ABOUT THE VARIABLE CAUSALITY DYNAMICAL SYSTEMS

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Abstract: *This paper deals with dynamical systems which models physical objects whose causal input-output ordering is changing during their evolution. Such a system is named Variable Causality Dynamical System (VCDS). VCDS are controlled from outside by a new input called causal ordering signal sharing the same set of state variables. In VCDS, all the variables, except the causal ordering signal, are gathered in two forms of so called global variables as current global variable and desired global variable. In this paper, different approaches of the causality concept are analyzed and there are proposed formal definitions for covariance and causality properties of variables and relations irrespective of the time domain. Two examples of VCDS are presented here, one is described by a nonlinear algebraic existence relation and the other by a proper linear differential equation*

Keywords: *Dynamical systems, Covariance, Causality, Inference.*

1. INTRODUCTION

As it is presented in [1], the causality concept is a very general one and it still is not precisely defined. Causality is a subject of debates in very different scientific communities from philosophy, biology, till technical and social sciences. The structure of the so-called causal theories and some fundamental forms of scientific inference are developed in [2]. According to this approach, a causal law is a statement that a change in the value of one variable is sufficient to produce a change in the value of another, without the operation of intermediate causes.

There are several different conceptions of cause as Positivist and Essentialists theories of causation, [3] which stresses the observations of regularities considering high correlations demonstrate or are synonymous with causation. Another approach the Essentialists theories of causation, where it is considered that cause should only be used to refer to variables that explain phenomena.

In the conception Stuart-Mill, [3], three main factors determine an inference to be causal: a) Cause has to precede the effect, b) The cause and effect have to be related c) Other explanations of the cause-effect relationship have to be eliminated. Several methods of causal inference analysis, in [3] are mentioned. There are many ways of knowing and different cultures uses different expectations and norms about causality, so much of the research process centers around what are the true causal or independent variables [4]. One important direction of modeling that rise the problem of causality is that of the bond graphs, [5], [6], [7].

The so called hybrid bond graph augment traditional bond graph by a variable causality switching element to facilitate models with mixed continuous/discrete, hybrid behavior.

An investigation of causal state theory and graphical causal models with applications in computational mechanics and the so-called ε-machines is developed in [8]. The problem of variable causality applied to power electronic converters is analyzed in [9]. In [10] it is argued that a regularity notion of causality can only be meaningfully defined for systems with linear interactions among their variables, with particular reference to the problem of causal inference in complex genetic systems. A new approach of dynamical systems conceived by Willems [11], [12], ignores the input-output causal ordering. It defines the so called the behavioral approach of dynamical systems formed by the triptych with the behavior of the system in the center, the behavioral equations and latent variables as side notions.

In our approach [1], the covariance and causality relations are defined in abstract space, irrespective of the time domain. To raise debates on them in the following the main aspects of this approach are retaken. Many applications on walking robots of the VCDS approach are developed including simulations in Mathlab environment proving the efficiency of this approach.

2. CAUSAL VARIABLES AND CAUSALITY RELATIONS

The fundamental notion in mathematical modeling of objects (physical or abstract) is that of variable. A variable V is the triple

$$V = \{v, \mathrm{V}, \mathsf{V}\},\ v \in \mathrm{V} \subseteq \mathsf{V}\ , \tag{1}$$

where: V named variable universe, is a set endowed with a well defined mathematical structure; V named variable domain, is a subset of V; v named the variable instant, is the generic element of V. The variable V is finite dimensional of the order p if it exists a one-to-one application $\mathsf{V} \to \mathsf{R}^p$. Let $\boldsymbol{R}(V_i, V_j)$ be a binary relation on the Cartesian product $\mathrm{V}_i \times \mathrm{V}_j$, called a relation between the variable V_i and V_j,

$$\boldsymbol{R}(V_i, V_j) \subseteq \mathrm{V}_i \times \mathrm{V}_j \subseteq \mathsf{V}_i \times \mathsf{V}_j\ . \tag{2}$$

Such a relation, as a crisp relation, can be expressed by its membership function (characteristic function),

$$\mu_R(v_i, v_j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in \boldsymbol{R}(V_i, V_j) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

A relation can be described by an equation defined as an equilibrium condition [13], on an equating space E

$$f_1(v_i, v_j) = f_2(v_i, v_j),\quad f_1, f_2 : \mathrm{V}_i \times \mathrm{V}_j \to \mathsf{E}\ . \tag{4}$$

If the equation space E is a finite m-dimensional linear space, then the equilibrium condition (4) can be expressed as an equation

$$R(v_i, v_j) = 0\ , \tag{5}$$

where $R(v_i, v_j) = f_1(v_i, v_j) - f_2(v_i, v_j) = 0$ $R : \mathrm{V}_i \times \mathrm{V}_j \to \mathsf{E}$ .The equilibrium equation (5) is called the existence equation for the relation (2), if each pair $(v_i, v_j) \in \boldsymbol{R}(V_i, V_j)$ verifies equation (2) and any solution (v_i, v_j) of $R(v_i, v_j) = 0$, belongs to $\boldsymbol{R}(V_i, V_j)$. In such a case, there is the equivalence

$$\boldsymbol{R}(V_i, V_j) \Leftrightarrow R(v_i, v_j) = 0 \tag{6}$$

Such a relation is an m-order relation, which involves m restrictions on its variables. A relation $\boldsymbol{R}(V_i, V_j)$ is well defined if its projection on each variable universe equals to the corresponding variable domain,

$$Pr\{\boldsymbol{R}(V_i, V_j) / \mathsf{V}_i\} = \mathrm{V}_i \tag{7}$$

$$Pr\{\boldsymbol{R}(V_i, V_j) / \mathsf{V}_j\} = \mathrm{V}_j \tag{8}$$

where

$$Pr\{\boldsymbol{R}(V_i, V_j) / \mathsf{V}_i\} = \{v_i \in \mathrm{V}_i,\ \exists v_j \in \mathrm{V}_j,\ \mu_{\boldsymbol{R}}(v_i, v_j) = 1\} \tag{9}$$

$$Pr\{\boldsymbol{R}(V_i, V_j) / \mathsf{V}_j\} = \{v_j \in \mathrm{V}_j,\ \exists v_i \in \mathrm{V}_i, \mu_{\boldsymbol{R}}(v_i, v_j) = 1\} \tag{10}$$

are the projection of $\boldsymbol{R}$ on the universes $\mathsf{V}_i, \mathsf{V}_j$ respectively. Two variables, V_i, V_j are covariant variables if there is a nonempty set of index $A \ni \alpha$ and a nonempty set variables

$$\boldsymbol{G}_A = \{G_\alpha\}_{\alpha \in A},\ G_\alpha = \{g_\alpha, \mathrm{G}_\alpha, \mathsf{G}_\alpha\} \tag{11}$$

called the set of intermediate variables, such a way for each intermediate variable G_α there are two families of functions $\mathbf{F}^{\alpha}_{X_i^{\alpha}}$, $\mathbf{F}^{\alpha}_{X_j^{\alpha}}$ called covering functions of the relation

$$\mathbf{F}^{\alpha}_{X_i^{\alpha}} = \{f^{\alpha}_{x_i^{\alpha}}\}_{x_i^{\alpha} \in \mathrm{X}_i^{\alpha} \subseteq \mathsf{X}_i^{\alpha}},\ f^{\alpha}_{x_i^{\alpha}} : \mathrm{G}_{\alpha} \to \mathrm{V}_i, \quad (12)$$

$$\mathbf{F}^{\alpha}_{X_j^{\alpha}} = \{f^{\alpha}_{x_j^{\alpha}}\}_{x_j^{\alpha} \in \mathrm{X}_j^{\alpha} \subseteq \mathsf{X}_j^{\alpha}},\ f^{\alpha}_{x_j^{\alpha}} : \mathrm{G}_{\alpha} \to \mathrm{V}_j \quad (13)$$

so that each family cover the relation

$$v_i = f^{\alpha}_{x_i^{\alpha}}(\mathrm{g}_{\alpha}),\ \bigcup_{x_i^{\alpha} \in \mathrm{X}_i^{\alpha}} f^{\alpha}_{x_i^{\alpha}}(\mathrm{G}_{\alpha}) = \mathrm{V}_i \quad (14)$$

$$v_j = f^{\alpha}_{x_j^{\alpha}}(\mathrm{g}_{\alpha}),\ \bigcup_{x_j^{\alpha} \in \mathrm{X}_j^{\alpha}} f^{\alpha}_{x_j^{\alpha}}(\mathrm{G}_{\alpha}) = \mathrm{V}_j\,. \quad (15)$$

The two families $\mathbf{F}^{\alpha}_{X_i^{\alpha}}$, $\mathbf{F}^{\alpha}_{X_j^{\alpha}}$ are called parameter families of functions. The triple $\{\alpha, f^{\alpha}_{x_i^{\alpha}}, f^{\alpha}_{x_j^{\alpha}}\}$ is called an instant of the covariant variables V_i, V_j and relations (14), (15) express one parametric representation of the two covariant variables. The parametric representation (14), (15) may be expressed also as

$$v_i = f_i^{\alpha}(\mathrm{g}_{\alpha}, x_i^{\alpha}) \quad (16)$$

$$v_j = f_j^{\alpha}(\mathrm{g}_{\alpha}, x_j^{\alpha}), \quad (17)$$

where x_i^{α}, x_j^{α} are instants of the new variables,

$$X_i^{\alpha} = \{x_i^{\alpha}, \mathrm{X}_i^{\alpha}, \mathsf{X}_i^{\alpha}\}, \quad (18)$$

$$X_j^{\alpha} = \{x_j^{\alpha}, \mathrm{X}_j^{\alpha}, \mathsf{X}_j^{\alpha}\} \quad (19)$$

called state variables. The covering conditions on the families of parametric functions $f^{\alpha}_{x_i^{\alpha}}, f^{\alpha}_{x_j^{\alpha}}$ will assure each variable V_i and V_j to be completely involved in the covariance, namely

$$\forall v_i \in \mathrm{V}_i, \exists\ (\alpha \in A, g_{\alpha} \in \mathrm{G}_{\alpha}, f^{\alpha}_{x_i^{\alpha}} \in \mathbf{F}^{\alpha}_{X_i^{\alpha}}) \quad (20)$$

$$v_i = f^{\alpha}_{x_i^{\alpha}}(g_{\alpha}) = f_i^{\alpha}(\mathrm{g}_{\alpha}, x_i^{\alpha}) \quad (21)$$

$$\forall v_j \in \mathrm{V}_j, \exists\ (\alpha \in A, g_{\alpha} \in \mathrm{G}_{\alpha}, f^{\alpha}_{x_j^{\alpha}} \in \mathbf{F}^{\alpha}_{X_j^{\alpha}}) \quad (22)$$

$$v_j = f^{\alpha}_{x_j^{\alpha}}(g_{\alpha}) = f_j^{\alpha}(\mathrm{g}_{\alpha}, x_j^{\alpha})\,. \quad (23)$$

So, for each pair $(v_i, v_j) \in \boldsymbol{R}(V_i, V_j)$, and each family $\mathbf{F}^{\alpha}_{X_i^{\alpha}}$ or $\mathbf{F}^{\alpha}_{X_j^{\alpha}}$, it exists a function whose graphic to contain this pair (v_i, v_j). These families are labeled by the variables $X_i^{\alpha}, X_j^{\alpha}$, which become state variables. If relations (20), (21) are true it is not necessary (22), (23) to be true too and vice-versa. One instant $\{\alpha, f^{\alpha}_{x_i^{\alpha}}, f^{\alpha}_{x_j^{\alpha}}\}$ of two covariant variables V_i, V_j defines a relation, called covariance relation,

$$\boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}(V_i, V_j) \subseteq \mathrm{V}_i \times \mathrm{V}_j \subseteq \mathsf{V}_i \times \mathsf{V}_j \quad (24)$$

$$\boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}(V_i, V_j) = \{(v_i, v_j), v_i = f^{\alpha}_{x_i^{\alpha}}(g_{\alpha}), v_j = f^{\alpha}_{x_j^{\alpha}}(g_{\alpha}), \forall g_{\alpha} \in \mathrm{G}_{\alpha})$$

whose membership function is,

$$\mu_{\boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}}(v_i, v_j) = 1 \ \text{ iff } (v_i, v_j) \in \boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}(V_i, V_j) \quad (25)$$

Because the relation $\boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}(V_i, V_j)$ could be different of $\mathrm{V}_i \times \mathrm{V}_j$, reflects the covariant character of the two variables. Two variables V_i, V_j are well covariant if for any instant $\{\alpha, f^{\alpha}_{x_i^{\alpha}}, f^{\alpha}_{x_j^{\alpha}}\}$ the set of correlated pairs $\boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}(V_i, V_j)$ is the same, which denoted $\boldsymbol{R}_{ij}(V_i, V_j)$

$$\boldsymbol{R}^{\alpha}_{X_i^{\alpha} X_j^{\alpha}}(V_i, V_j) = \boldsymbol{R}_{ij}(V_i, V_j),\ \forall \{\alpha, f^{\alpha}_{x_i^{\alpha}}, f^{\alpha}_{x_j^{\alpha}}\} \quad (26)$$

Between two well covariant variables, V_i, V_j it exists a nonempty binary relation

$$\boldsymbol{R}_{ij}(V_i, V_j) \subseteq \mathrm{V}_i \times \mathrm{V}_j \subseteq \mathsf{V}_i \times \mathsf{V}_j \quad (27)$$

but does not exist a function type dependency between them. The covariance relation (26) can be expressed by an equilibrium equation

$$R_{ij}(v_i, v_j) = 0,\ (v_i, v_j) \in \mathrm{V}_i \times \mathrm{V}_j \subseteq \mathsf{V}_i \times \mathsf{V}_j \quad (28)$$

but has no physical meaning to withdraw from it the function type dependencies, $v_i = f_i(v_j)$ or $v_j = f_j(v_i)$. Two variables V_i, V_j are called independent variables if it is not possible to establish a covariance relation between them. The variables V_i, V_j are independent if one of the three conditions takes place:

$$A = \varnothing \ \text{ or } \ \boldsymbol{G}_A = \varnothing \quad (29)$$

$$\mathbf{F}^{\alpha}_{X_i^{\alpha}} = \varnothing, \forall \alpha \in A \quad (30)$$

$$\mathbf{F}^{\alpha}_{X_j^{\alpha}} = \varnothing, \forall \alpha \in A \quad (31)$$

Two any variables V_i, V_j which could be covariant but for which a covariance relation is not established yet are called uncharacterized variables. Uncharacterized variables are considered to be independent variables. Two covariant variables V_i, V_j characterized by $A, \boldsymbol{G}_A, \mathbf{F}^{\alpha}_{X_i^{\alpha}}, \mathbf{F}^{\alpha}_{X_j^{\alpha}}$ are called causal variables if

$$A = \{\alpha_1, \alpha_2\}\ ,\ \alpha_1 = ij\ ,\ \alpha_2 = ji \tag{32}$$

$$\boldsymbol{G}_A = \{\mathrm{G}_{\alpha_1}, \mathrm{G}_{\alpha_2}\} = \{\mathrm{G}_{ij}, \mathrm{G}_{ji}\} = \{\mathrm{V}_i, \mathrm{V}_j\} \tag{33}$$

$$\mathbf{F}^{\alpha}_{X_i^{\alpha}} \neq \varnothing,\ \ \mathbf{F}^{\alpha}_{X_j^{\alpha}} \neq \varnothing\ . \tag{34}$$

The parametric representation (16), (17) takes the following two explicit forms (35), (36) and (39), (40).

For $\alpha = \alpha_1 = ij$

$$v_i = f_i^{ij}(v_i, x_i^{ij})\ ,\ x_i^{ij} \in X_i^{\alpha_1} = X_i^{ij} \tag{35}$$

$$v_j = f_j^{ij}(v_i, x_j^{ij})\ ,\ x_j^{ij} \in X_j^{\alpha_1} = X_j^{ij}\ , \tag{36}$$

which illustrates the so called causality $V_i \to V_j$ where x_i^{ij}, x_j^{ij} are the state instants of the respectively variables $\mathrm{V}_i, \mathrm{V}_j$ in this causality ordering. The state variables are,

$$X_i^{\alpha_1} = \{x_i^{\alpha_1}, \mathrm{X}_i^{\alpha_1}, \mathsf{X}_i^{\alpha_1}\} = \{x_i^{ij}, \mathrm{X}_i^{ij}, \mathsf{X}_i^{ij}\} = X_i^{ij}\ , \tag{37}$$

$$X_j^{\alpha_1} = \{x_j^{\alpha_1}, \mathrm{X}_j^{\alpha_1}, \mathsf{X}_j^{\alpha_1}\} = \{x_j^{ij}, \mathrm{X}_j^{ij}, \mathsf{X}_j^{ij}\} = X_j^{ij} \tag{38}$$

For $\alpha = \alpha_2 = ji$

$$v_i = f_i^{ji}(v_j, x_i^{ji})\ ,\ x_i^{ji} \in \mathrm{X}_i^{\alpha_2} = \mathrm{X}_i^{ji} \tag{39}$$

$$v_j = f_j^{ji}(v_j, x_j^{ji})\ ,\ x_j^{ji} \in \mathrm{X}_j^{\alpha_2} = \mathrm{X}_j^{ji}\ , \tag{40}$$

which illustrates the so called causality $V_j \to V_i$ where x_i^{ji}, x_j^{ji} are the state instants of the respectively variables V_i, V_j in this causality ordering. The state variables are,

$$X_i^{\alpha_2} = \{x_i^{\alpha_2}, \mathrm{X}_i^{\alpha_2}, \mathsf{X}_i^{\alpha_2}\} = \{x_i^{ji}, \mathrm{X}_i^{ji}, \mathsf{X}_i^{ji}\} = X_i^{ji}\ , \tag{41}$$

$$X_j^{\alpha_2} = \{x_j^{\alpha_2}, \mathrm{X}_j^{\alpha_2}, \mathsf{X}_j^{\alpha_2}\} = \{x_j^{ji}, \mathrm{X}_j^{ji}, \mathsf{X}_j^{ji}\} = X_j^{ji} \tag{42}$$

Each state instant specify a function from the sets $\mathbf{F}^{\alpha}_{X_i^{\alpha}}, \mathbf{F}^{\alpha}_{X_j^{\alpha}}$.

As state variables internally characterize variables involved in a causal relation, the state variables attached to a variable must be the same irrespective of the causality ordering, so

$$X_i^{ij} = X_i^{ji} = X_i = \{x_i, \mathrm{X}_i, \mathsf{X}_i\} \tag{43}$$

$$X_j^{ij} = X_j^{ji} = X_j = \{x_j, \mathrm{X}_j, \mathsf{X}_j\} \tag{44}$$

and the parametric equations (35), (36), (39), (40) take the following specific forms depending on the causality ordering. For the causality ordering $V_i \to V_j$, (35), (36) become

$$v_i = f_i^{ij}(v_i, x_i)\ ,\ x_i \in X_i \tag{45}$$

$$v_j = f_j^{ij}(v_i, x_j)\ ,\ x_j \in X_j\ , \tag{46}$$

where v_i acts as cause or input and v_j is an effect of the cause v_i. This effect v_j depends on the instant x_j of the state variable X_j attached to it.

For the causality ordering $V_j \to V_i$, (39), (40) become

$$v_i = f_i^{ji}(v_j, x_i)\ ,\ x_i \in X_i \tag{47}$$

$$v_j = f_j^{ji}(v_j, x_j)\ ,\ x_j \in X_j\ , \tag{48}$$

where v_j acts as cause or input and v_i is an effect of the cause v_j. This effect v_i depends on the instant x_i of the state variable X_i attached to it. Equations (46), (47) are input-state-output equations and relations (45) (48) are transition state equations. Any binary relation between two causal variables is called a causal relation. . For n-ary relations,

$$\boldsymbol{R}(V_1, V_2 ..., V_n) = \boldsymbol{R}(V)\ ,\ V = \{V_1, V_2 ..., V_n\} \tag{49}$$

In addition, n variables $V_1, V_2 ..., V_n$ are covariant, independent or causal if any two variables of them have the above-defined properties.

3. CAUSAL ORDERING IN DYNAMICAL SYSTEMS

Several definitions of the system notion there are well known from any System Theory textbook [14], [15], [16]. According to the definition issued from thermodynamics, "a system is a part (a fragment) of the universe for which one inside and one outside can be delimited from behavioral point of view",[17].

The mathematical model of a physical system is a pair $\boldsymbol{S} = \{\mathrm{V}, \boldsymbol{R}\}$ where $V = \{V_i\}_{i=1:n}$ is a set of variables and $\boldsymbol{R} = \boldsymbol{R}(V)$ is a causal relation between them. If p_i is the order of the variable V_i, then there are $p = \sum_{i=1}^{n} p_i$ scalar components of the variables involved in system.

Suppose that the relation $\boldsymbol{R}$ is of the order m. The system acts as a restriction among its variables and delimits its inside. All the other variables and relations different of V and $\boldsymbol{R}$, denoted $\bar{V}$ and $\bar{\boldsymbol{R}}$ belong to the outside of the system denoted by $\bar{\boldsymbol{S}} = \{\bar{V}, \bar{\boldsymbol{R}}\}$. This is the so-called an un-oriented system and it is similar to the model of Willems in his behavioral approach of systems [11], [12], [13].

As the model represents a physical system, the relation $\boldsymbol{R}$, called also the existence relation, is true as far as the physical system exists.

At this stage, looking at the un-oriented system $\boldsymbol{S}$, we can observe only the instants of the variables $v=\{v_i\}_{i=1:n}$.

One instant v is a realization of the physical system and it verifies the existence relation $\boldsymbol{R}$. The un-oriented system looks like an isolated one from the universe it belongs to, so any variable $\mathrm{W} \in \bar{\boldsymbol{S}}$ is independent with respect to any variable $\{V_i\}_{i=1:n}$ of $\boldsymbol{S}$.

But, any physical system whose model is $\boldsymbol{S}$, as a part of the universe, is not isolated one, it has changes of energy, material and information with the outside. Let W_k be a variable from the outside of $\boldsymbol{S}$, $W_k = \{w_k, \mathrm{W}_k, \mathsf{W}_k\} \in \bar{\boldsymbol{S}}$.

A variable $W_k \in \bar{\boldsymbol{S}}$ is assigned to a variable $V_i \in \boldsymbol{S}$, $V_i = \{v_i, \mathrm{V}_i, \mathsf{V}_i\} \in \boldsymbol{S}$, denoted $W_k \to V_i$, if $\mathsf{W}_k = \mathsf{V}_i$ $\mathrm{W}_k \subseteq \mathrm{V}_i$, $v_i = w_k$, that means the instant v_i takes the value of the instant w_k. Through this assignment process, a new variable $U_j \in \boldsymbol{S} \cap \bar{\boldsymbol{S}}$ is created,

$$U_r = \{u_r, \mathrm{U}_r, \mathsf{U}_r\} = \{w_k, \mathrm{V}_i, \mathsf{V}_i\}. \tag{50}$$

The variable U_j is a cause for the system $\boldsymbol{S}$ or an input variable. The set of variables $\{W_k\}_{k=1,2,..}$ assigned to the system $\boldsymbol{S}$, could be independent variables, which is not a necessary condition. Let

$$Y_k = \{y_k, \mathrm{Y}_k, \mathsf{Y}_k\} \in \bar{\boldsymbol{S}} \tag{51}$$

be a variable from outside. If $\mathsf{Y}_k = \mathsf{V}_j$, $Y_k = V_j$, $y_k = v_j$ then the variable $V_j \in \boldsymbol{S}$ is assigned to the outside world so it becomes an output of the system $\boldsymbol{S}$.

4. VARIABLE CAUSALITY DYNAMICAL SYSTEMS

There are many physical systems where the causal ordering is controlled from outside expressed by a new variable $q \in Q = \{q_\beta\}_{\beta \in B}$ and where the terminal variables $V = \{V_1, V_2 ..., V_n\}$ are some times inputs, and other times they are outputs. Let $X = \{X_1, X_2 ..., X_n\}$ be the state variables attached to each terminal variable in such a way to reestablish the univocity of the selected variables as being outputs with respect to the variables selected as to be inputs. They are internal variables.

In the case of VCDS description, there are no explicitly defined input and output variables. All the variables, the terminal variables $V = \{V_1, V_2 ..., V_n\}$ and the internal variables $X = \{X_1, X_2 ..., X_n\}$, satisfy the existence relation of the system named System Existence Relation (SER) $\boldsymbol{R}(V_1, V_2 ..., V_n) = \boldsymbol{R}(V)$.

As the system exist, the SER is true according to the causality ordering, defined as a new variable, available at that time instant. Particularly let we consider that any variable, written without time index (t, k), is interpreted, depending of the context, as being its value at the current continuous time or discrete time. Let we denote by

$$\xi = \{V_1, V_2 ..., V_n, X_1, X_2 ..., X_n\} \tag{52}$$

the current value of the so called global variable of the system and denote by

$$\hat{\xi} = \{\hat{V}_1, \hat{V}_2 ..., \hat{V}_n, \hat{X}_1, \hat{X}_2 ..., \hat{X}_n\} \tag{53}$$

the desired value of the global variable. The VCDS evolution equation is of the form,

$$\xi = f(\hat{\xi}, q) \tag{54}$$

For algebraic dynamical systems, the equation (22) takes the form,

$$\xi_{t+\varepsilon} = f(\xi_t, \hat{\xi}_t, q_t) \tag{55}$$

5. NONLINEAR VCDS SYSTEM WITH FOUR PAIRS OF STATES

Let us consider a sytem about it is known having a causal relation $\boldsymbol{R}_{ij}(V_i, V_j)$ as in Fig.1

In the causality ordering $V_i \rightarrow V_j$ the causal relation can be covered by two functions, so the label set X_i contains two elements, let we denote them as s_j^1, s_j^2 so $x_j \in \mathrm{X}_j = \{s_j^1, s_j^2\}$.

Also, for the causality ordering $V_j \rightarrow V_i$ the causal relation can be covered by two functions, so the label set X_i contains two elements, let we denote them as s_i^1, s_i^2 so $x_i \in \mathrm{X}_i = \{s_i^1, s_i^2\}$.

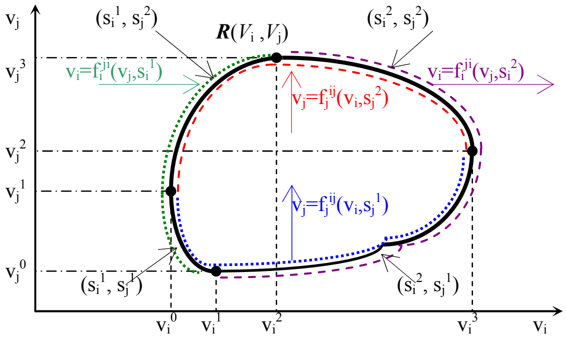

Fig 1. Example of a nonlinear VCDS with 4 pairs of ststes.

For the causality ordering $V_i \rightarrow V_j$, $\alpha = \alpha_1 = ij$, the state transition equation, whose solution is the state x_i is given by ,

$$v_i = f_i^{ij}(v_i, x_i), \; x_i \in \mathrm{X}_i = \{s_i^1, s_i^2\} \tag{56}$$

and the input-state-output equation is

$$v_j = f_j^{ij}(v_i, x_j), \; x_j \in \mathrm{X}_j = \{s_j^1, s_j^2\}, \tag{57}$$

For the causality ordering $V_j \rightarrow V_i$, $\alpha = \alpha_2 = ji$, the state transition equation, whose solution is the state x_j is given by

$$v_j = f_j^{ji}(v_j, x_j), \; x_j \in \mathrm{X}_j = \{s_j^1, s_j^2\} \tag{58}$$

and the input-state-output equation is

$$v_i = f_i^{ji}(v_j, x_i), \; x_i \in \mathrm{X}_i = \{s_i^1, s_i^2\} \tag{59}$$

In this example, for any terminal pair (v_i, v_j) there is a unique state pair $(x_i, x_j) \in \mathrm{X}_i \times \mathrm{X}_j$ which completely characterises the VCDS behaviour. As a whole there are four such a state pairs.

Explicite expression of VCDS

The actual and desired global variables are

$$\xi = \{V_i, V_j, X_i, X_j\} \qquad \hat{\xi} = \{\hat{V}_i, \hat{V}_j, \hat{X}_i, \hat{X}_j\}$$

and the VCDS evolution equation is

$$\xi = f(\hat{\xi}, q),$$

where, the expression of f is , as follows:

For the causality ordering $V_i \rightarrow V_j$, $q = ij$

$$v_i = \hat{v}_i, x_j = \hat{x}_j, \; v_j = f_j^{ij}(v_i, x_j), \; x_j \in \mathrm{X}_j = \{s_j^1, s_j^2\}$$

$$x_i = \begin{cases} s_i^1, & \left[v_i \in [v_i^0, v_i^1) \,\&\, x_j = s_j^1\right] \vee \left[v_i \in (v_i^0, v_i^2] \,\&\, x_j = s_j^2\right] \\ s_i^2, & \left[v_i \in [v_i^2, v_i^3) \,\&\, x_j = s_j^1\right] \vee \left[v_i \in (v_i^2, v_i^3] \,\&\, x_j = s_j^2\right] \end{cases}$$

For the causality ordering $V_j \rightarrow V_i$, $q = ji$

$$v_j = \hat{v}_j, \; x_i = \hat{x}_i, \; v_i = f_i^{ji}(v_j, x_i), \; x_i \in \mathrm{X}_i = \{s_i^1, s_i^2\}$$

$$x_j = \begin{cases} s_j^1, & \left[v_j \in (v_j^0, v_j^1] \,\&\, x_i = s_i^1\right] \vee \left[v_j \in [v_j^0, v_j^2) \,\&\, x_i = s_i^2\right] \\ s_j^2, & \left[v_j \in (v_j^1, v_j^3] \,\&\, x_i = s_i^1\right] \vee \left[v_j \in [v_j^2, v_j^3) \,\&\, x_i = s_i^2\right] \end{cases}$$

6. LINEAR DIFFERENTIAL VCDS SYSTEM WITH COMMON STATE VARIABLES

Let us consider a proper linear differential system of the second order, with V_i, V_j as terminal variables, described by a causal relation defined by the equilibrium equation, $a_2\ddot{v}_j + a_1\dot{v}_j + a_0 v_j - b_2\ddot{v}_i + b_1\dot{v}_i + b_0 v_i = 0$, $a_2 \neq 0, \; b_2 \neq 0$.

For the causality ordering $V_i \rightarrow V_j$, $q = \boldsymbol{0}$, $v_i = \hat{v}_i$, a state realization is,

$$\dot{x}_1 = -\frac{a_1}{a_2} \cdot x_1 + x_2 + (b_1 - b_2\frac{a_1}{a_2}) \cdot \hat{v}_i$$

$$\dot{x}_2 = -\frac{a_0}{a_2} \cdot x_1 + (b_0 - b_2\frac{a_0}{a_2}) \cdot \hat{v}_i$$

$$v_j = \frac{1}{a_2} \cdot x_1 + \frac{b_2}{a_2} \cdot \hat{v}_i \Leftrightarrow a_2 v_j - b_2\hat{v}_i + x_1 = 0$$

$$\dot{x} = A^{ij} \cdot x + B^{ij} \cdot \hat{v}_i, \qquad v_j = C^{ij} \cdot x + D^{ij} \cdot \hat{v}_i$$

$x = [x_1 \; x_2]^T$, $X = \{x, \mathrm{X}, \mathrm{X}\}$, $\mathrm{X} = \mathrm{X} = \mathrm{R}^2$

$$A^{ij} = \begin{bmatrix} -\frac{a_1}{a_2} & 1 \\ -\frac{a_0}{a_2} & 0 \end{bmatrix} \; B^{ij} = \begin{bmatrix} b_1 - b_2\frac{a_1}{a_2} \\ b_0 - b_2\frac{a_0}{a_2} \end{bmatrix} \; C^{ij} = [\frac{1}{a_2} \; 0] \; D^{ij} = \frac{b_2}{a_2}$$

For the causality ordering $V_j \rightarrow V_i$, $q = \boldsymbol{1}$, $v_j = \hat{v}_j$ state realization is,

$$\dot{x}_1 = -\frac{b_1}{b_2} x_1 + x_2 - (a_1 - a_2\frac{b_1}{b_2})\hat{v}_j$$

$$\dot{x}_2 = -\frac{b_0}{b_2}x_1 - (a_0 - a_2\frac{b_0}{b_2})\hat{v}_j$$

$$v_i = -\frac{1}{b_2}x_1 + \frac{a_2}{b_2}\hat{v}_j \Leftrightarrow a_2\hat{v}_j - b_2 v_i + x_1 = 0$$

$$\dot{x} = A^{ji}\cdot x + B^{ji}\cdot \hat{v}_j \ , \qquad v_i = C^{ji}\cdot x + D^{ji}\cdot \hat{v}_j ,$$

$$x = [x_1\ x_2]^T ,\ X = \{x, \mathsf{X}, \mathsf{X}\} ,$$

$$\mathsf{X} = \mathsf{X} = \mathsf{R}^2 \quad A^{ji} = \begin{bmatrix} \frac{-b_1}{b_2} & 1 \\ \frac{-b_0}{b_2} & 0 \end{bmatrix} \quad B^{ji} = \begin{bmatrix} -(a_1 - a_2\frac{b_1}{b_2}) \\ -(a_0 - a_2\frac{b_0}{b_2}) \end{bmatrix}$$

$$C^{ji} = [\frac{-1}{b_2}\ 0] \quad D^{ji} = \frac{a_2}{b_2}$$

Consider the actual and desired global variable defined for terminal variables only,

$$\xi = \{V_i, V_j\}\ \xi = [v_i, v_j]^T ,\quad \hat{\xi} = \{\hat{V}_i, \hat{V}_j\} ,\ \hat{\xi} = [\hat{v}_i, \hat{v}_j]^T$$

The VCDS is,

$$\dot{x} = A(q)\cdot x + B(q)\cdot\hat{\xi} ,\quad \xi = H(q)\cdot x + G(q)\cdot\hat{\xi}$$

$$A(q) = (1-q)\cdot A^{ij} + q\cdot A^{ji} ,$$
$$B(q) = (1-q)\cdot B^{ij} + q\cdot B^{ji}$$

$$H(q) = \begin{bmatrix} -q\cdot\frac{1}{b_2} & 0 \\ (1-q)\frac{b_2}{a_2} & 0 \end{bmatrix} \quad G(q) = \begin{bmatrix} 1-q & q\cdot\frac{a_2}{b_2} \\ (1-q)\cdot\frac{1}{a_2} & q \end{bmatrix}$$

Generally for an un-oriented differential system whose existence relation

$$\sum_{k=1}^{n} a_k\cdot v_j^{(k)} - \sum_{k=1}^{n} b_k\cdot v_i^{(k)} = 0 ,\ a_n \neq 0 ,\ b_n \neq 0 ,$$

contains only two terminal variables V_i, V_j, two causal ordering can be established. For each of them the above existence relation is realized by state equations, considering a common state variable n-dimensional vector x.

For the causality ordering $V_i \to V_j$, $q = 0$, $v_i = \hat{v}_i$, a state realization is of the form,

$$\dot{x} = A^{ij}\cdot x + B^{ij}\cdot\hat{v}_i , \qquad v_j = C^{ij}\cdot x + D^{ij}\cdot\hat{v}_i$$

and for the causality ordering $V_j \to V_i$, $q = 1$, $v_j = \hat{v}_j$, a state realization is of the form,

$$\dot{x} = A^{ji}\cdot x + B^{ji}\cdot\hat{v}_j , \qquad v_i = C^{ji}\cdot x + D^{ji}\cdot\hat{v}_j$$

The VCDS state equations have now the are,

$$\dot{x} = A(q)\cdot x + B(q)\cdot\hat{\xi} \ , \qquad \xi = H(q)\cdot x + G(q)\cdot\hat{\xi}$$

$$A(q) = (1-q)\cdot A^{ij} + q\cdot A^{ji} ,$$
$$B(q) = (1-q)\cdot B^{ij} + q\cdot B^{ji}$$

$$H(q) = \begin{bmatrix} q\cdot C^{ji} \\ (1-q)\cdot C^{ij} \end{bmatrix} ,\ G(q) = \begin{bmatrix} 1-q & q\cdot D^{ji} \\ (1-q)\cdot D^{ij} & q \end{bmatrix}$$

Example. Let the Existence Relation be

$$16\cdot\ddot{v}_j + 1.6\cdot\dot{v}_j + v_j - \ddot{v}_i - 0.2\cdot\dot{v}_i - v_i = 0$$

This involves two causal orderings characterized by the following transfer functions and state equations,

$$H_{ij}(s) := \frac{V_i(s)}{V_j(s)} = \frac{s^2 + 0.2s + 1}{16s^2 + 1.6s + 1}$$

$$A^{ij} = \begin{bmatrix} -0.1 & -0.125 \\ 0.5 & 0 \end{bmatrix} ,\ B^{ij} = [0.25\ 0]^T ,$$

$$C^{ij} = [0.025\ 0.4688]\ \ D^{ij} = 0.0625$$

$$H_{ji}(s) := \frac{V_j(s)}{V_i(s)} = \frac{16s^2 + 1.6s + 1}{s^2 + 0.2s + 1}$$

$$A^{ji} = \begin{bmatrix} -0.2 & -0.25 \\ 4 & 0 \end{bmatrix} ;\ B^{ji} = [2\ 0]^T ;$$

$$C^{ji} = [-0.8\ -1.875] ;\ D^{ji} = 16 .$$

The responses of the corresponding VCDS to step desired global variable $\hat{\xi}$ and pulse variation of the causality ordering variable $q = q(t)$ is represented in Fig.2. for the actual global variables ξ.

The evolution of the shared state variables is illustrated in Fig.3.

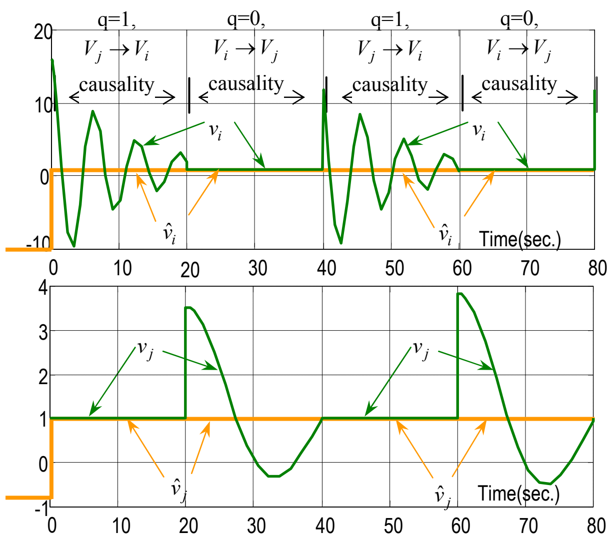

Fig 2. Step response of terminal variables.

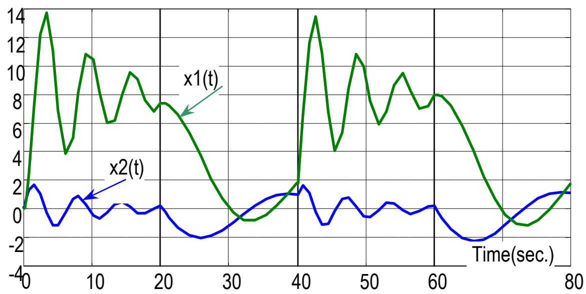

Fig 3. State variables evolution.

7. CONCLUSIONS

This paper deals with dynamical systems which models physical objects whose causal input-output ordering is changing during their evolution. Such a system is named Variable Causality Dynamical System (VCDS).

VCDS are controlled from outside by a new input called causal ordering signal sharing the same set of state variables. In VCDS, all the variables, except the causal ordering signal, are gathered in two forms of so called global variables as current global variable and desired global variable. There are proposed formal definitions for covariance and causality properties of variables and relations irrespective of the time domain.

Applications in walking robots of the VCDS approach developed including simulations in Mathlab environment prove the advantages of this approach.

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