 5 is a discussion of the suboptimal cheap control problem. The simulation results are provided in Section 6 and conclusions in Section 7.

## 2. THE SDRE CONTROLLER

The formulation of the SDRE controller for non-affine systems is presented in this section. Consider the nonlinear time-invariant non-affine in the control system as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x},\mathbf{u}), \tag{1}$$

where equilibrium point of the system is $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $\mathbf{x} \in \Re^n$ is the state vector; and $\mathbf{u} \in \Re^m$ is the input vector of the system.

**Assumption 1.** The vector-valued functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x},\mathbf{u})$ are continuous smooth nonlinear vectors with respect to their arguments in $C^k$, $k \geq 1$ and at least one time differentiable functions exist for all **x** and **u** in region $\mathbf{\Omega}$.

**Assumption 2.** The vector-valued functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x},\mathbf{u})$ are uniformly bounded by $t \to \infty$ and the derivative of the vector $\mathbf{g}(\mathbf{x},\mathbf{u})$ with respect to $\mathbf{u}$ is nonzero $\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}} \neq \mathbf{0}$ for all $\mathbf{x} \in \mathbf{\Omega}$.

An optimal control scheme is applied by minimizing the following performance index:

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt, \tag{2}$$

in which **Q** is weighting matrix for states and symmetric positive semi definite, and **R** is weighting matrix for control inputs and symmetric positive definite.

The next step for implementing the method is state-dependent coefficient (SDC) parameterization to form the equation of the system in Eq. (1) as:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{u} \tag{3}$$

where $\mathbf{A}(\mathbf{x}) : \mathbf{\Omega} \to \Re^{n\times n}$ and $\mathbf{B}(\mathbf{x},\mathbf{u}) : \mathbf{\Omega} \to \Re^{n\times m}$.

**Preposition 1** (Cimen, 2012). For the vector $\mathbf{f}(\mathbf{x})$ and $n > 1$ with equilibrium point $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, there exists a non-unique SDC parameterization in the form of $\mathbf{A}(\mathbf{x})\mathbf{x}$.

(Cloutier *et al.,* 1999) proposed the following definition for non-linear-affine-in-control systems. Definition 1 is the extended form of the definition introduced by (Cloutier *et al.,* 1999).

**Definition 1.** The SDC representation in Eq. (3) is a stabilizable parameterization of nonlinear system in Eq. (1) in the region of $\mathbf{\Omega}$, if the pair of $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x},\mathbf{u})\}$ is pointwise stabilizable in the linear sense for all $\mathbf{u}$ and $\mathbf{x} \in \mathbf{\Omega}$.

By shaping the Hamiltonian as:

$$\mathbf{H} = \frac{1}{2}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) + \boldsymbol{\lambda}^T\left(\mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{g}(\mathbf{x},\mathbf{u})\right), \tag{4}$$

and applying the optimality conditions as:

$$\frac{\partial \mathbf{H}}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}}; -\frac{\partial \mathbf{H}}{\partial \mathbf{x}} = \dot{\boldsymbol{\lambda}}; \frac{\partial \mathbf{H}}{\partial \mathbf{u}} = \mathbf{0}, \tag{5}$$

and by rearrangement and use of mathematical operations, the control law is obtained in the form of:

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{x}, \tag{6}$$

where $\mathbf{k}(\mathbf{x},\mathbf{u})$ is the symmetric positive solution of the state-dependent Riccati equation as:

$$\begin{aligned}&\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{k}(\mathbf{x},\mathbf{u}) - \\ &\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u}) + \mathbf{Q} = \mathbf{0}.\end{aligned} \tag{7}$$

Two steps are necessary for stability analysis of the proposed non-affine structure. The continuity of $\mathbf{k}(\mathbf{x},\mathbf{u})$ must be proven for its arguments and the stability of the control algorithm must be proven. (Wernli & Cook, 1975) proved both of these and provided theorems with extraordinarily beautiful proofs for the power series approximation method.

In the present study, another approach is used to show the continuity of $\mathbf{k}(\mathbf{x},\mathbf{u})$ and the stability of the algorithm.

**Lemma 1** (Mean value theorem). Assume that $f(x,y):\Re^n \times \Re \rightarrow \Re$ has a derivative at each point of an open set $\Re^n \times (a,b)$, and assume that it is continuous at end points $y=a$ and $y=b$. There is a point $\pi \in (a,b)$ such that (Arefi & Motlagh, 2011)

$$f(x,b)-f(x,a)=f'(x,\pi)(b-a). \tag{8}$$

Based on Assumption 1, and the mean value theorem in the Lemma 1, it can be defined:

$$\mathbf{g}(\mathbf{x},\mathbf{u})=\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{u}^*, \tag{9}$$

where $\mathbf{u}$ is a point between zero and $\mathbf{u}^*$. Applying Eq. (9), the control law is rewritten as:

$$\mathbf{u}^*=-\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{x}. \tag{10}$$

In (Arefi & Motlagh, 2011), vector $\mathbf{f}(\mathbf{x})$ and matrix $\mathbf{B}(\mathbf{x},\mathbf{u})$ were unknown and the controller was an adaptive neural network. In the present work, $\mathbf{f}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x},\mathbf{u})$ are known.

**Theorem 1.** The nonlinear time-invariant non-affine system (1) with its performance criteria (2) and conditions based on Assumptions 1 and 2 can be stabilized using control law (10), for which $\mathbf{k}(\mathbf{x},\mathbf{u})$ is the positive definite solution of the SDRE (7).

**Proof.** Consider the Lyapunov function as:

$$\mathbf{V}=\mathbf{x}^T\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{x}, \tag{11}$$

and the time derivative of that as:

$$\dot{\mathbf{V}}=\mathbf{x}^T\begin{pmatrix}\mathbf{A}^T(\mathbf{x})\mathbf{k}(\mathbf{x},\mathbf{u})+\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{A}(\mathbf{x})- \\ \mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u})- \\ \mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u})\end{pmatrix}\mathbf{x}. \tag{12}$$

This must be negative to guarantee stability. Using Eq. (7) and substituting, the following equation:

$$\dot{\mathbf{V}}=-\mathbf{x}^T\left(\mathbf{Q}+\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u})\right)\mathbf{x}, \tag{13}$$

obtains a negative value, since $\mathbf{R}$ is positive definite, $\mathbf{Q}$ is positive semi-definite, $\mathbf{k}(\mathbf{x},\mathbf{u})$ is positive definite, and $\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{B}(\mathbf{x},\mathbf{u})$ is positive whether or not $\mathbf{B}(\mathbf{x},\mathbf{u})$ is positive, which guarantees the stability of the system. ■

There are three methods to obtaining the solution of the SDRE; exact solution, power series approximation, and online control update. The exact solution can provide the answer by adding the integral of control structure (Cloutier & Stansbery, 2004)

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{u}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}(\mathbf{x}) & \mathbf{B}(\mathbf{x},\mathbf{u})\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}\tilde{\mathbf{u}}, \tag{14}$$

or solving the control law and the system equations simultaneously.

The details of power series approximation method are referred to (Wernli & Cook, 1975). The OCU formulation, a solution method for a non-affine SDRE, calculates most equations online during control implementation (Beeler, 2004). The correspondence SDRE and the control law, Eqs. (15) and (16) can be solved repeatedly for updating the optimal gain. The SDRE becomes the algebraic Riccati equation at each step as:

$$\begin{aligned}&\mathbf{k}_{n+1}\mathbf{A}(\mathbf{x}_n)+\mathbf{A}^T(\mathbf{x}_n)\mathbf{k}_{n+1}-\\ &\mathbf{k}_{n+1}\mathbf{B}(\mathbf{x}_n,\mathbf{u}_n)\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x}_n,\mathbf{u}_n)\mathbf{k}_{n+1}+\mathbf{Q}=\mathbf{0},\end{aligned} \tag{15}$$

and the control law in the form of:

$$\mathbf{u}_{n+1}=-\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x}_n,\mathbf{u}_n)\mathbf{k}_{n+1}\mathbf{x}_n. \tag{16}$$

In present study, the control law and the SDRE are solved simultaneously to gain a suboptimal solution. This formulation is capable of controlling the systems in the form of Eq. (1); however, to apply this method, full-state feedback is required. The system in this scheme is shown in Fig. 1.

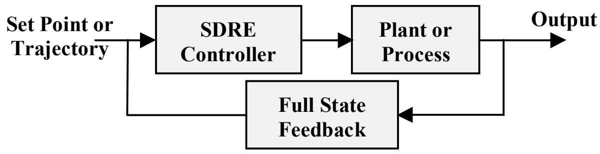


Fig. 1. Control system in full state feedback mode.

If full-state feedback is not available, an observer is required to provide an estimation of all unavailable states.

## 3. SDRE OBSERVER

Consider the following system:

$$\dot{\mathbf{x}}=\mathbf{A}(\mathbf{x})\mathbf{x}+\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{u}^*, \tag{17}$$

$$\mathbf{y}=\mathbf{C}(\mathbf{x})\mathbf{x}, \tag{18}$$

where $\mathbf{y}\in\Re^r$ is the vector of measured states and $\mathbf{C}(\mathbf{x}):\mathbf{\Omega}\rightarrow\Re^{r\times n}$.

**Definition 2** (Mracek & Cloutier, 1998). The pair $\{\mathbf{C}(\mathbf{x}),\mathbf{A}(\mathbf{x})\}$ are an observable (detectable) parameterization of the nonlinear system Eqs. (17) and (18) in the region of $\mathbf{\Omega}$, if that pair is pointwise observable (detectable) in the linear sense for all $\mathbf{x}\in\mathbf{\Omega}$.

The SDRE observer has been presented for affine-in-control systems by (Mracek *et al.,* 1996). Since $\mathbf{B}(\hat{\mathbf{x}},\mathbf{u})$ and $\mathbf{u}^*$ are not directly effective in the design, the formulation can extend to this type of system. The observer framework is defined as:

$$\dot{\hat{\mathbf{x}}}=\mathbf{A}(\hat{\mathbf{x}})\hat{\mathbf{x}}+\mathbf{B}(\hat{\mathbf{x}},\mathbf{u})\mathbf{u}^*+\mathbf{\Gamma}(\hat{\mathbf{x}})\left(\mathbf{y}-\mathbf{C}(\hat{\mathbf{x}})\hat{\mathbf{x}}\right), \tag{19}$$

and the observer gain is:

$$\mathbf{\Gamma}(\hat{\mathbf{x}})=\mathbf{p}(\hat{\mathbf{x}})\mathbf{C}^T(\hat{\mathbf{x}})\mathbf{W}^{-1}, \tag{20}$$

in which $\mathbf{\Gamma}(\hat{\mathbf{x}}):\mathbf{\Omega}\rightarrow\Re^{n\times r}$, $\hat{\mathbf{x}}$ is the estimated state vector, $\mathbf{W}:\mathbf{\Omega}\rightarrow\Re^{r\times r}$ is the weighting matrix for output, and $\mathbf{p}(\hat{\mathbf{x}}):\mathbf{\Omega}\rightarrow\Re^{n\times n}$ is the dual SDRE gain as:

$$\mathbf{A}(\hat{\mathbf{x}})\mathbf{p}(\hat{\mathbf{x}})+\mathbf{p}(\hat{\mathbf{x}})\mathbf{A}^T(\hat{\mathbf{x}})- \\ \mathbf{p}(\hat{\mathbf{x}})\mathbf{C}^T(\hat{\mathbf{x}})\mathbf{W}^{-1}\mathbf{C}(\hat{\mathbf{x}})\mathbf{p}(\hat{\mathbf{x}})+\mathbf{E}=0, \quad (21)$$

and $\mathbf{E}:\mathbf{\Omega}\rightarrow\Re^{n\times n}$ is the weighting matrix for estimated states.

$\mathbf{B}(\hat{\mathbf{x}},\mathbf{u})$ and $\mathbf{u}^*$ do not interfere in the solution of the observer design and the details of the observer, theorems, lemmas and more aspects, such as robustness design, can be found in (Mracek *et al.,* 1996; Friedland, 1998; Beikzadeh & Taghirad, 2012; Iratni *et al.,* 2012). The combined SDRE controller with observer is shown in Fig. 2.

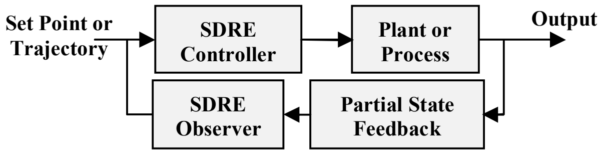


Fig. 2. The combined SDRE controller with observer.

The system presented in Fig. 2 is more general than the system in Fig. 1, where full state-feedback is available.

## 4. TUNING THE OBSERVER BASED CONTROLLER: **R**, **Q**, **W** AND **E** SELECTION

One advantages of the SDRE method is the systematic steps used to complete the design of the controller. The **R** and **Q** matrices are the tuning parameters for obtaining the desired performance. The **Q** matrix is related to the states and the **R** matrix to the inputs. The values for elements of **Q** must be increased to decrease the errors for regulation or tracking. One approach to determining the **R** matrix is to define the elements of that, equal to the inverse of the maximum power of the relevant actuator. Since some models are mathematical, it should be defined by iteration. The combination of the SDRE controller and observer makes selection more complicated. The following steps for tuning the controller and observer are suggested:

- Choose and tune **Q** and **R** assuming availability of full-state feedback (without using the estimator for tuning).
- Increase **Q** until the error of regulation or tracking is satisfied; then change **R** based on the control signal that is generatable.
- Apply the observer in the control loop, increase the **E** matrix, and change **W** until the error of estimation is satisfying.

It should be noted that error in estimation increases the control input and has an undesirable effect on the results. The gain of the observer should be tuned so that the estimation error is within allowable bounds.

## 5. SUBOPTIMAL CHEAP CONTROL PROBLEM

High gain linear systems are those in which the norm of the feedback control matrix has a high magnitude that is usually one or more orders of magnitude greater than that of the norm of the system matrix (Gajic & Lim, 2001). A cheap control problem is characterized by a small penalty factor applied to the control term for the quadratic performance criterion, usually one or more orders of magnitude smaller that the state penalty term (Gajic & Lim, 2001). In this case, the usual hypotheses, even for linear case, are no longer appropriate (Jameson & O'Malley, 1975).

To show the high gain properties of a system, the feedback control matrix is considered as:

$$\mathbf{A}_{cl}=\mathbf{A}(\mathbf{x})-\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u}), \quad (22)$$

and the Frobenius norm is selected to show the magnitude of the system as:

$$\|\mathbf{A}_{cl}\|_F=\sqrt{trace(\mathbf{A}_{cl}^T\mathbf{A}_{cl})}. \quad (23)$$

This system can be interpreted as having a singular **R** matrix or positive semi-definite **R,** because it fails to deliver a proper solution. To avoid this problem in real-life cases, the states of system must be chosen carefully; however, sometimes the plant or process leaves no choice except a system with both fast and slow variables. In this case, the suboptimal cheap control problem can be used as a remedy.

For high gain systems, the quadratic performance criteria is rewritten as (Gajic & Lim, 2001):

$$J=\frac{1}{2}\int_0^\infty\left(\mathbf{x}^T\mathbf{Q}\mathbf{x}+\varepsilon^2\mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt, \quad (24)$$

where $\varepsilon$ is a small positive value. The correspondent control law is:

$$\mathbf{u}^*=-\frac{1}{\varepsilon^2}\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x},\mathbf{u})\mathbf{k}(\mathbf{x},\mathbf{u})\mathbf{x}. \quad (25)$$

Using this approach, all the input signals are enhanced to compensate for the regulation speed of the slow variables of the system. When a part of a system varies slowly, some relevant inputs must be enhanced. In that case, imposing $\varepsilon$ onto relevant rows of $\mathbf{R}$ is suggested to avoid an unnecessary increase in the magnitude of the other input signals.

This topic presented in details for linear systems in (Gajic & Lim, 2001); however, the supporting theorems and aspects of the problem have yet to be investigated for the nonlinear case.

## 6. ILLUSTRATIVE EXAMPLES

### *6.1. Mathematical model with* **B(u)**

The first case study for this part was chosen from (Arefi & Motlagh, 2011) and is as in the following form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1-2x_1x_2 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{1}{\sqrt{|u|+0.1}} \end{bmatrix}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}d \quad (26)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \quad (27)$$

in which $d = 0.5\sin(2t)$ is the external disturbance. The output of the system is $y = x_1$ and an observer-based controller is needed to provide full-state feedback for the controller. The control problem is to track desired trajectory $y_{des} = 0.5\sin(t)$. It is assumed that $n$ derivatives of $y_{des}$ are available (Arefi & Motlagh, 2011). The following parameters were chosen based on proposed method of selecting weighting matrices:

$$R = 0.1, \quad (28)$$

$$\mathbf{Q} = 100 \times \mathbf{I}_{2\times2}, \quad (29)$$

$$W = 1, \quad (30)$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 \\ 0 & 100000 \end{bmatrix}, \quad (31)$$

and the initial conditions are selected as $\mathbf{x(0)} = \begin{bmatrix} -0.2 & 0.4 \end{bmatrix}^T$ and $\hat{\mathbf{x}}\mathbf{(0)} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T$. The simulation in Fig. 3 and 4 show the responses of states. Fig. 5 shows the error of trajectory tracking and Fig. 6 shows the control input. Fig. 7 shows a better presentation of control input at commencement.

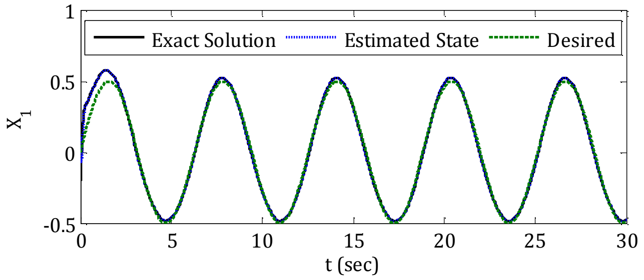


Fig. 3. First state, example 1: Actual state, estimated state and desired trajectory.

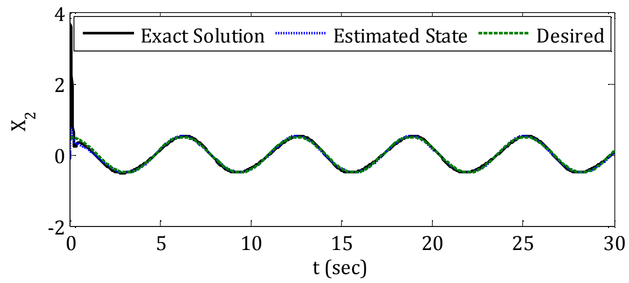


Fig. 4. Second state, example 1: Actual state, estimated state and desired trajectory.

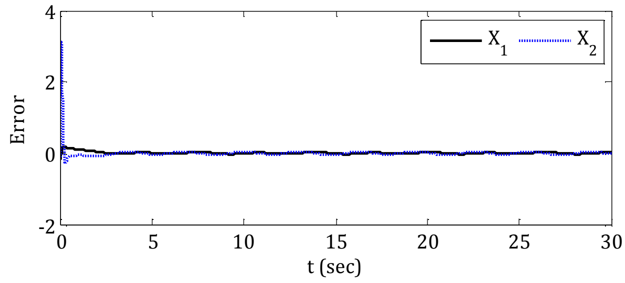


Fig. 5. Error of states in tracking, example 1.

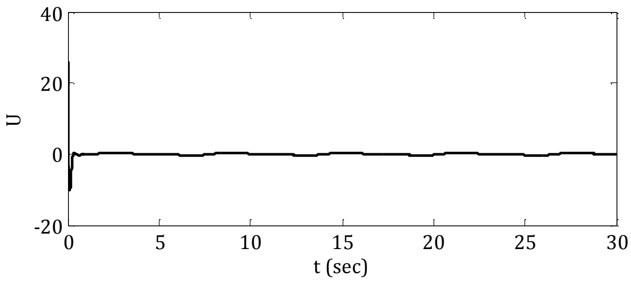


Fig. 6. Control input, example 1.

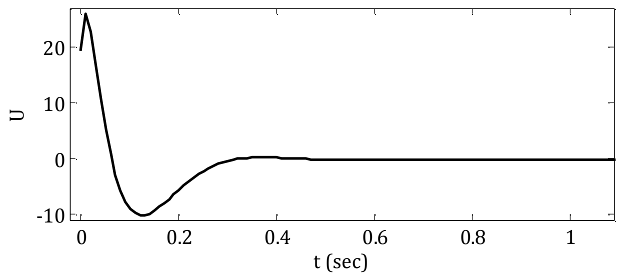


Fig. 7. The starting of control input, example 1.

The results of (Arefi & Motlagh, 2011) using an observer-based adaptive neural network control confirm the results of the proposed method. Both control inputs have chatter-like motion at the beginning of the simulation caused by the observer; however, the magnitude of the control signal of SDRE was lower than the control effort in (Arefi & Motlagh, 2011). (Arefi & Motlagh, 2011) proposed an adaptive neural network algorithm that does not need the priori knowledge of control gain direction. The neural network controller was employed for approximation of the unknown nonlinear functions. The SDRE controlled the system with acceptable results. The error can be decreased, although it may cause a large jump at the beginning of the control signal. Generally, perfect tracking (initial condition and beginning of trajectory are the same) is suggested for this controller. The term $R^{-1}$ in the control law Eq. (10) multiplied in the error, causing an initial error that could cause big jump or chatter-like motion in the control signal.

*6.2. Mathematical model with* **B(x,u)**

The case study here was chosen form (Essounbouli & Hamzaoui, 2006) and uses $\mathbf{B}(\mathbf{x},\mathbf{u})$ as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ x_1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0.1u^3 + 0.1(1+x_2^2)u + \sin(0.1u) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d \qquad (32)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \qquad (33)$$

where $d = 0.2(\sin(t) + \sin(2t))$ is the external disturbance and the desired trajectory is $y_{des} = \sin(t)$. The same condition availability of $n$ derivatives of $y_{des}$ is considered.

There is a difference between this model and the previous one. Vector $\mathbf{g}(\mathbf{x},\mathbf{u})$ in this example has term $\sin(0.1u)$ that has no additional $u$ to be factored for forming $\mathbf{B}(\mathbf{x},u)u$. There are two approaches for dealing with this kind of situation: the first one is to divide and multiply the term by the $u$ that forms $\frac{\sin(0.1u)}{u}u$. This operation may lead to singularity problems. The second approach is to use the Taylor series expansion that forms $\sin(0.1u) = \left(\frac{1}{10} + \frac{u^2}{6000} + \ldots\right)u$. In this example, the second approach is applied. (Essounbouli & Hamzaoui, 2006) solved this based on the availability of all-state feedback; however, in the present study, simulation is done using the knowledge of only $y = x_1$ to show use of an observer. The weighting matrices are tuned and selected as:

$$R = 1, \qquad (34)$$

$$\mathbf{Q} = 100 \times \mathbf{I}_{2\times2}, \qquad (35)$$

$$W = 0.1, \qquad (36)$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 \\ 0 & 100 \end{bmatrix}, \qquad (37)$$

The initial conditions are $\mathbf{x}(\mathbf{0}) = [0.5 \quad 0.5]^T$ and for estimation are $\hat{\mathbf{x}}(\mathbf{0}) = [0.4 \quad 0.6]^T$. The following results were attained in 20 s. Figs 8 and 9 show the states and Fig. 10 shows the error from the desired trajectory. The control signal is shown in Fig. 11.

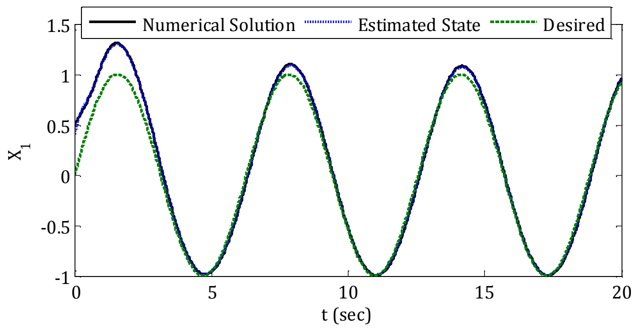


Fig. 8. First state, example 2: Actual state, estimated state and desired trajectory.

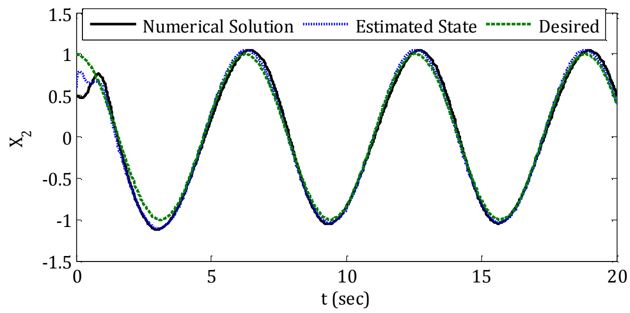


Fig. 9. Second state, example 2: Actual state, estimated state and desired trajectory.

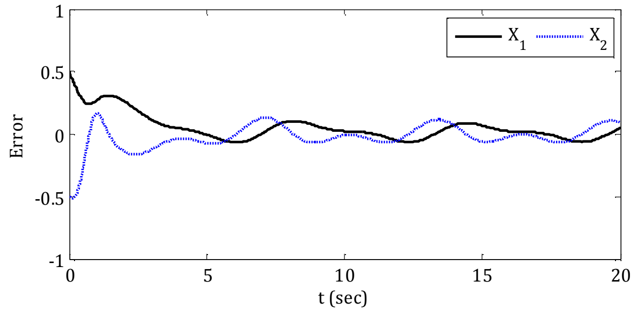


Fig. 10. Error of states in tracking, example 2.

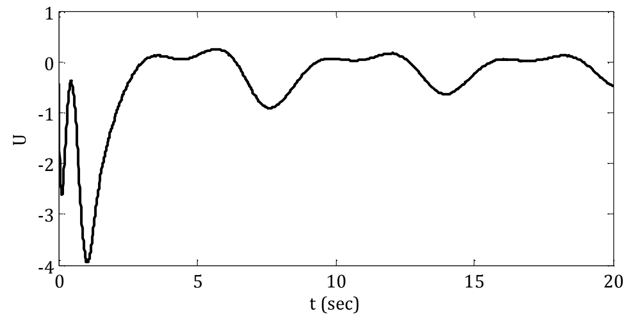


Fig. 11. Control input, example 2.

The error of states can be decreased by increasing the **Q** matrix. The jump at the start of control input is cause by initial error, which can be greatly decreased with perfect tracking. The value of external disturbance in this example is also notable. The magnitude of the control signal decreased after 3 s in comparison with that for (Essounbouli & Hamzaoui, 2006).

*6.3. Mathematical high order example: A 5D model*

Consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{u}, \qquad (38)$$

$$\mathbf{y} = \mathbf{C}(\mathbf{x})\mathbf{x}, \qquad (39)$$

where

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_4^2 & 0 \\ -x_1 & 0 & 0 & x_4^2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (40)$$

$$\mathbf{B}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 + x_3 + u_1 \sin(u_2) & 0 \\ 0 & 0 \\ 0 & 1 + x_2 + u_2 e^{-u_1} \end{bmatrix}, \quad (41)$$

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (42)$$

This case study was chosen from (Banks, Lewis & Tran, 2003) and was modified to a non-affine system. The initial conditions for states are $\mathbf{x}(\mathbf{0}) = [0.4 \quad -0.2 \quad 0.1 \quad -0.1 \quad 0.5]^T$ and for estimation are $\hat{\mathbf{x}}(\mathbf{0}) = [0.3 \quad -0.1 \quad 0 \quad 0 \quad 0.4]^T$. The parameters of the controller are tuned to:

$$\mathbf{R} = 0.1 \times \mathbf{I}_{2\times2}, \quad (43)$$

$$\mathbf{Q} = 10 \times \mathbf{I}_{5\times5}, \quad (44)$$

$$\mathbf{W} = 0.1 \times \mathbf{I}_{2\times2}, \quad (45)$$

$$\mathbf{E} = 100 \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{bmatrix}, \quad (46)$$

and the results were obtained in 10 s of simulation. Figs. 12-16 show the variations of states. The control signals are also shown in Fig. 17.

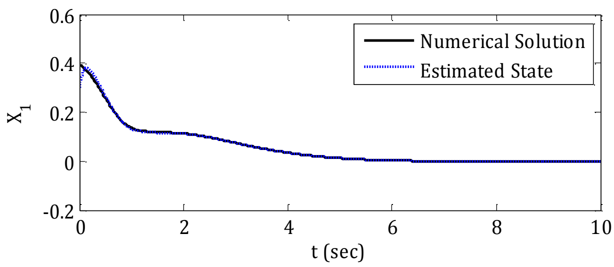


Fig. 12. The variation of first state.

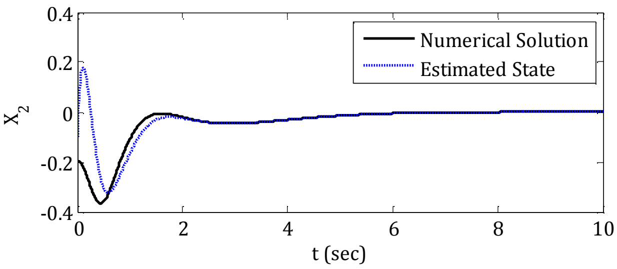


Fig. 13. The variation of second state.

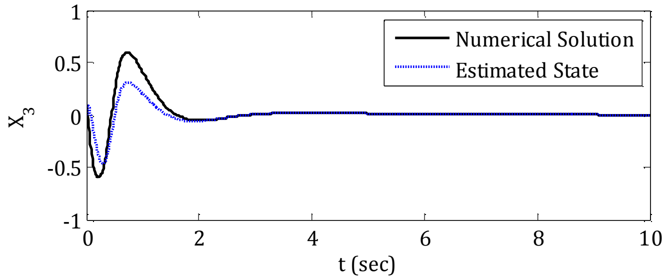


Fig. 14. The variation of third state.

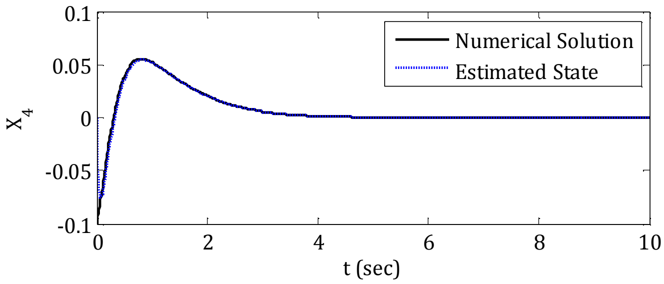


Fig. 15. The variation of fourth state.

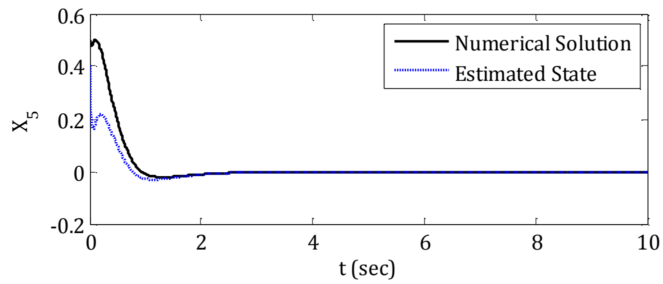


Fig. 16. The variation of fifth state.

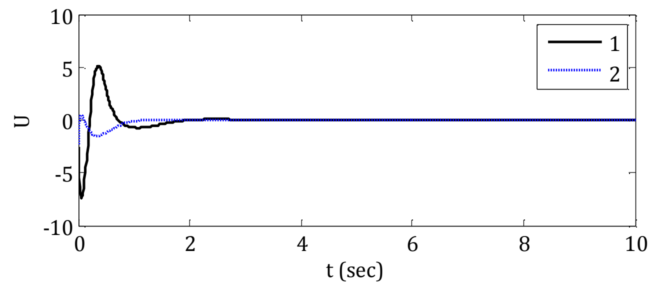


Fig. 17. Control input signals.

The **Q** and **R** matrices must be changed to improve performance, which affects the magnitude of the control signals.

### *6.4. Suboptimal Cheap Control Example*

The 5D example in Section 6.3 is changed to a high gain system for this example. All system and weighting matrices remain the same, except $\mathbf{B}(\mathbf{x},\mathbf{u})$:

$$\mathbf{B}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 + x_3 + u_1 \sin(u_2) & 0 \\ 0 & 0 \\ 0 & \varepsilon(1 + x_2 + u_2 e^{-u_1}) \end{bmatrix}, \quad (47)$$

where $\varepsilon = 0.01$. The high gain properties of the system from Eqs. (22) and (23) are shown in Fig. 18.

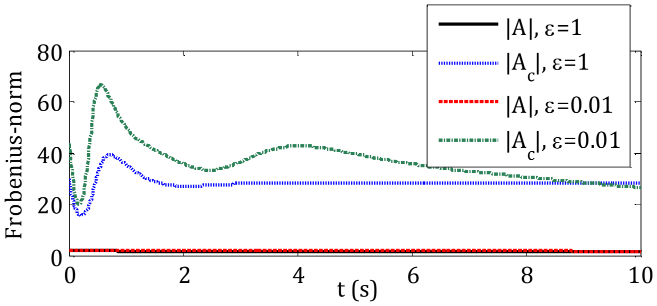


Fig. 18. High gain properties of the system.

The cheap control algorithm results an enormous input signal in comparison with the signal for the unperturbed system. To observe the system, the second input signal only requires the cheap control scheme. Instead of using Eqs. (24) and (25), the proposed approach is used by consideration **R** as:

$$\mathbf{R} = 0.1 \times \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{bmatrix}. \quad (48)$$

The norm of states errors gained is the same at the cost of increasing the second input signal Figs. 19 and 20.

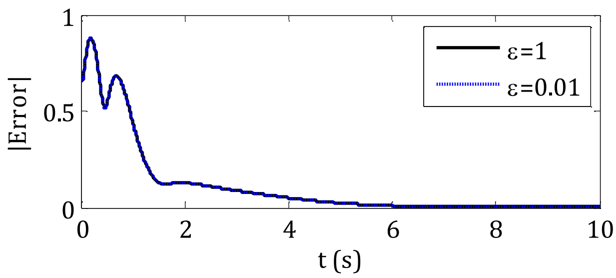


Fig. 19. Norm of the states error.

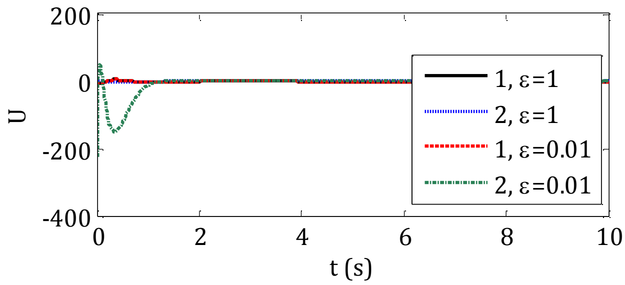


Fig. 20. Control inputs for high gain example.

Generally, this approach generates a large signal at the beginning of the control signals. To remedy this, a fixed constraint is proposed to limit the signal $\mathbf{u}_{\max,\min} = \pm 50$; this definitely increases the norm of the error Figs. 21 and 22.

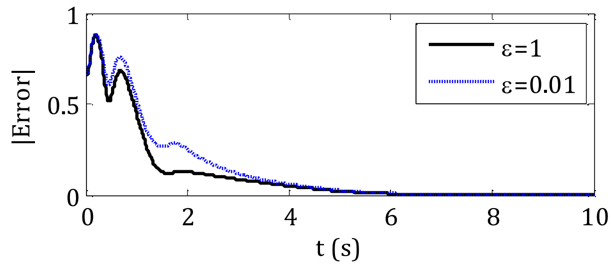


Fig. 21. Norm of the states error.

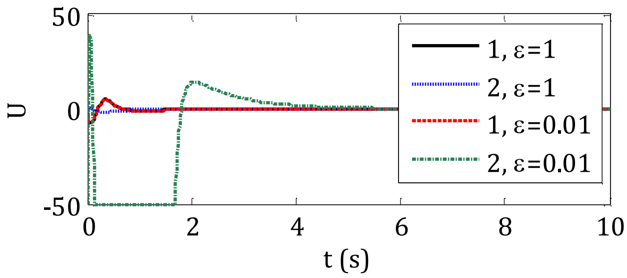


Fig. 22. Control inputs for high gain example.

## 7. CONCLUSIONS

The present study formulated the state-dependent Riccati equation as a nonlinear closed loop optimal controller for a class of non-affine systems based on a SDRE observer. The SDRE observer provided the unavailable states for full-state feedback. A brief discussion of sub-optimal cheap control problem was provided. Four illustrative examples, each focusing on a specific point, were solved to show the adaptability and capability of the SDRE controller. The first case study was compared with the results of previous research on an observer-based adaptive neural controller and the proposed SDRE method was shown to be well-confirmed with less control effort. The second example used a robust adaptive fuzzy controller and similar results were obtained. The overall results demonstrate that the good performance of the controller and the simple and systematic steps for completing the design procedure are the advantages of this technique.


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