Robust Block Control of Fractional-order Systems via Nonlinear Sliding Mode Techniques

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Abstract: This paper presents two novel fast converging robust controllers for Caputo derivative based fractional-order nonlinear systems. These fractional-order systems are high-relative-degree with model uncertainties and external disturbances. First, a new fractional-order model is derived from the original model based on block transformation strategy. Employing the block transformation technique makes the high-relative-degree systems versatile for sliding mode controllers design. In the second step, two different nonlinear sliding manifolds are proposed to reach a short time convergence. Subsequently, appropriate nonlinear sliding mode control laws are developed to assure the robustness and fast converging behaviors. It is worthy to notify that the mentioned sliding manifolds guarantee the fractional-order system last state convergence, and the other states convergence can be assured by control gains of the block transformation. The stability of closed-loop system for both controllers is achieved by the fractional-order stability theorems. Finally, two comprehensive numerical simulations are carried out to indicate the superiority and effectiveness of the suggested robust controllers.

Keywords: Fractional-order system, Block transformation strategy, Nonlinear sliding mode control, Fast convergence.

1. INTRODUCTION

Fractional calculus idea was established in the 17th century which discusses about non-integer integrations and derivatives. The basic ideas in this field are generalizations of the common ideas in integer calculus. The fractional calculus has been taken into account as an exclusive theoretical subject with no practical applications for nearly 300 years (Podlubny, 1999). Nowadays, researchers have been interested in the application of fractional calculus in various branches of science such as thermal systems modelling (Gabano and Poinot, 2011), electromechanical systems (Jesus and Tenreiro Machado, 2012) and biological systems (Petras and Magin, 2011). Also, designing fractional-order controllers is another interesting application.

Sliding mode control is a famous nonlinear control technique which presents high precision and robust behaviour against model uncertainties and external disturbances (Edwards and Spurgeon, 1998). In the conventional sliding mode control, an arbitrary linear manifold is considered as a sliding surface and a control law is planned in such a way that the system state trajectories reach this manifold. In last three decades, this technique is employed for different integer-order systems such as: robot manipulators (Islam and Liu, 2011), DC-DC boost converters (Wai and Shih, 2011), electrical motors (Chi and Xu, 2009), and so on. In addition, nowadays the sliding mode control is applied for governing the fractional-order systems (Tavazoei and Haeri, 2008; Hosseinnia et al., 2010; Aghababa, 2012a; Yin, 2013; Djari et al., 2014; Shoja-Majidabad et al., 2014a). However, the main drawback of sliding mode scheme is that the closed-loop system errors cannot reach equilibrium point in a finite time, while accomplishing fast time convergence is more worthwhile in practice. In recent years, a new control strategy called nonlinear or terminal sliding-mode control is proposed to reach a faster convergence with high precision tracking. This technique utilizes a nonlinear sliding manifold instead of the linear one. Successively, various application examples of nonlinear sliding mode control have been developed for integer-order systems in literature (Zhihong and O'Day, 1999; Yu et al., 2005; Hui and Li, 2009; Chang et al., 2008; Feng et al., 2002; Jin, 2009). Unfortunately, most of the nonlinear sliding mode controllers are developed only for second-order (two-relative-degree) systems (Chiu, 2012). Besides, majority of the mentioned works are designed for integer-order systems and a few works does exist for fractional-order ones (Aghababa, 2012b, 2013).

Inspired by the above discussions, enlarging the application of nonlinear sliding mode controllers on fractional-order systems seems more significant. In this paper, two new fractional-order nonlinear sliding mode controllers are combined with block transformation technique for fast governing the Caputo derivative based systems. Initially, the block transformation technique is applied to arrange the system dynamics in new coordinates, and then the sliding mode controllers are designed. Both methods employ a nonlinear integral manifold (a sign function for the first controller and a fractional power for the second one). The fast converging behaviour is obtained using the proposed nonlinear sliding surfaces and the block transformation technique control gains. Employing the block transformation technique makes the suggested controllers versatile for higher-order applications. In addition, the influences of model uncertainties and external disturbances are fully taken into account. Also asymptotic stability of the closed-loop system is proofed using fractional-order nonlinear stability theorems. Generally, this paper presents the following main contributions: (1) Applying the block transformation technique for fractional-order systems. (2) Designing fast converging nonlinear sliding mode controllers for fractional-order systems.

The rest of this paper is organized as follows: Some fractional calculus preliminaries are presented in Section 2. In Section 3, a Caputo derivative based uncertain fractional-order system dynamics and their block transformations are expressed. Designing the conventional fractional-order sliding mode technique and two new nonlinear sliding mode controllers are discussed in Sections 4 and 5. In Sections 6, the efficiency of proposed controllers is highlighted through two numerical simulations. Finally, this paper terminates with some conclusions in Section 7.

2. FRACTIONAL CALCULUS

The main definitions, properties and theorems of applied fractional calculus are expressed in this section.

Definition 1 (Li and Deng, 2007)**:** The fractional integration of function $f(t)$ with respect to t can be given as follows:

$$I_{0,t}^{\alpha} f(t) = D_{0,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (Li and Deng, 2007)**:** The α -th order Caputo fractional derivative of ($f(t) \in C^m[0,t]$) function $f(t)$ can be described by

$${}_C D_{0,t}^{\alpha} f(t) = D_{0,t}^{-(m-\alpha)} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{1-m+\alpha}} d\tau$$

where $m-1 < \alpha < m$, $m \in N$.

Definition 3 (Li and Deng, 2007)**:** The Riemann-Liouville (RL) fractional derivative of α -th order of function $f(t)$ is defined as follows:

$$\begin{aligned} {}_{RL} D_{0,t}^{\alpha} f(t) &= D^m D_{0,t}^{-(m-\alpha)} f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-m+\alpha}} d\tau \end{aligned}$$

where $m-1 \le \alpha < m$, $m \in N$.

Property 1 (Li and Deng, 2007)**:** If $f(t) \in C^m[0,\infty)$, $m-1 < \alpha < m$ and $m \in N$, then

(a) ${}_C D_{0,t}^{\alpha} D_{0,t}^{-\alpha} f(t) = f(t)$ holds for $m = 1$.

(b) ${}_{RL} D_{0,t}^{\alpha} D_{0,t}^{-\alpha} f(t) = f(t)$.

Property 2 (Shoja-Majidabad et al., 2014b)**:** If $s(t) \in C^1[0,T]$ for some $T > 0$, $\alpha_i \in (0,1)$ $(i = 1,2)$ and $\alpha_1 + \alpha_2 \in (0,1]$, then

$${}_C D_{0,t}^{\alpha_1} {}_C D_{0,t}^{\alpha_2} s(t) = {}_C D_{0,t}^{\alpha_2} {}_C D_{0,t}^{\alpha_1} s(t) = {}_C D_{0,t}^{\alpha_1+\alpha_2} s(t)$$

Theorem 1 (Li et al., 2009, 2010)**:** Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system

$${}_C D_{0,t}^{\alpha} x(t) = f(t, x(t)) \tag{1}$$

where $f(t, x(t))$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$ and $\alpha \in (0,1)$. Assume that there exists a Lyapunov function $V(t, x(t))$ satisfying

$$\alpha_1 \|x\|^a \le V(t, x(t)) \le \alpha_2 \|x\| , \ \frac{d}{dt} V(t, x(t)) \le -\alpha_3 \|x(t)\| .$$

where α_1 , α_2 , α_3 and a are positive constants. Then the equilibrium point of the system (1), $x = 0$, is asymptotically stable.

3. SYSTEM DESCRIPTION AND BLOCK TRANSFORMATION

In this section, a canonical fractional-order system dynamic model and its block transformation is presented.

Consider a class of Caputo derivative based fractional-order dynamical system with model uncertainty and external disturbance as follows:

$$\begin{cases} {}_C D^{\alpha} x_1(t) = x_2(t) \\ {}_C D^{\alpha} x_i(t) = x_{i+1}(t) \qquad i = 2,3,\ldots,n-1 \\ {}_C D^{\alpha} x_n(t) = f(X,t) + \Delta f(X,t) + d(t) + u(t) \end{cases} \tag{2}$$

where $\alpha \in (0,1)$ is the order of system, $X = [x_1, x_2, \ldots, x_n]^T$ is the state vector, $f(X,t)$ is a known nonlinear function of X and t , $\Delta f(X,t)$ describes the model uncertainty term which is unknown, and $u(t)$ is the system control input.

Assumption 1: The uncertainty term $\Delta f(X,t)$ is assumed to be bounded as

$$\left| {}_C D_{0,t}^{1-\alpha} \Delta f(X,t) \right| \le \gamma_f ,$$

where γ_f is a given and positive constant.

Assumption 2: The external disturbance $d(t)$ is supposed to be bounded by

$$\left| {}_C D_{0,t}^{1-\alpha} d(t) \right| \le \gamma_d \tag{3}$$

where γ_d is a known and positive constant.

Step 1: Let's define first new variable as follows:

$$z_1(t) = x_1(t) - x_{1d}(t) \tag{4}$$

where $x_{1d}(t)$ is the first state desired signal. By taking ${}_C D^\alpha$ derivative from both sides of (4) and using (2), we can get

$${}_C D^\alpha z_1(t)={}_C D^\alpha x_1(t)-{}_C D^\alpha x_{1d}(t)=x_2(t)-x_{2d}(t) \quad (5)$$

To stabilize equation (5) dynamic, the first virtual control input can be selected as

$$x_2(t)=x_{2d}(t)-b_1 z_1(t)+z_2(t) \quad (6)$$

where b_1 is a positive constant, and $z_2(t)$ is a new variable which is necessary for next block calculation of the block transformation technique. Hence, the first state closed-loop dynamic will be as

$${}_C D^\alpha z_1(t)=-b_1 z_1(t)+z_2(t) \quad (7)$$

Step 2: From (6) the new variable $z_2(t)$ can be obtained as

$$z_2(t)=x_2(t)-x_{2d}(t)+b_1 z_1(t) \quad (8)$$

In this stage, by taking ${}_C D^\alpha$ derivative from (8) along the equations (3) and (7), results in

$${}_C D^\alpha z_2(t)=x_3(t)-x_{3d}(t)+g_2(z_1,z_2)\ ,$$

where $g_2(z_1,z_2)={}_C D^\alpha(g_1(z_1)+b_1 z_1(t))=b_1(-b_1 z_1(t)+z_2(t))$ and $g_1(z_1)=0$.

Choosing the second virtual control input $z_3(t)$ as

$$x_3(t)=x_{3d}(t)-g_2(z_1,z_2)-b_2 z_2(t)+z_3(t) \quad (9)$$

results the following dynamic

$${}_C D^\alpha z_2(t)=-b_2 z_2(t)+z_3(t) \quad (10)$$

where b_2 is a positive constant.

Step 3: From (9) the new variable $z_3(t)$ can be attained as

$$z_3(t)=x_3(t)-x_{3d}(t)+g_2(z_1,z_2)+b_2 z_2(t) \quad (11)$$

By applying the derivative ${}_C D^\alpha$ on (11) and using the equations (3), (7) and (10), one can get

$${}_C D^\alpha z_3(t)=x_4(t)-x_{4d}(t)+g_3(z_1,z_2,z_3)\ ,$$

where $g_3(z_1,z_2,z_3)={}_C D^\alpha(g_1(z_1,z_2)+b_2 z_2(t))=b_1(-b_1(-b_1 z_1(t)$ $+z_2(t))+(-b_2 z_2(t)+z_3(t))+b_2(-b_2 z_2(t)+z_3(t))$.

Selecting the third virtual control input $z_4(t)$ in the form of

$$x_4(t)=x_{4d}(t)-g_3(z_1,z_2,z_3)-b_3 z_3(t)+z_4(t) \quad (12)$$

yields the following dynamic

$${}_C D^\alpha z_3(t)=-b_3 z_3(t)+z_4(t)\ .$$

This procedure can be proceed for the variables $z_4,z_5,...,z_{n-1}$.

At the last step, after calculating ${}_C D^\alpha z_n(t)$, the original system (3) can be represented in the new coordinates ($z_1,z_2,...,z_n$) as

$$\begin{cases} {}_C D^\alpha z_1(t)=-b_1 z_1(t)+z_2(t) \\ {}_C D^\alpha z_i(t)=-b_i z_i(t)+z_{i+1}(t) \quad ,i=2,3,...,n-1 \\ {}_C D^\alpha z_n(t)=f(X,t)+\Delta f(X,t)-{}_C D^\alpha x_{nd}(t)+ \\ \qquad g_n(z_1,z_2,\cdots,z_n)+d(t)+u(t) \end{cases} \quad (13)$$

where $g_n(z_1,z_2,...,z_n)$ is a linear function of the transformed new variables, and is calculable by the following recursive equation:

$$g_{j+1}(z_1,z_2,\cdots,z_{j+1})={}_C D^\alpha(g_j(z_1,z_2,\cdots,z_j)+b_j z_j(t))$$
$$g_1(z_1)=0 \quad ,j=1,2,\ldots,n-1$$

In (13), the coefficients $b_1,b_2,\cdots,b_{n-1}>0$ are called control gains and are crucial for tuning the new state convergence. For more details see the following remark.

Remark 3: For the system (13) that is constrained to $z_n(t)=0$ by a control law, the system dynamics reduce to

$$\begin{cases} {}_C D^\alpha z_1(t)=-b_1 z_1(t)+z_2(t) \\ {}_C D^\alpha z_i(t)=-b_i z_i(t)+z_{i+1}(t) \quad ,i=2,3,...,n-1 \\ {}_C D^\alpha z_{n-1}(t)=-b_{n-1} z_{n-1}(t) \end{cases} \quad (14)$$

it is evident that the above linear system is stable and ensures that $\lim_{t\to\infty} z_i(t)=0$, Also the convergence rate of states $z_1,z_2,...,z_{n-1}$ is adjustable by the coefficients $b_1<b_2<\cdots<b_{n-1}$.

From the equations (6), (9) and the reduced model (14), zero convergence of the transformed states ($z_i(t)\to 0$) yields the original system states convergence to the desired values ($x_i(t)\to x_{id}(t)$).

4. CONVENTIONAL SLIDING MODE CONTROLLER DESIGN

Consider the conventional sliding surface as follows:

$$s(t)=e_n(t)+\sum_{i=1}^{n-1} c_i e_i(t) \quad (15)$$

where $e_i(t)=x_i(t)-x_{id}(t)$, ${}_C D^\alpha x_{id}(t)=x_{(i+1)d}(t)$ and $c_1,c_2,...,c_{n-1}$ are selected in such a way that all roots of the polynomial $P(s)=s^{n-1}+c_{n-1}s^{n-2}+\cdots+c_1$ be in the left half of S-plane.

Now, by taking ${}_C D^\alpha$ from (15), one can obtain

$${}_C D^\alpha s(t)={}_C D^\alpha e_n(t)+\sum_{i=1}^{n-1} c_i\, {}_C D^\alpha e_i(t) \quad (16)$$

The system equations (3) can be substituted into (16), yielding the simplified expression

$$ {}_C D^{\alpha} s(t) = {}_C D^{\alpha} x_n(t) - {}_C D^{\alpha} x_{nd}(t) + \sum_{i=1}^{n-1} c_i e_{i+1}(t)\,, $$

and completely

$$ {}_C D^{\alpha} s(t) = f(X,t) + \Delta f(X,t) - {}_C D^{\alpha} x_{nd}(t) + \sum_{i=1}^{n-1} c_i e_{i+1}(t) + d(t) + u(t) $$

Now selecting the control law

$$ \begin{aligned} u(t) = & -f(X,t) + {}_C D^{\alpha} x_{nd}(t) - \sum_{i=1}^{n-1} c_i e_{i+1}(t) \\ & - D^{-(1-\alpha)}(\eta s(t) + (\gamma_f + \gamma_d)\operatorname{sgn}(s(t)))\quad,\quad \eta > 0 \end{aligned} \tag{17} $$

will provide the system states $X(t)$ asymptotic decaying into the sliding surface $s(t) = 0$. In (17), η is known as proportional gain of the fractional exponential reaching law ${}_C D^{\alpha} s(t) = -D^{-(1-\alpha)}(\eta s(t) + (\gamma_f + \gamma_d)\operatorname{sgn}(s(t)))$.

By taking fractional derivative ${}_C D^{(1-\alpha)}$ from both sides of the fractional exponential reaching law or assuming $\alpha = 1$, the conventional reaching law will be as $\dot{s}(t) = -\eta s(t) - (\gamma_f + \gamma_d)\operatorname{sgn}(s(t))$ which is common in integer calculus (Bandyopadhyay et al., 2009). Figure 1 shows the sliding surface convergence regime based on discussed reaching laws.

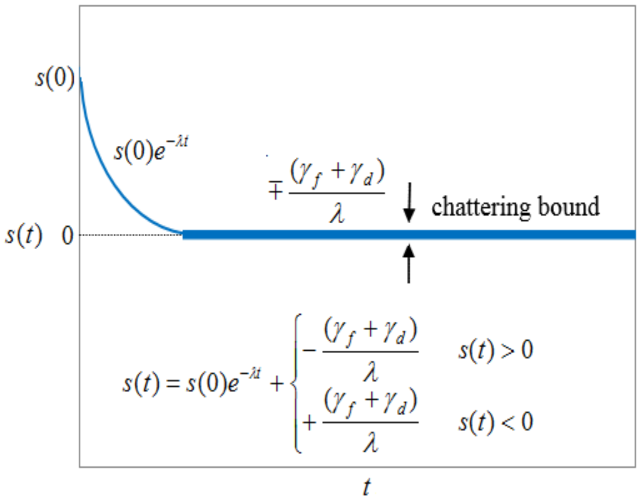


Fig. 1. Sliding surface $s(t)$ convergence regime.

5. NONLINEAR SLIDING MODE CONTROLLER DESIGN

In this section, two novel nonlinear sliding surfaces are suggested, and proper control laws are designed to provide the closed-loop system stability and fast convergence.

5.1. Sign Integral Nonlinear Sliding Mode

For the transformed system (13), let define the sign integral terminal sliding surface as follows:

$$ \begin{aligned} s(t) &= z_n(t) + \lambda_1 \int_0^t \operatorname{sgn}(z_n(\tau))d\tau \\ &= z_n(t) + \lambda_1 D^{-1}\operatorname{sgn}(z_n(t)) \end{aligned} \tag{18} $$

where λ_1 is a positive constant which can increase or decrease sliding variable reaching time and overshoot. Figure 2 presents the variable $z_n(t)$ convergence regime approximately (short lines with $+\lambda_1 t$ or $-\lambda_1 t$ ramp).

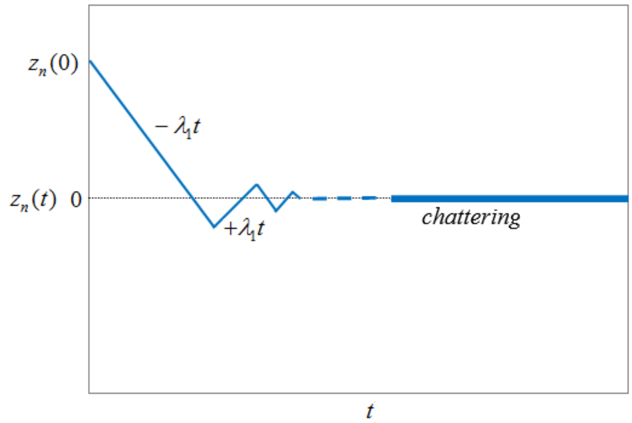


Fig. 2. New variable $z_n(t)$ convergence regime.

Taking ${}_C D^{\alpha}$ derivative from the previous equation yields

$$ {}_C D^{\alpha} s(t) = {}_C D^{\alpha} z_n(t) + \lambda_1 {}_C D^{\alpha} D^{-1}\operatorname{sgn}(z_n(t))\,. $$

Employing the Caputo derivative definition ${}_C D^{\alpha} s(t) = D^{-(1-\alpha)} D^1 s(t)$, results in

$$ {}_C D^{\alpha} s(t) = {}_C D^{\alpha} z_n(t) + \lambda_1 D^{-(1-\alpha)}\operatorname{sgn}(z_n(t)) \tag{19} $$

By substituting the transformed system dynamics (13) in (19), one can get

$$ \begin{aligned} {}_C D^{\alpha} s(t) = & f(X,t) + \Delta f(X,t) - {}_C D^{\alpha} x_{nd}(t) + g_n(z_1, z_2, \cdots, z_n) \\ & + \lambda_1 D^{-(1-\alpha)}\operatorname{sgn}(z_n(t)) + d(t) + u(t) \end{aligned} \tag{20} $$

Theorem 2: Consider the transformed fractional-order system (12), choosing the robust block controller as

$$ \begin{aligned} u(t) = & -f(X,t) + {}_C D^{\alpha} x_{nd}(t) - g_n(z_1, z_2, \cdots, z_n) \\ & - \lambda_1 D^{-(1-\alpha)}\operatorname{sgn}(z_n(t)) - D^{-(1-\alpha)}(\eta_1 s(t) \\ & + (\gamma_f + \gamma_d)\operatorname{sgn}(s(t))) \end{aligned} \tag{21} $$

will result the system trajectories convergence to the sliding surface $s(t)$ in a short time. Where $\eta_1 > 0$ is known as proportional gain of the fractional exponential reaching law.
Proof: Choosing the Lyapunov candidate in the form of $V(t) = |s(t)|$ and evaluating its time derivative, results

$\dot{V}(t) = \text{sgn}(s(t))\dot{s}(t)$.

From Property 2, one can obtain

$\dot{V}(t) = \text{sgn}(s(t))_C D^{1-\alpha}\, _C D^{\alpha} s(t)$.

Using (20), we have

$$\dot{V}(t) = \text{sgn}(s(t))_C D^{1-\alpha}(f(X,t) + \Delta f(X,t) - _C D^{\alpha} x_{nd}(t) + g_n(z_1, z_2, \cdots, z_n) + \lambda_1 D^{-(1-\alpha)} \text{sgn}(z_n(t)) + d(t) + u(t))$$

Applying the control law (21), yields

$$\begin{aligned}\dot{V}(t) &= \text{sgn}(s(t))_C D^{1-\alpha}(\Delta f(X,t) + d(t) - D^{-(1-\alpha)}(\eta_1 s(t) + (\gamma_f + \gamma_d)\,\text{sgn}(s(t)))) \\ &= \text{sgn}(s(t))(_C D^{1-\alpha}(\Delta f(X,t) + d(t)) - \eta_1 s(t) + (\gamma_f + \gamma_d)\,\text{sgn}(s(t)))\end{aligned}$$

Using $\text{sgn}(s(t))s(t) = |s(t)|$, one can get $\dot{V}(t) \le -\eta_1 |s(t)|$.

Hence, the states of the system will converge to $s(t) = 0$ asymptotically.

On the sliding surface ($s(t) = 0$), to show that the sliding motion transpires in a short time, we have

$z_n(t) + \lambda_1 D^{-1} \text{sgn}(z_n(t)) = 0$,

taking time derivative, results in

$\dot{z}_n(t) + \lambda_1 \text{sgn}(z_n(t)) = 0$,

$$\frac{z_n(t)\dot{z}_n(t)}{|z_n(t)|} = -\lambda_1 \quad \rightarrow \quad \frac{d|z_n(t)|}{dt} = -\lambda_1 ,$$

then, the reaching time can be calculated in the following form $t_{reach} = \dfrac{|z_n(0)|}{\lambda_1}$.

t_{reach} is tuneable by declaring a proper value for λ_1 . □

5.2. Fractional Integral Nonlinear Sliding Mode

Consider the second nonlinear sliding manifold as follows:

$$s(t) = z_n(t) + \lambda_2 \int_0^t z_n^{q/p}(\tau) d\tau = z_n(t) + \lambda_2 D^{-1} z_n^{q/p}(t) \qquad (22)$$

where p and q are both positive odd integers which should satisfy q/p . In (22), λ_2 is a positive constant which can increase or decrease sliding variable reaching time and overshoot.

Applying $_C D^{\alpha}$ derivative on the previous equation, results in

$_C D^{\alpha} s(t) = _C D^{\alpha} z_n(t) + \lambda_2 D^{-(1-\alpha)} z_n^{q/p}(t)$.

Now, by applying the Caputo derivative definition, one can get

$$_C D^{\alpha} s(t) = f(X,t) + \Delta f(X,t) - _C D^{\alpha} x_{nd}(t) + g_n(z_1, z_2, \cdots, z_n) + \lambda_2 D^{-(1-\alpha)} z_n^{q/p}(t) + d(t) + u(t) \qquad (23)$$

Theorem 3: Consider the transformed fractional-order system (13), if the system controlled by the robust control law (24), the system states will converge to the sliding surface $s(t)$ in a short time.

$$\begin{aligned}u(t) = &-f(X,t) + _C D^{\alpha} x_{nd}(t) - g_n(z_1, z_2, \cdots, z_n) \\ &- \lambda_2 D^{-(1-\alpha)} z_n^{q/p}(t) - D^{-(1-\alpha)}(\eta_2 s(t) \\ &+ (\gamma_f + \gamma_d)\,\text{sgn}(s(t)))\end{aligned} \qquad (24)$$

where $\eta_2 > 0$ is known as proportional gain of the fractional exponential reaching law.

Proof: By defining the Lyapunov function as $V(t) = |s(t)|$ and evaluating its time derivative and using (23), one can write

$$\dot{V}(t) = \text{sgn}(s(t))_C D^{1-\alpha}(f(X,t) + \Delta f(X,t) - _C D^{\alpha} x_{nd}(t) + g_n(z_1, z_2, \cdots, z_n) + \lambda_2 D^{-(1-\alpha)} z_n^{q/p}(t) + d(t) + u(t)) \qquad (25)$$

Inserting the second control law (24) in (25), results in

$$\begin{aligned}\dot{V}(t) &= \text{sgn}(s(t))_C D^{1-\alpha}(\Delta f(X,t) + d(t) - D^{-(1-\alpha)}(\eta_2 s(t) + (\gamma_f + \gamma_d)\,\text{sgn}(s(t)))) \\ &= \text{sgn}(s(t))(_C D^{1-\alpha}(\Delta f(X,t) + d(t)) - \eta_2 s(t) + (\gamma_f + \gamma_d)\,\text{sgn}(s(t)))\end{aligned}$$

Hence we can get

$\dot{V}(t) \le -\eta_2 |s(t)|$,

which guarantees the system states asymptotically convergence to $s(t) = 0$.

On the sliding surface ($s(t) = 0$), the dynamic of $z_n(t)$ can be expressed in the following form

$z_n(t) + \lambda_2 D^{-1} z_n^{q/p}(t) = 0$.

Taking time derivative from the above equation, gives us

$$\dot{z}_n(t) + \lambda_2 z_n^{q/p}(t) = 0 \qquad (26)$$

The solution of (26) for the reaching time t_{reach} is given by $t_{reach} = \dfrac{|z_n(0)|^{(1-q/p)}}{\lambda_2 (1 - q/p)}$, which shows that the sliding motion occurs in a finite time, and this time is tuneable by choosing proper parameters (λ_2 and q/p).

Remark 5: In order to have a smooth control signal and hold the continuously differentiable condition (C^1), the function $\text{sgn}(\bullet)$ in the sliding surface (18) and control laws (21), (24) can be modified in the following forms:

$$\mathrm{sgn}(z_n(t)) = \frac{z_n(t)}{|z_n(t)| + \delta}, \tag{27}$$
$$\mathrm{sgn}(s(t)) = \frac{s(t)}{|s(t)| + \varepsilon},$$

where $\delta, \ \varepsilon > 0$ and should be enough small.

Remark 6: It is worthwhile to notify that the actual state x_i is a function of transformed state z_i and the other states $z_1, z_2, \ldots, z_{i-1}$, then only fast convergence of z_i will not result x_i convergence, and the other transformed states are related in this process. Therefore, performance of actual states ($x_1, x_2, \ldots, x_n$) should be checked instead of the transformed states ($z_1, z_2, \ldots, z_n$) in the controller parameters tuning (λ_i, b_i, δ, ε, q/p).

6. SIMULATION RESULTS

In this section, two numerical simulations are performed to show the usefulness and efficiency of the suggested nonlinear sliding mode controllers. The first case study is a gyro system and the second one is Arneodo system. Numerical simulations are performed using MATLAB toolbox called Ninteger (Valério, 2005).

6.1. Fractional-order Gyro System

The dynamic model of fractional-order gyro system with model uncertainties and external disturbances is expressed as follows:

$$\begin{cases} {}_C D^{\alpha} x_1 = x_2 \\ {}_C D^{\alpha} x_2(t) = -100\dfrac{(1-\cos(x_1))^2}{\sin^3(x_1)} - 0.5x_2 - 0.05x_2^3 + \sin(x_1) \\ \qquad + 35.5\sin(25t)\sin(x_1) + \Delta f(X,t) + d(t) + u(t) \end{cases}$$

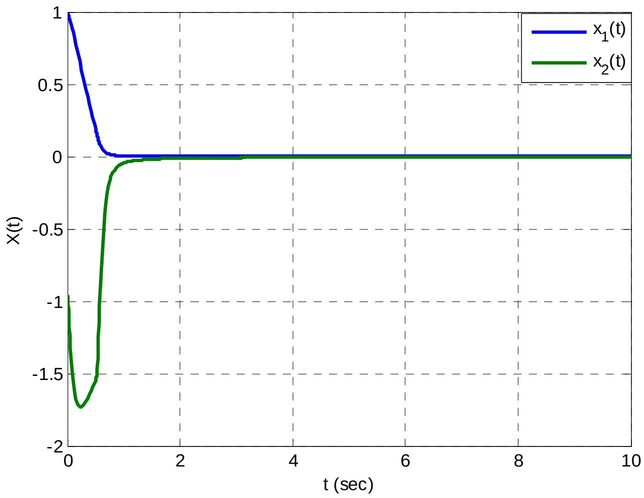
where x_1 represent the rotation angle, x_2 is rotation velocity, $100\dfrac{(1-\cos(x_1))^2}{\sin^3(x_1)}$ is the nonlinear resilience term, $0.5x_2$ and $0.05x_2^3$ are linear and nonlinear damping terms, and $35.5\sin(25t)$ is the parametric excitation.

In this case the fractional-order and uncertainty terms are considered as $\alpha = 0.98$ and $\Delta f(X,t) + d(t) = 0.25\cos(2\pi t)x_2 + 0.1\sin(2t)$ for the simulation. The initial conditions of the system is selected as: $x_1(0) = 1$ and $x_2(0) = -1$. Also common value parameters between two strategies (18)-(21) and (22)-(24) are given as:

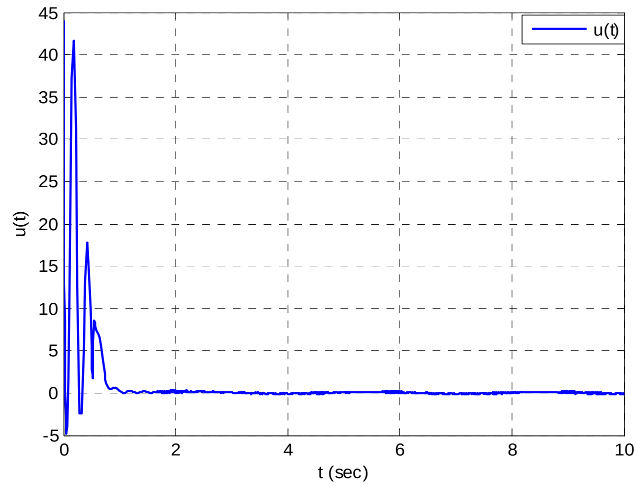
$$\lambda_1 = \lambda_2 = 7, \quad \eta_1 = \eta_2 = 1.5, \quad b_1 = 10$$

Also the distinctive value parameters are selected as: $\delta = \varepsilon = 0.01$ for (21), and $\varepsilon = 0.02$, $q/p = 1/3$ for (24).

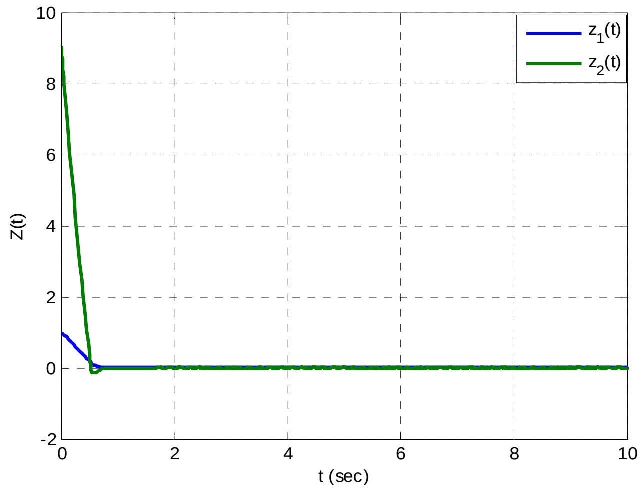
The system state trajectories (x_1, x_2), control signal, states of block transformation (z_1, z_2) and sliding surface are shown in Figure 3. The right hand sides figures are belong to the controller with sign integral sliding manifold (18), and the responses of controller with fractional integral sliding manifold (22) are depicted in the left hand side. Figure 3, confirms that the system states, sliding manifold and transformed states are converged to zero in a short time. By comparing the Figures 3(a)-(e) with (c)-(g), it can be seen that the convergence speed of the actual and transformed states is different which testifies the idea of Remark 6.



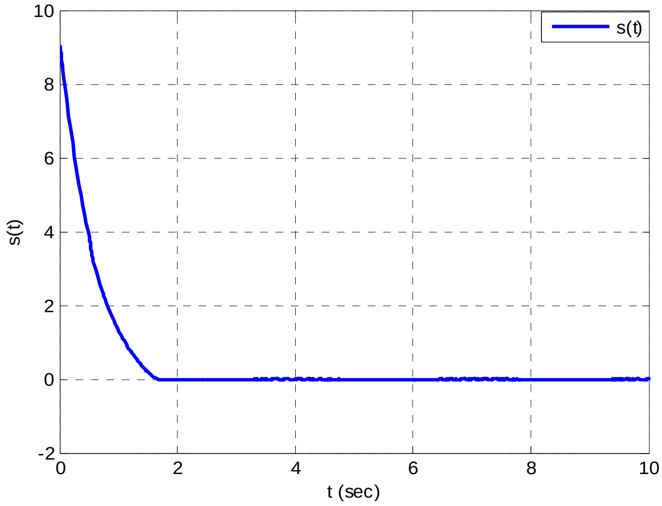
(a)



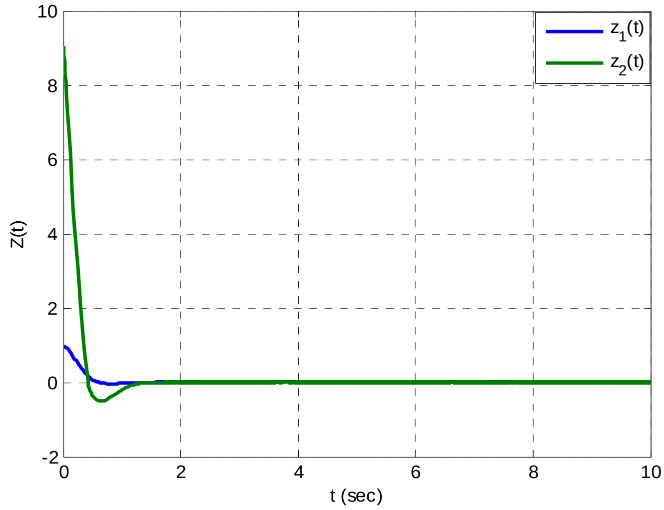
(b)



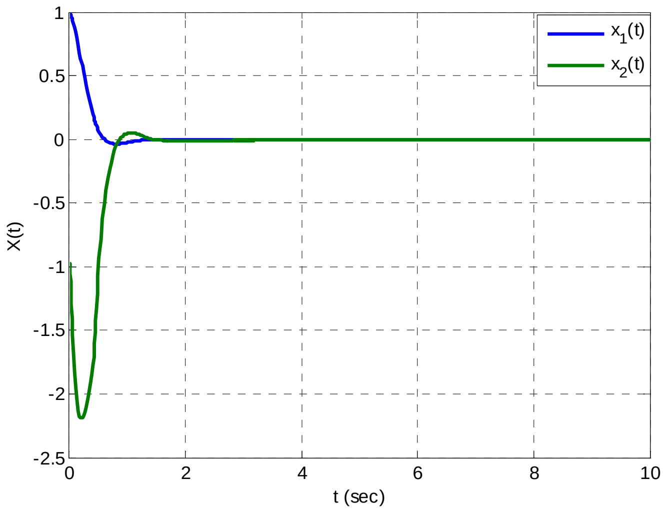
(c)



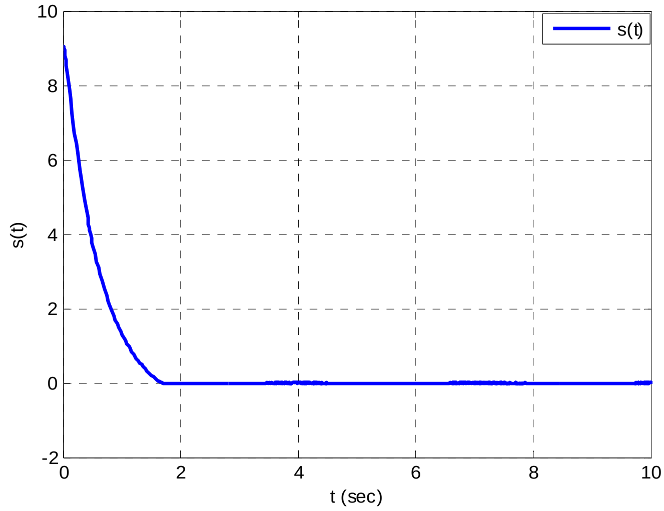
(d)



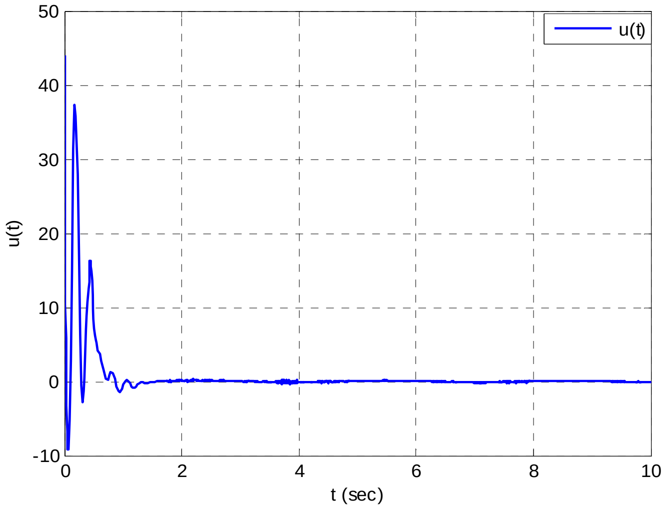
(g)



(e)



(h)



(f)

Fig. 3. Responses of system state trajectories (x_1 , x_2), control signal, block transformation states (z_1 , z_2) and sliding surface. **a-b-c-d:** The control law (21) with sign integral sliding manifold (18). **e-f-g-h:** The control law (24) with fractional integral sliding manifold (22).

6.2. Fractional-order Arneodo System

The performances of proposed controllers are tested on the fractional-order Arneodo system in this section. The dynamic equations of the uncertain system are presented as:

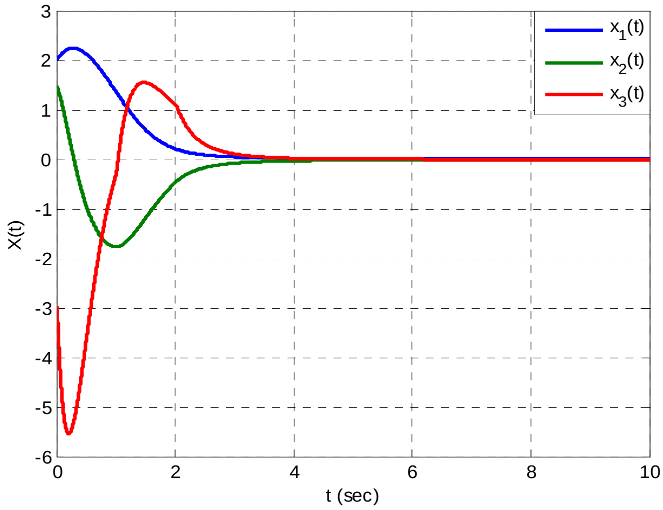
$$\begin{cases} {}_C D^{\alpha} x_1 = x_2 \\ {}_C D^{\alpha} x_2 = x_3 \\ {}_C D^{\alpha} x_3(t) = 5.5x_1 + x_1^3 - 3.5x_2 - x_3 + \Delta f(X,t) + d(t) + u(t) \end{cases}$$

The initial conditions and uncertainty term are selected as $x_1(0) = 2$, $x_2(0) = 1.5$, $x_3(0) = -3$ and $\Delta f(X,t) + d(t) = 0.1\cos(t)x_3 - 0.15\sin(t)$. The similar parameters of both controllers are declared as:

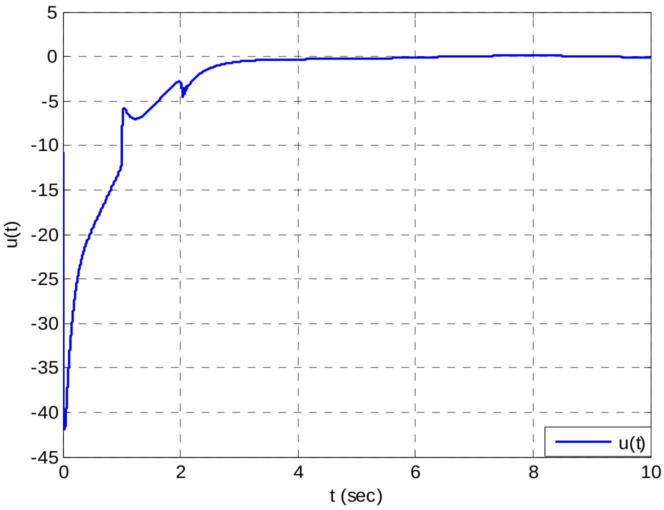
$$\alpha = 0.98, \quad \lambda_1 = \lambda_2 = 4, \quad \eta_1 = \eta_2 = 1.5, \quad b_1 = 2, \quad b_2 = 4$$

Besides, the distinctive value parameters considered as: $\delta = \varepsilon = 0.01$ for (18)-(21), and $\varepsilon = 0.02$, $q/p = 1/5$ for (22)-(24).

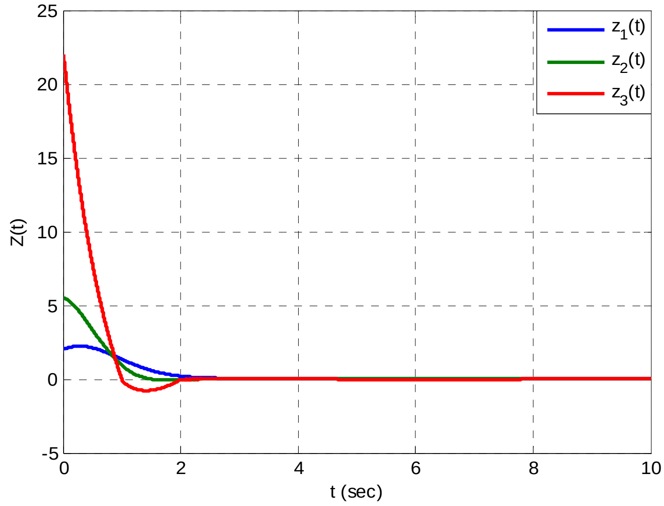
Figure 4 indicates the system state trajectories (x_1, x_2, x_3), control signal, states of block transformation (z_1, z_2, z_3) and sliding surface. From Figure 4(a) and (e), it is evident that the chaotic behaviour of the system has been suppressed and system states converge to zero for both controllers. Also fast convergence of the sliding surface is clear from Figure 4(d) and (h).



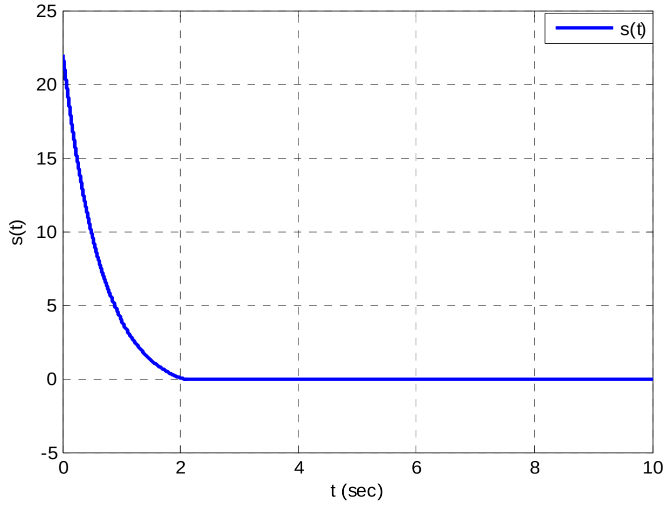
(a)



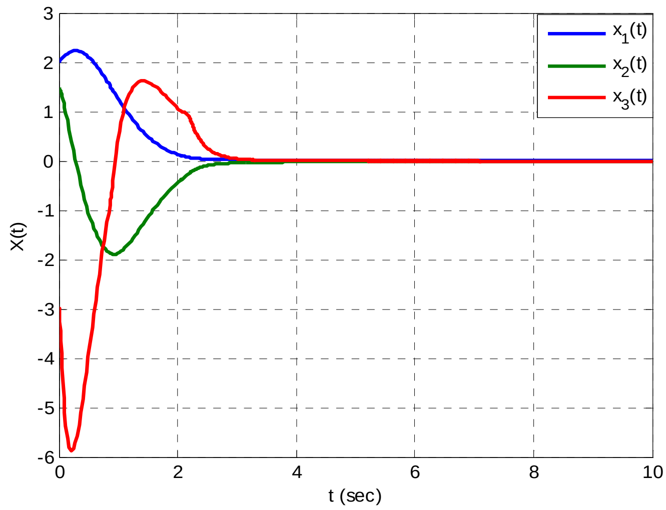
(b)



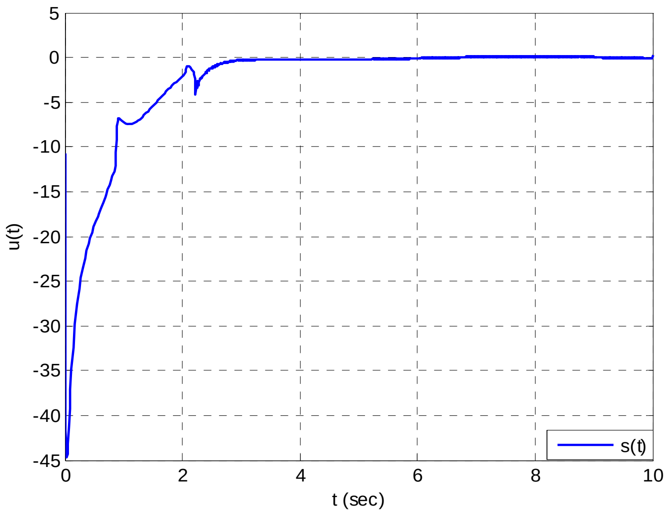
(c)



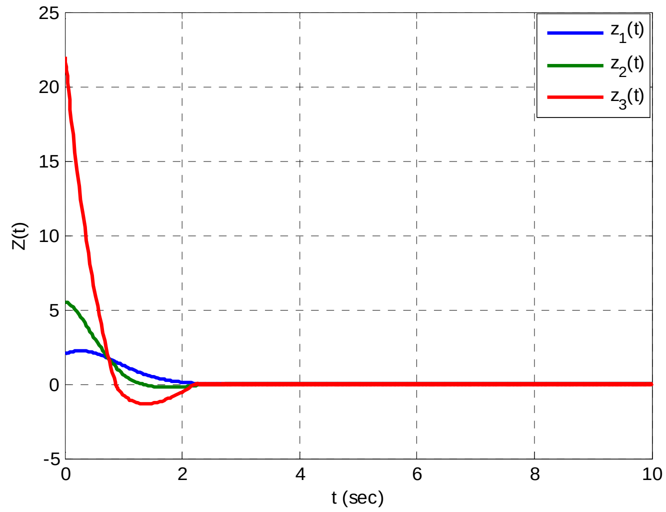
(d)



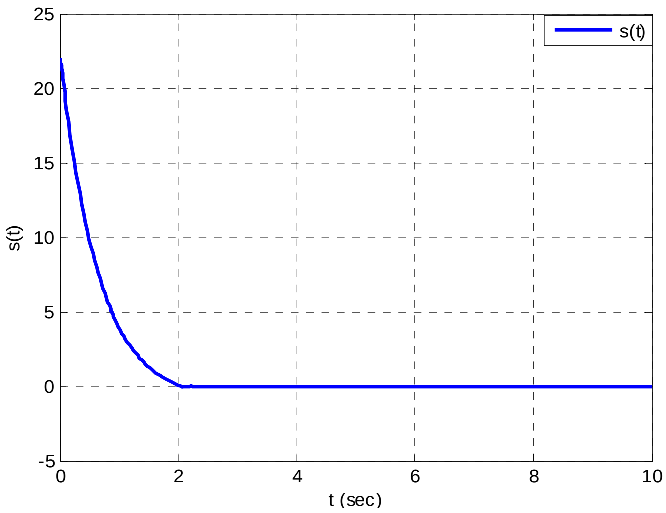
(e)



(f)



(g)



(h)

Fig. 4. Responses of system state trajectories (x_1 , x_2 , x_3), control signal, block transformation states (z_1 , z_2 , z_3) and sliding surface. **a-b-c-d:** The control law (21) with sign integral sliding manifold (18). **e-f-g-h:** The control law (24) with fractional integral sliding manifold (22).

Existence of $\mathrm{sgn}(z_n(t))$ function in the sliding manifold (18), is main cause of control law (21) chattering. Although, replacing $\mathrm{sgn}(z_n(t))$ by the non-chatter function (27) is a simple remedy for mentioned problem, but using the non-chatter function will cause small tracking error which is evident from Figures 3(a) and 4(a). Meanwhile, the mentioned problem is not much prominent in the sliding surface (22) and control law (24).

7. CONCLUSIONS

In this paper, the problem of designing fast converging robust controllers for a Caputo derivative based nonlinear fractional-order uncertain system is investigated. This fractional-order system have high-relative-degree with model uncertainties and external disturbances. We proposed two novel types of nonlinear sliding surfaces in order to have a fast zero convergence. Hence, two new nonlinear fractional-order sliding mode controllers are suggested. The suggested controllers guarantee the fractional-order system last state convergence, and the other states convergence are assured by control gains of the block transformation. The asymptotic stability of the proposed control schemes is proved using the fractional-order stability theorems. Computer simulations reveal the performance of presented robust controllers in a short time convergence for uncertain fractional-order gyro and Arneodo systems.

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