SPECIFIC H_∞ SYNTHESIS FOR TIME DELAY SYSTEMS

Alexandru Serbanescu[1], Corneliu Popeea[2]

[1]*ARS T&TT, Oranje Straat 7, 2514 JB Den Haag, The Netherlands*

[2]*University Politehnica Bucharest, Automatic Control and Computers Faculty, Splaiul Independentei 313, Bucharest, Romania*

Abstract: The problems arising in Time Delay Systems (TDS) have increased complexity with respect to those related to Linear Delay Free Systems. Linear Matrix Inequalities (LMI) proved to be a useful instrument for solving several control problems (see [1]). The present paper is approaching the dynamic output feedback H_∞ control problem for a class of TDS, where both the state dynamics and the output depends on the delayed states. Two new procedures are proposed to solve the optimal and the sub-optimal H_∞ control problems.

Keywords: Time Delay Systems, H_∞ synthesis, Linear Matrix Inequalities (LMI), Semidefinite Programming, Projection Lemma.

1. INTRODUCTION

Starting with the beginning of the 1980's, the H_∞ control problem has been extensively studied. For the delay free linear systems several results approaching the problem of output H_∞ feedback synthesis have been published in the 1990's. For a general theoretic framework for this problem and other related issues see [1]. Gahinet [2] and Iwasaki [3] extended the general control problem using the Bounded Real Lemma (BRL) and linear matrix inequalities (LMI). Necessary and sufficient conditions for the existence of an H_∞ controller were given in terms of three LMI's.

The delay is frequently a source of instability and encountered in various engineering systems such as chemical processes, hydraulics, power plants or combustion engines. As a result the stability of TDS received much attention in the last years. Since these systems include perturbations, it is important to study the H_∞ problem for this class of systems. There have been published several results in the last years regarding the control for TDS, see [4], [5], [6], [7], [8] and [9]. For an analytical approach, see [10]. Most of the approaches that have numerical relevance (can be applied in general situations), are Lyapunov based methods. As a general pattern a Lyapunov functional* is defined for the system and the final results are expressed in terms of LMI's. The resulting LMI's are dependent on the way the Lyapunov functional is chosen.

The purpose of this paper is to offer necessary and sufficient conditions for the existence of an H_∞ output feedback controller, for a class of TDS described in (1.5) and to propose a procedure that will converge towards an optimum controller, or in other words to approach as much as possible the minimum value for the infinite norm.

First a sufficiency condition for the asymptotic stability of the closed loop system is proposed. Then the existence conditions for an H_∞ output controller are expressed in terms of three LMI's. A description of how to construct such a controller starting from the solution of this set of LMI's is presented.

A previous result proposed by Jeung in [11], approach the problem of H_∞ control by extrapolating the classical pattern for delay free systems to TDS. Due to the fact that TDS are more complex, by simply applying the classical methodology for the H_∞ synthesis does not provide the best results. The approach presented in this paper starts in a similar way, but for a class of TDS with multiple delays described by (1.5) and using a different Lyapunov functional proposed in a more recent paper by Park in [4]. In the final stages of the synthesis process however, it becomes TDS oriented trying to achieve the best possible results for this class of systems.

Due to the increased complexity of TDS the classical methodology proposed in Jeung [11] for the sub-optimal problem might simply not work in many situations. The major reason for that is the fact that the presence of the delay makes the LMI's more complex, and as a result we cannot optimize in the same time with respect to all variables. As a consequence, an iterative process is necessary and this brings up a new manner to solve the H_∞ synthesis, which consist in finding first a feasible set of weight matrices for the Lyapunov functional, by solving in two steps the equivalent problem and then computing a controller K by solving the initial problem. Afterwards the initial problem is iteratively solved with respect of the weight matrices and with respect of the controller parameters in an alternate manner till no relevant progress is achieved or the attenuation of the perturbations becomes small enough. An analysis of several aspects is presented in the last chapter of this paper. In the end, two convergent procedures are proposed, one for approaching the minimum H_∞ norm as much as possible and the other one for solving the sub-optimal problem. This results are delay independent.

Let us consider the general control configuration in Fig. 1, where the system is

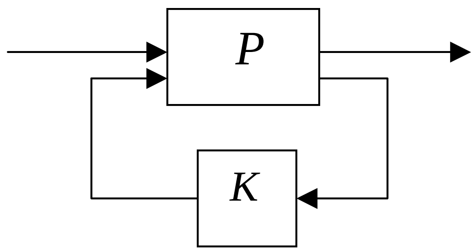

Figure 1

described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} \tag{1.1}$$

and the controller is

$$u = K(s)y.$$

The closed loop transfer function from w to z will be given by a linear fractional transformation

$$H_{zw} = F_l(P, K),$$

where

$$F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \tag{1.2}$$

* In the literature there are two main approaches one using Lyapunov-Krasovskii functionals and the other using Lyapunov-Razumikhin functions. The last one is considered to be more conservative since it is using the Khargonekar Lemma, and it is recommended only when the previous one fails. For more details see Dugard [8].

Considering the system in Fig. 1, the optimal H_∞ control problem is to find all the dynamic controllers *K* which minimize

$$\|F_l(P,K)\|_\infty = \max_\omega \bar{\sigma}(F_l(P,K)(jw)) \qquad (1.3)$$

or, expressed in a in time domain interpretation

$$\|F_l(P,K)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} \qquad (1.4)$$

The purpose of the H_∞ control is to minimize the effect of some perturbations, seen as an exogenous input, over a certain set of quality output in the worse possible scenario. The H_∞ synthesis methodology actually guarantees an upper bound on the propagation of the perturbation in the worst case. The objective is to minimize this upper bound (the optimal H_∞ control) or to bring it under a certain limit (the sub-optimal case).

The problem formulation itself is independent with respect to the presence of the delay.

In this paper we will consider a system with concentrated state delays described by the (1.5)

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \\ \qquad + \sum_{i=1}^{L} A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t) \\ z(t) = C_z x(t) + \\ \qquad + \sum_{i=1}^{L} C_{z\_i} x(t-\tau_i) + D_{11} w(t) + D_{12} u(t) \\ y(t) = C_y x(t) + \sum_{i=1}^{L} C_{y\_i} x(t-\tau_i) + D_{21} w(t) \\ x(t) = \phi(t), \quad t \in [-\max(\tau_1, \ldots, \tau_L, 0)] \end{cases} \qquad (1.5)$$

where $x \in \mathbf{R}^n$ is the state vector, $w \in \mathbf{R}^{m1}$ is the exogenous input, $u \in \mathbf{R}^{m2}$ is the control input,

$z \in \mathbf{R}^{p1}$ is the controlled output and $y \in \mathbf{R}^{p2}$ is the measurement output. The controller will be defined by:

$$\begin{cases} \dfrac{d}{dt}x_k(t) = A_k x_k + B_k y \\ u = C_k x_k + D_k y \end{cases} \qquad (1.6)$$

As we can see, the controlled and the measured outputs depend on delayed states. In many real situations it is expected that some of the matrices in the triplets A_i, C_{z_i}, C_{y_i} might be 0 but that will not affect our results.

2. PROBLEM FORMULATION

Given the system described by (1.5) and the controller (1.6) where $x_k \in \mathbf{R}^k$, the closed loop system will be described by the system of equations (2.1)

$$\begin{cases} \dfrac{d\xi}{dt} = A_{cl}\xi(t) + A_{cl\_1}\xi(t-\tau_1) + \ldots \\ \qquad \ldots + A_{cl\_L}\xi(t-\tau_L) + B_{cl}w(t) \\ z(t) = C_{cl}\xi(t) + C_{cl\_1}\xi(t-\tau_1) + \ldots \\ \qquad \ldots + C_{cl\_L}\xi(t-\tau_L) + D_{cl}w(t) \end{cases}$$

(2.1)

where $\xi(t)$ is the joint state $\xi(t) = \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix}$ and

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A + B_2 D_k C_y & B_2 C_k \\ B_k C_y & A_k \end{bmatrix}, \\ A_{cl\_i} &= \begin{bmatrix} A_i + B_2 D_k C_{y\_i} & 0 \\ B_k C_{y\_i} & 0 \end{bmatrix}, i = 1:L, \\ B_{cl} &= \begin{bmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{bmatrix} \\ C_{cl} &= \begin{bmatrix} C_z + D_{12} D_k C_y & D_{12} C_k \end{bmatrix}, \\ C_{cl\_i} &= \begin{bmatrix} C_{z\_i} + D_{12} D_k C_{y\_i} & 0 \end{bmatrix}, i = 1:L. \\ D_{cl} &= \begin{bmatrix} D_{11} + D_{12} D_k D_{21} \end{bmatrix}. \end{aligned} \qquad (2.2)$$

In order to derive the existence conditions for the controller we have to separate the unknown matrices. We can gather all the controller parameters in one matrix

$$K := \begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix} \qquad (2.3)$$

and define

$$A_{00} = \begin{bmatrix} A & 0 \\ 0 & 0_k \end{bmatrix}, \quad A_{0\_i} = \begin{bmatrix} A_i \\ 0_{k\times n} \end{bmatrix}, \quad i = 1:L$$

$$B_{00} = \begin{bmatrix} B_2 & 0 \\ 0 & I_k \end{bmatrix}, \quad B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$C_{z0} = \begin{bmatrix} C_z & 0 \end{bmatrix} C_{00} = \begin{bmatrix} C_y & 0 \\ 0 & I_k \end{bmatrix},$$

$$C_{0\_i} = \begin{bmatrix} C_{y\_i} \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} I_n & 0 \end{bmatrix},$$

$$D_{10} = \begin{bmatrix} D_{12} & 0 \end{bmatrix}, \quad D_{20} = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix}.$$

(2.4)

We can express now the state matrices (2.2) in terms of (2.3) and (2.4)

$$\begin{aligned} A_{cl} &= A_{00} + B_{00}KC_{00}, \\ A_{cl\_i} &= (A_{0\_i} + B_{00}KC_{0\_i})E, \\ B_{cl} &= B_{10} + B_{00}KD_{20}, \\ C_{cl} &= C_{z0} + D_{10}KC_{00,} \\ C_{cl\_i} &= (C_{z\_i} + D_{10}KC_{0\_i})E, \\ D_{cl} &= D_{11} + D_{10}KD_{20}. \end{aligned}$$

(2.5)

All the matrices defined in (2.4) contain only plant data. From (2.5) we can see that the state matrices of the closed loop system have an afine dependency on the controller data *K*. In order to design a stabilizing controller *K* which minimizes the H_∞ norm of the closed loop system (2.1) a few additional results are necessary.

Lemma 1. (Schur complement). For any symmetric matrix $L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{bmatrix}$, the following are equivalent

1) $L < 0$
2) $L_{11} < 0, L_{22} - L_{12}^T L_{11}^{-1} L_{12} < 0$
3) $L_{22} < 0, \quad L_{11} - L_{12} L_{22}^{-1} L_{12}^T < 0.$

Lemma 2. (Projection Lemma). Considering the matrices $F \in \mathbf{R}^{\mathbf{n \times m}}$, $G \in \mathbf{R}^{\mathbf{k \times n}}$, $Q \in \mathbf{R}^{\mathbf{n \times n}}$, $Q = Q^T$ and $K \in \mathbf{R}^{\mathbf{m \times k}}$, and assuming that $\operatorname{rank}(F) < n$ and $\operatorname{rank}(G) < n$, the LMI $FKG + G^T K^T F^T + Q < 0$ has a solution *K* if and only if $F_\perp^T Q F_\perp < 0$ and $\left(G^T{}_\perp\right)^T Q \left(G^T{}_\perp\right) < 0$, where $F_\perp$ and $G^T{}_\perp$ are any basis of the orthogonal subspaces of F and G^T. For proof, see [3].

3. STABILITY AND H_∞ NORM BOUND

Given the closed loop system described by (2.1), let us consider the following Lyapunov functional

$$V(\xi,t) = \xi(t)^T P \xi(t) + \\ + \sum_{i=1}^{L} \int_{-\tau_i}^{0} \xi(t+s)^T E^T P_i E \xi(t+s) ds, \qquad (3.1)$$

where $P = P^T > 0, P_1 = P_1^T > 0, \ldots, P_L = P_L^T > 0$ and *E* is defined in (2.4).

If $\dfrac{dV(\xi,t)}{dt} < 0$ is satisfied for every $\xi \in R^n$, then the system described by (2.1) is stable, i.e. $\xi(t) \to 0$ as $t \to \infty$.

If we consider

$$\tilde{\xi}(t) = \begin{bmatrix} \xi(t) \\ E\xi(t-\tau_i) \\ \vdots \\ E\xi(t-\tau_L) \end{bmatrix} \qquad (3.2)$$

from (3.1) and (2.1) it results $\dfrac{dV(\xi,t)}{dt} = \tilde{\xi}(t)^T W \tilde{\xi}(t)$, where W is defined in (3.3).

$$W = \begin{bmatrix} A_{cl}^T P + PA_{cl} + \sum_{i=1}^{L} E^T P_i E & PA_{cl\_1} & \cdots & PA_{cl\_L} \\ A_{cl\_1}^T P & -P_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{cl\_L}^T P & 0 & \cdots & -P_L \end{bmatrix}.$$

(3.3)

Theorem 3.1. If there is a set of symmetric positive definite matrices $P \in R^{(n+k)\times(n+k)}$ and $P_1, \ldots, P_L \in \mathbf{R}^{n \times n}$ such that the LMI $W < 0$ is satisfied the system described by (2.1) is asymptotically stable.

The H_∞ norm of the system (2.1) is defined as $\|H_{2w}\|_\infty = \max\limits_{w(t)\neq 0} \dfrac{\|z(t)\|_2}{\|w(t)\|_2}$, where the L_2 norm is defined as $\|w(t)\|_2^2 = \int_0^\infty w(t)^T w(t)\, dt$.

Considering zero initial conditions, i.e. $x(t) = 0, \forall t \le 0$, and the functional defined in (3.1), if we presume that for all $t \ge 0$ the following relation holds

$$\frac{d}{dt}V(\xi) + z^T z - \gamma^2 w^T w < 0, \tag{3.4}$$

then $\|H_{wz}\|_\infty < \gamma$.

For proof, if we integrate from 0 to T we get

$$\begin{aligned} &V(\xi(T)) - V(\xi(0)) + \\ &+ \int_0^T (z^T z - \gamma^2 w^T w) < 0 \end{aligned} \tag{3.5}$$

and due to the fact that $\xi(0) = 0$ and $V(\xi(T)) \ge 0$ it results $\dfrac{\|z\|_2}{\|w\|_2} < \gamma$.

If $T \to \infty$ then $V \to 0$, since the system is stable.

If we define

$$\tilde{C} = \begin{bmatrix} C_{cl} & C_{cl\_1} & \cdots & C_{cl\_L} & D_{cl} \end{bmatrix} \tag{3.6}$$

then the inequality (3.4) is equivalent with the LMI (3.7).

$$W_{\gamma^2} = \\ = \begin{bmatrix} A_{cl}^T P + PA_{cl} + \sum_{i=1}^{L} E^T P_i E & PA_{cl\_1} & \cdots & PA_{cl\_L} & PB_{cl} \\ A_{cl\_1}^T P & -P_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ A_{cl\_L}^T P & \vdots & \ddots & -P_L & 0 \\ B_{cl}^T P & 0 & \cdots & 0 & -\gamma^2 I_{m1} \end{bmatrix} + \\ + \tilde{C}^T \tilde{C} < 0. \tag{3.7}$$

Since

$$W_{\gamma^2} < 0 \Rightarrow$$

$$\Rightarrow W + \gamma^2 \begin{bmatrix} PB_{cl} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} B_{cl}^T P & 0 & \cdots & 0 \end{bmatrix} < 0 \Rightarrow$$

$$\Rightarrow W < 0. \tag{3.8}$$

from $W_{\gamma^2} < 0$ it results $W < 0$ and as a result (3.7) implies also the stability of the closed loop system.

The LMI (3.7) is depending on γ^2. If $W_{\gamma^2} \le 0$ also $\dfrac{1}{\gamma} W_{\gamma^2} \le 0$ and if we denote by $R = \gamma^{-1}P, R_1 = \gamma^{-1}P_1, \ldots, R_L = \gamma^{-1}P_L$, then after applying the Schur complement (3.7) becomes equivalent with (3.9).

$$W_\gamma = \\ \begin{bmatrix} A_{cl}^T R + RA_{cl} + \sum_{i=1}^{L} E^T R_i E & RA_{cl\_1} & \cdots & RA_{cl\_L} & RB_{cl} & C_{cl}^T \\ A_{cl\_1}^T R & -R_1 & & & & C_{cl\_1}^T \\ \vdots & & \ddots & & & \vdots \\ A_{cl\_L}^T R & & & -R_L & & C_{cl\_L}^T \\ B_{cl}^T R & & & & -\gamma I_{m1} & D_{cl}^T \\ C_{cl} & C_{cl\_1} & \cdots & C_{cl\_L} & D_{cl} & -\gamma I_{p1} \end{bmatrix} < 0 \tag{3.9}$$

Theorem 3.2. If there is a set of symmetric positive definite matrices $R \in R^{(n+k)\times(n+k)}$ and $R_1, \ldots, R_L \in \mathbf{R}^{n \times n}$ and a controller K such that $W_\gamma < 0$, the system described by (2.1) is asymptotically stable and has an L_2 gain smaller then γ.

If the delays are not present, the *Theorem* 3.2 becomes equivalent with the Bounded Real Lemma. Further on we will use in (3.9) P instead of R since it makes no difference as long as they are both variable.

4. THE H_∞ CONTROLLER

The inequality (3.9) is not linear in terms of both the weight matrix P and the controller parameters. We can see that easily since

$PA_{cl} = P(A_{00} + B_{00}KC_{00})$. As a result the direct approach by trying to solve (3.9) is not possible. We will express the existence conditions for an H_∞ controller in terms of three LMI's. Let us consider the inequality (3.9) and the notations (2.4) and (2.5). After applying the Schur complement the LMI (3.9) can be rewritten as

$$\Sigma + \Lambda\Pi K\Theta^T + \Theta K^T\Pi^T\Lambda < 0, \tag{4.1}$$

where Π and Θ are defined in (4.2), Λ is defined in (4.3) and Σ is defined in (4.4).

$$\Pi = \begin{bmatrix} B_2 & 0_{n\times k} \\ 0_{k\times m2} & I_k \\ \multicolumn{2}{c}{0_{n\times(m2+k)}} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{0_{n\times(m2+k)}} \\ \multicolumn{2}{c}{0_{m1\times(m2+k)}} \\ D_{12} & 0_{p1\times k} \\ \multicolumn{2}{c}{0_{n\times(m2+k)}} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{0_{n\times(m2+k)}} \end{bmatrix}, \qquad \Theta = \begin{bmatrix} C_y^T & 0_{n\times k} \\ 0_{k\times p2} & I_k \\ C_{y1}^T & 0_{n\times k} \\ \vdots & \\ C_{yL}^T & 0_{n\times k} \\ D_{21}^T & 0_{m1\times k} \\ \multicolumn{2}{c}{0_{p1\times(p2+k)}} \\ \multicolumn{2}{c}{\left.\begin{matrix} 0_{n\times(p2+k)} \\ \vdots \\ 0_{n\times(p2+k)} \end{matrix}\right\} L} \end{bmatrix} \tag{4.2}$$

$$\Lambda = \mathrm{diag}\left\{ P \quad \overbrace{I_n \cdots I_n}^{L} \quad I_{m1} \quad I_{p1} \quad \overbrace{I_n \cdots I_n}^{L} \right\} \tag{4.3}$$

By applying the *Lemma* 2 (Projection Lemma) we can completely eliminate the controller matrices. The inequality (4.1) will become equivalent with the following set of LMI's

$$\begin{aligned} &\Pi_\perp^T \Lambda^{-1}\Sigma\Lambda^{-1}\Pi_\perp < 0 \\ &\Theta_\perp^T \Sigma\Theta_\perp < 0 \end{aligned} \tag{4.5}$$

where $\Pi_\perp$ and $\Theta_\perp$ are any basis of the orthogonal subspaces of the matrices Π and Θ.

To simplify the conditions (4.5) we will denote

$$P = \begin{bmatrix} Y & N \\ N^T & \tilde{Y} \end{bmatrix}, P^{-1} = \begin{bmatrix} X & M \\ M^T & \tilde{X} \end{bmatrix} \tag{4.6}$$

where $X, Y \in R^{n\times n}$, $M, N \in R^{n\times k}$, and $\tilde{X}, \tilde{Y} \in R^{k\times k}$. We will consider also the following notation

$$\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}_\perp = \begin{bmatrix} O_{B2} \\ O_{D12} \end{bmatrix} \tag{4.7}$$

and

$$\begin{bmatrix} C_y^T \\ C_{y\_1}^T \\ \vdots \\ C_{y\_L}^T \\ D_{21}^T \end{bmatrix}_\perp = \begin{bmatrix} O_C \\ O_{C1} \\ \vdots \\ O_{CL} \\ O_{D21} \end{bmatrix} \tag{4.8}$$

where $O_{B2} \in R^{n\times(n+p1-m2)}, O_{D12} \in R^{p_1\times(n+p1-m2)}$, $O_C, O_{C1}, ..., O_{CL} \in R^{n\times((L+1)n+m1-p2)}$ and $O_{D21} \in R^{m2\times((L+1)n+m1-p2)}$.

We can construct now $\Pi_\perp$ (see (4.9)) and $\Theta_\perp$ (see (4.10)), and simplify (4.1) to obtain the (4.11)-(4.13) equivalent set of LMI's.

$$\Sigma = \begin{bmatrix} A_{00}^T P + PA_{00} & PA_{0\_1} & \cdots & PA_{0\_L} & PB_{10} & C_{z\_0}^T & E^T & \cdots & E^T \\ A_{0\_1}^T P & -P_1 & & & & C_{z\_1}^T & & & \\ \vdots & & \ddots & & & \vdots & & & \\ A_{0\_L}^T P & & & -P_L & & C_{z\_L}^T & & & \\ B_{10}^T P & & & & -\gamma I_{m1} & D_{11}^T & & & \\ C_{z\_0} & C_{z\_1} & \cdots & C_{z\_L} & D_{11} & -\gamma I_{p1} & & & \\ E & & & & & & -P_1^{-1} & & \\ \vdots & & & & & & & \ddots & \\ E & & & & & & & & -P_L^{-1} \end{bmatrix} \tag{4.4}$$

$$\Pi_{\perp} = \\ = \begin{bmatrix} O_{B2} & & & & & & \\ 0_{k\times(n+p1-m2)} & & & & & & \\ 0_{n\times(n+p1-m2)} & I_n & & & & & \\ \vdots & & \ddots & & & & \\ 0_{k\times(n+p1-m2)} & & & I_n & & & \\ 0_{m1\times(n+p1-m2)} & & & & I_{m1} & & \\ O_{D12} & & & & & & \\ 0_{n\times(n+p1-m2)} & & & & & I_n & \\ \vdots & & & & & & \ddots \\ 0_{k\times(n+p1-m2)} & & & & & & & I_n \end{bmatrix} \quad (4.9)$$

$$\Theta_{\perp} = \begin{bmatrix} O_C & & & \\ 0_{k\times((L+1)n+m1-p2} & & & \\ O_{C1} & & & \\ \vdots & & & \\ O_{CL} & & & \\ O_{D21} & & & \\ 0_{p1\times((L+1)n+m1-p2} & I_{p1} & & \\ 0_{n\times((L+1)n+m1-p2)} & & I_n & \\ \vdots & & & \ddots \\ 0_{n\times((L+1)n+m1-p2} & & & & I_n \end{bmatrix} \quad (4.10)$$

$$\begin{bmatrix} S_X & \begin{matrix} O_{B1}^T B_1 + \\ + O_{D12}^T D_{11} \end{matrix} & O_{B1}^T X \cdots O_{B1}^T X \\ \begin{matrix} B_1^T O_{B1} + \\ + D_{11}{}^T O_{D12} \end{matrix} & -\gamma\ I_{m1} & \\ XO_{B1} & & -P_1^{-1} \\ \vdots & & & \ddots \\ XO_{B1} & & & & -P_L^{-1} \end{bmatrix} < 0 \quad (4.11)$$

$$\begin{bmatrix} S_Y & S_{Y\_C} \\ S_{Y\_C}^T & -\gamma\ I_{p1} \end{bmatrix} < 0 \quad (4.12)$$

$$S_X = O_{B1}^T X A^T O_{B1} + O_{B1}^T A X O_{B1} + \\ + O_{D12}^T C_z X O_{B1} + O_{B1}^T X C_z^T O_{D12} + \\ + \sum_{i=1}^{L} (O_{B1}^T A_i + O_{D12}^T C_{z\_i}) P_i^{-1} (O_{B1}^T A_i + O_{D12}^T C_{z\_i})^T - \\ - Y O_{D12}^T O_{D12}$$

$$S_Y = \\ = (O_C^T A^T Y O_C + \sum_{i=1}^{L} O_{Ci}^T A_i^T Y O_C + O_{D21}^T B_1^T Y O_C) + \\ + (O_C^T A^T Y O_C + \sum_{i=1}^{L} O_{Ci}^T A_i^T Y O_C + O_{D21}^T B_1^T Y O_C)^T + \\ + \sum_{i=1}^{L} (O_C^T P_i O_C - O_{Ci}^T P_i O_{Ci}) - \gamma\ O_{D21}^T O_{D21},$$

$$S_{Y\_C}^T = C_z O_C + \sum_{i=1}^{L} C_{z\_i} O_{Ci} + D_{11} O_{D21}. \quad (4.13)$$

If we take a look to the size of the LMI's (4.11) and (4.12) they are actually of comparable dimensions, $(L+1)n + m1 + \quad + p1 - m2$ and $(L+1)n + m1 + p1 - p2$.

Theorem 4.1. If there is a set of positive definite matrices $P_1, \ldots, P_L$, X and Y such that the LMI's (4.11) and (4.12) are satisfied, and

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} > 0 \quad (4.14)$$

there exists a γ sub-optimal H_∞ controller for the system (1.5). The order of the controller will be given by

$$\mathrm{rank}(I_n - XY) = k \le n. \quad (4.15)$$

If in addition we minimize γ , the solution will converge towards the optimal one.

Proof. The equivalence between (3.8) and (4.11) and (4.12) have been proven. Regarding now the LMI (4.14), there exist a symmetric positive definite matrix P satisfying (4.6), if (4.14) is satisfied. For proof we define $Z = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$ and from $P^{-1}P = I$ we get

$$\begin{bmatrix} XY + MN^T & XN + M\tilde{Y} \\ M^T Y + \tilde{X}N^T & M^T N + \tilde{X}\tilde{Y} \end{bmatrix} = \qquad (4.16)$$
$$= \begin{bmatrix} I_n & 0_{n\times k} \\ 0_{k\times n} & I_k \end{bmatrix},$$

or

$$\begin{aligned} &XY + MN^T = I_n \\ &XN + M\tilde{Y} = 0_{n\times k}, \\ &M^T N + \tilde{X}\tilde{Y} = I_k. \end{aligned} \qquad (4.17)$$

Using now (4.17) if we consider the following transformation

$$Z^T PZ = \begin{bmatrix} X & M \\ I & 0 \end{bmatrix} \begin{bmatrix} Y & N \\ N^T & \tilde{Y} \end{bmatrix} \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} XY + MN^T & XN + M\tilde{X} \\ Y & N \end{bmatrix} \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$$
$$= \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \qquad (4.18)$$

we can see that $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \Rightarrow P \geq 0$ if Z has maximal rank , condition satisfied in a generic case. If we consider the strict inequality then the rank condition is not necessary anymore.

5. Controller Synthesis and Numerical Results

The *Theorem* 4.1 provides the necessary conditions for the existence of an H_∞ controller. Due to the structure of the LMI (4.11) the problem is not linear in both the pair *X*, *Y* and the weight (or multiplier) matrices $P_i, i = 1:L$. As a result the optimal value for γ cannot be searched by optimizing in the same time with respect to all of these matrices. One possible approach would be to do it sequentially. This means to fix first the matrices $P_i, i = 1:L$ and find a feasible pair *X* and *Y* that minimize γ. Then given the computed matrices *X* and *Y*, find a set of matrices $P_i, i = 1:L$ that minimize γ and repeat these steps till no major progress is achieved. Once the unknown matrices *X* and *Y* are computed we can determine the *P* matrix from the following equality

$$\begin{bmatrix} Y & I \\ N^T & 0 \end{bmatrix} = P \begin{bmatrix} I & X \\ 0 & M^T \end{bmatrix}, \qquad (5.1)$$

where the matrices *M* and *N* can be computed from $MN^T = I - XY$ using a Singular Value Decomposition (SVD). Given now the $P, P_i, i = 1:L$ matrices we can go back and compute the controller matrices as any solution to the LMI (4.1).

One of the packages in use for solving SDP is SDPT3, which works with Matlab5x and was developed by K.G.Toh 1998 (see [12]). For more about SDP see [13],[14].

Let us consider de following system

$$\begin{aligned} &A = -1, A_1 = -0.5, \\ &C_z = 1, C_{z\_1} = 0.5 \\ &C_y = 0.2, C_{y\_1} = 2, \\ &B_1 = 1, B_2 = 1, \\ &D_{11} = 0.2, D_{12} = 0.1, D_{21} = 0.1. \end{aligned} \qquad (5.2)$$

We can choose as an initial value for $P_1 = 1$ and solve the following SDP problem

$$\text{SDP\_1} \begin{cases} \min\limits_{X,Y}(\gamma) \\ LMI's(4.11), (4.12), (4.14) \end{cases} \qquad (5.3)$$

In this case the SDP algorithm converged and we obtained $X = 2.162,\ Y = 1.28,\ \gamma = 0.781$. If we try now to compute the controller matrices, using the previously computed values for *X* and *Y*, by solving

$$\text{SDP\_2} \begin{cases} \min\limits_{K}(\gamma) \\ LMI's(4.1) \end{cases}, \qquad (5.4)$$

we get $A_k = -1.54, B_k = 0.3, C_k = 1.75{,}21$ $D_k = -0$ and $\gamma = 1.976$. As we can see, in (5.4) we achieved a bigger value for the min value of γ compared with (5.3). This is due to the fact that we solved actually different SDP problems in which the optimum is influenced by the arbitrary choice of the $P_i, i = 1:L$ matrices and the way we computed the P matrix. In consequence a different approach for the optimal and sub-optimal H_∞ problems is necessary, and we will present in the end of this section two procedures to solve them.

Table 1

	SDP1 a)	SDP2 b)	SDP3 c)		SDP2 d)
	γ	γ	γ	P_1	γ
1	0.781	1.976	1.339	0.197	1.339
2	0.649	1.318	1.278	0.133	1.278
3	0.629	1.279	1.258	0.099	1.258
4	0.613	1.258	1.245	0.078	1.245
5	0.598	1.245	1.236	0.063	1.236
6	0.582	1.236	1.229	0.050	1.229
7	0.563	1.229	1.223	0.039	1.223
8	0.54	1.223	1.219	0.032	1.219
9	0.515	1.219	1.215	0.026	1.215
10	0.488	1.216	1.213	0.021	1.213
11	0.459	1.213	1.211	0.017	1.211
12	0.425	1.211	1.208	0.013	1.208
13	0.374	1.208	1.206	0.009	1.206
14	0.32	1.205	1.204	0.007	1.204
15	0.271	1.206	1.205	0.057	1.205

In order to find the optimum $P_i, i = 1:L$ matrices we can change the LMI (4.11) through a series of a Schur complement transformations in (5.5). With the values for X and Y computed in the first step we can solve the following problem

$$\text{SDP\_3} \begin{cases} \min\limits_{P_i, i=1:L} (\gamma) \\ LMI's(4.11),(5.5) \end{cases}. \tag{5.6}$$

The SDP algorithm converged and we get $P_1 = 0.183, \gamma = 1.333$. If we solve now again the SDP_1 with the new value for P_1 we get $X = 6.452, Y = 0.992, \gamma = 0.645$ and if solve again SDP_3 we get $P_1 = 0.121, \gamma = 1.271$.

Both SDP problems 1 and 3 are convex and the values of γ are decreasing at each step but for each problem itself. As we can see, for numerical reasons, the values of γ achieved by solving the SDP_3 are not smaller then the ones obtained from SDP_1.

In consequence a possible procedure to find a value for γ as small as possible could be to solve successively the SDP_1 and SDP_3 till no significant progress is achieved. By doing so for the system (5.2) we get the values in Table 1.

We solved also the SDP_2 to see the impact of each step on the final result. In Fig. 2 we can see the graphic of the values presented above

As we can see from the *Table* 1 the value of P_1, the weight of the delay in the cost functional, is decreasing toward zero. Due to numerical aspects the value of γ resulted in SDP_2 is decreasing up to point and then is growing again making no sense to iterate further more.

$$\begin{bmatrix} T_X & O_{B1}^T B_1 + O_{D12}^T D_{11} & O_{B1}^T A_1 + O_{D12}^T C_{z\_1} & \cdots & O_{B1}^T A_L + O_{D12}^T C_{z\_L} \\ B_1^T O_{B1} + D_{11}^T O_{D12} & -\gamma\, I_{m1} & & & \\ A_1^T O_{B1} + C_{z\_1}^T O_{D12} & & -P_1 & & \\ \vdots & & & \ddots & \\ A_L^T O_{B1} + C_{z\_L}^T O_{D12} & & & & -P_L \end{bmatrix} < 0.$$

where

$$T_{X0} = O_{B1}^T X A^T O_{B1} + O_{B1}^T A X O_{B1} + O_{D12}^T C_z X O_{B1} + O_{B1}^T X C_z^T O_{B1} + \sum_{i=1}^{L} (O_{B1}^T X) P_i (X O_{B1}) \tag{5.5}$$

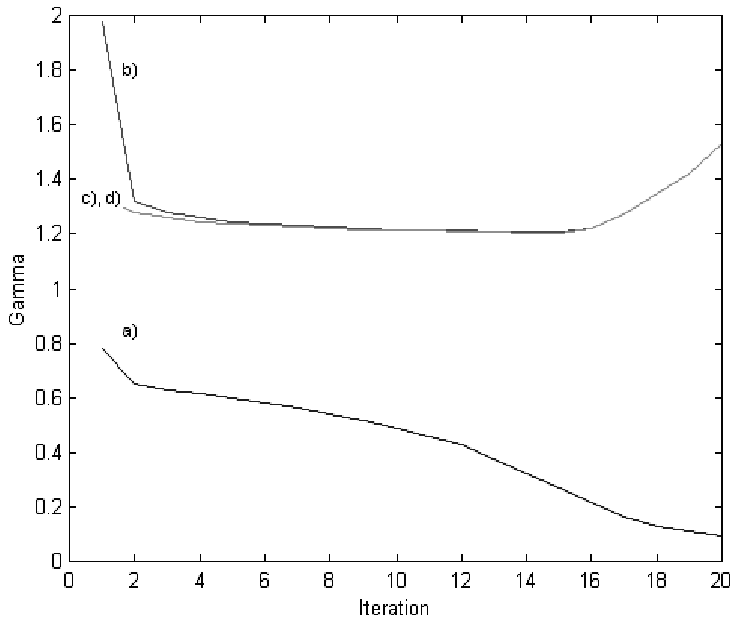

Figure 2 (**SDP_1** - a), **SDP_2** - b), **SDP_3** –c), **SDP_2** - d) sec.)

Both SDP problems SDP_1 and SDP_3 are convex. As a result optimizing at one step using the matrices computed in the previous one should provide always a smaller γ. The monotone decreasing values for γ should ensure the convergence of the procedure. However due to numerical reasons related to the way the SDP problems are solved, at a certain step the values computed by solving the SDP_3 are not smaller then those resulted from solving the SDP_1. In spite of that, due to the fact that each procedure provides a better set of matrices, the values of γ are decreasing from one step to another for each SDP. As long as the value of γ computed by solving the initial problem SDP_2 is decreasing by an amount bigger then a reasonable value, it makes sense to continue iterating.

The above-presented approach has a few inconveniences. One of them is that the real indication over when we should stop is given by SDP_2 and as a result it has to be solved. A more efficient way to proceed is to solve first the SDP_1 in order to get a feasible set of matrices X and Y and then construct P and solve the SDP_2. At this point we have a feasible controller K and we can optimize with respect of the P and the $P_i, i = 1:L$ matrices by solving the following problem

$$\text{SDP\_4}\begin{cases} \min\limits_{P, P_i, i=1:L} (\gamma) \\ LMI's\ (4.1) \end{cases}. \tag{5.7}$$

Table2.

	SDP2 a)	**SDP4** b)	
	γ	γ	P_1
1	1.9764	1.4804	0.2710
2	1.3663	1.3024	0.1642
3	1.2972	1.2944	0.1596
4	1.2944	1.2900	0.1524
5	1.29	1.2827	0.1403
6	1.2826	1.2701	0.1190
7	1.2698	1.2503	0.0842
8	1.2491	1.2280	0.0409
9	1.2241	1.2126	0.0151
10	1.2086	1.2047	0.0057
11	1.2033	1.2018	0.0022
12	1.2013	1.2007	0.00086
13	1.2005	1.2003	0.00034
14	1.2002	1.2001	0.00014
15	1.2001	1.2001	0.00006

The convexity of both SDP problems, ensure the convergence of the procedure. By repeatedly solving the SDP_2 and SDP_4 till no further progress we can get a value of γ close to the optimal one.

If we apply this procedure to the system described in (5.2) we get the values in Table 2. In analogy with the previous optimization procedure we can see a similar behavior in the evolution of the values of the matrix P_1, which in this case is also approaching 0. On the other hand the minimum value of γ achieved in the second case is smaller and this is somehow natural, since we directly optimize the initial problem instead of an equivalent one.

In Fig. 4 we can see the graphic of the values displayed in Table 2.

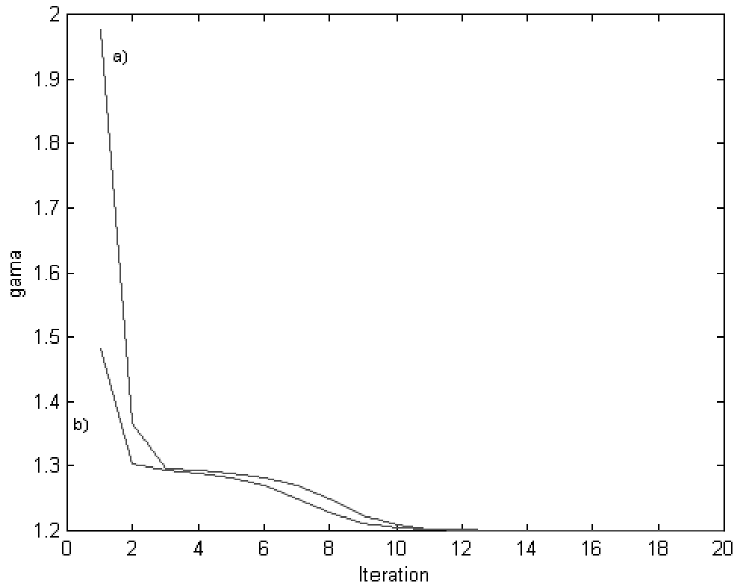

Figure 3 ((**SDP_2** - a) , **SDP_4** - b))

At each iteration we compute first the controller, and then we optimize with respect to the P, $P_i, i = 1 : L$ matrices. The min value of γ which is the closest to the real one for that specific controller, and for a certain iteration, is given by solving the SDP_4. Since solving the SDP_2 at step d) in the first procedure and at step a) in the second one provides us with a value of γ computed in a similar way, we can compare these values at successive iterations in Fig. 4. What we can see is that in the case of the second procedure the starting point is worst and the convergence is slower in the early stages, while after iteration no 8 it converges faster and to a minimum value, which is smaller then the one achieved in the first procedure. It is somehow normal since we optimized directly the initial problem and the influence of the numerical errors is smaller. The main advantages of the second procedure are a smaller computational effort, due to the fact that we get the controller matrices directly, and better final results.

In the case of the optimal H_∞ problem we can propose a third algorithm in which we solve first both the SDP_1 and SDP_3 to get a good starting point and then we repeatedly solve the SDP_2 and SDP_4 till no significant progress is achieved. It is somehow obvious that solving the SDP_3 at the initial stage, and by doing so optimizing with respect of $P_i, i = 1 : L$, provides a better starting point for the second faze where all P and $P_i, i = 1 : L$ have been optimized once, and as a consequence it is an improvement with respect to the previous approach. In Fig. 5 we can see the values for γ in this last approach, plotted with a dot line, compared with the ones presented in Fig. 4. One could ask why not to solve directly the inequality (4.1). The reason why is because (4.1) is not linear in terms of both, the controller matrices and *P*, and we do not have a receipt to chose an initial feasible value for one of them.

To sum up, we propose the following procedure to solve the H_∞ optimal problem.

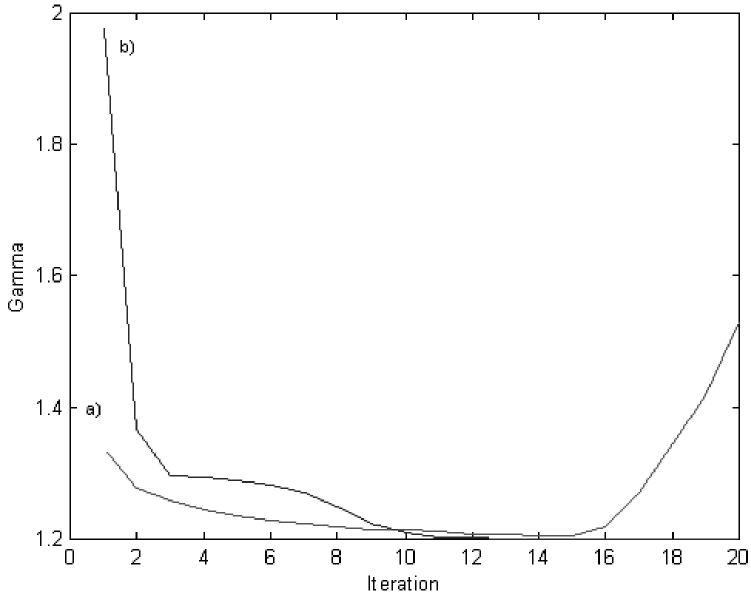

Figure 4 ((**SDP_2** first proc. - a), **SDP_2** second proc. - b))

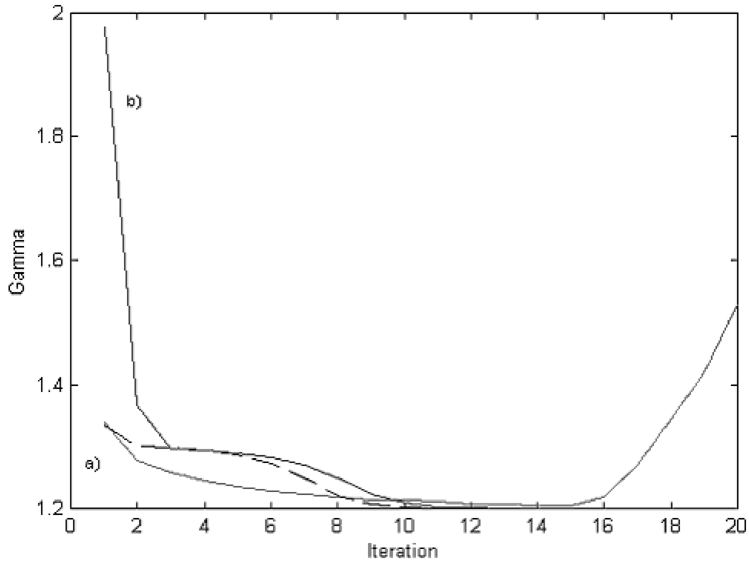

Figure 5 ((**SDP_2** first proc. - a), **SDP_2** second proc. - b), --- line third proc.)

Procedure I (H_∞ optimal).

1) Solve the *SDP*_1 and compute the *X* and *Y* matrices.

2) Solve the *SDP*_3 and compute the $P_i, i = 1 : L$ matrices.

3) Compute the *P* matrix.

4) Solve the *SDP*_2 and compute the *K* matrix.

5) Solve the *SDP*_4 and compute the *P* and $P_i, i = 1 : L$, $\gamma_{current} = \gamma$, $\gamma_{previous} = \gamma$.

6) While $\gamma_{previous} - \gamma_{current} > eps$

6.1) Solve the *SDP*_2 and compute the *K* matrix.

6.2) Solve the *SDP*_4 and compute the *P* and $P_i, i = 1 : L$, $\gamma_{previous} = \gamma_{current}$, $\gamma_{current} = \gamma$.

7) Return γ.

In the case of the H_∞ sub-optimal control problem, simply checking the solvability of the LMI set (4.11), (4.12) and (4.14) it's not a solution since the optimizing with respect to the $P_i, i = 1 : L$ matrices is also necessary. We can propose in this case a similar approach to the optimal case. The only difference is that the algorithm will stop if no progress is achieved or if the current value of γ becomes smaller then a requested value. The following procedure is proposed to solve the H_∞ sub-optimal problem.

Procedure II (H_∞ sub-optimal).

1) Solve the *SDP*_1 and compute the *X* and *Y* matrices.

2) Solve the *SDP*_3 and compute the $P_i, i = 1 : L$ matrices.

3) Compute the *P* matrix.

4) Solve the *SDP*_2 and compute the *K* matrix. If $\gamma < \gamma_{requested}$ STOP.

5) Solve the *SDP*_4 and compute the *P* and $P_i, i = 1 : L$, $\gamma_{current} = \gamma$, $\gamma_{previous} = \gamma$.

6) While ($\gamma_{previous} - \gamma_{current} > eps$) and ($\gamma < \gamma_{requested}$)

6.1) Solve the *SDP*_2 and compute the *K* matrix.

6.2) Solve the *SDP*_4 and compute the *P* and $P_i, i = 1 : L$, $\gamma_{previous} = \gamma_{current}$, $\gamma_{current} = \gamma$.

7) If $\gamma < \gamma_{requested}$ return A_k, B_k, C_k, D_k. Else go to 8).

8) Return, 'The problem doesn't has a solution'.

6. CONCLUSIONS

The LMI's are a powerful and also practical tool for the study of many control problems among which the H_∞ control.

We approached the problem for a general class of TDS in which both the state dynamics and the output depend on the delayed states. In the case of the dynamic output feedback the synthesis problem cannot be solved directly and an equivalent set of LMI's is used. This is a classical approach for the delay free systems, but one of the problems related to TDS is that we cannot optimize in the same time with respect of the pair *X, Y* and the weight matrices $P_i, i = 1 : L$, see (4.1)-(4.6). Moreover, if we get the controller matrix as a solution to the LMI (4.1) in the unknown *K* we also need an alternating iterative process in order to get a value of γ that is as close as possible to the optimal one. Computational procedures are proposed for solving both the optimal and the sub-optimal problems which offer both, a good start and good final results in the last stages of the iterative process.

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