SLIDING MODE BASED DIGITAL SERVO-SYSTEM DESIGN

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Abstract: *This paper is dedicated to the design, realization and analysis of the tracking control systems based on digital sliding modes. It is shown that combination of some basic principles of classical as well as modern control theory in design and realizations may results in high-quality tracking system. Tracking accuracy, influence of unmodeled dynamics, disturbances and static friction are demonstrated by digital simulation of illustrative example.*

Keywords: *Servo-system, Tracking control, Sliding modes, PI compensation.*

1. INTRODUCTION

Servo-systems - elements of many industrial and other control systems - are implemented in diverse fields of machine construction (machine tools, manipulators), robotics, navigation (aviation and astronautics), etc. Their common task is to track motion trajectories with a required accuracy.

In the linear control theory, the servo-system accuracy is defined both in transient as well as in steady-state mode. The transient mode is characterized by the speed of response and the degree of oscillatory behavior (overshoot). A servo-system must have fast response without overshoot. Duration of transient mode is, in general, very short. Basic accuracy in tracking systems is determined in a steady-state. This is a dynamical state since system is in motion.

In electromechanical servo-systems, considered in this paper, accuracy is defined by the error constants (the system class). A high quality tracking system must be of the class 3 or at least of the class 2 with large gain. Thus, small tracking error is obtained. These features are hardly attained by the conventional servo-systems.

The current theoretical research [1-12] and the practical implementations of the sliding mode control systems have exhibited superiority over the conventional solutions, under certain conditions (good structural plant identification, measurability of state coordinates, fulfillment of the invariance conditions [4]).

In the practice of electromechanical systems, a plant can be good structurally identified, but due to the control system simplicity some inertial modes (motor electrical time constant, viscous friction), present nonlinearities and similar imperfections are usually neglected. The neglected dynamics may lead to the parasitic motions (chattering), which cannot be tolerated in electromechanical systems. In the previous decade a great effort has been done in chattering alleviation. Significant number of papers has been published [1,2,5,6].

This paper proposes a servo-system design procedure with the digital sliding mode control in combination with the conventional proportional-integral (PI) algorithm. Section 2 gives some information about accuracy of servo-system with sliding mode. The reduced mathematical model of servo-system is derived in section 3, which is used in the control system design, in section 4. The fifth section analyzes a possibility of reduction of load disturbance effects. An illustrative example of control design and the simulation results are presented in section 6. Some concluding remarks and reference list are given at the end of the paper.

2. SLIDING MODE SYSTEMS ACCURACY

The steady-state accuracy in tracking of standard reference signals of sliding mode based servo-systems has been analyzed in [7]. The main results of this research will be presented here. A typical realization of variable structure system (VSS) is given in Fig. 1. In general case, switching function $g(t)$ in the VSS controller is evaluated using the error signal, $e(t) = r(t) - y(t) = r(t) - x_1(t)$, some of its time-derivatives as well as plant canonical controllable state coordinates $\mathbf{x}^{(i)}$. Note that plant is of n^{th}-order.

First, it is assumed that the error signal and its $n-1$ time-derivatives are available, and the sliding mode occurs in the given system. Then, the well known relation

$$g(t) = \sum_{i=1}^{n} c_i e^{(i-1)}(t) = \sum_{i=1}^{n} c_i (r^{(i-1)}(t) - x_1^{(i-1)}(t)) = 0 \tag{1}$$

is the system motion after reaching the sliding surface $g(t)$=0. Laplace transform of (1) yields

$$G(s) = \sum_{i=1}^{n} c_i s^{i-1} R(s) - \sum_{i=1}^{n} c_i s^{i-1} X_1(s) - \sum_{i=2}^{n} c_i \sum_{k=1}^{i-1} s^{i-k-1} e^{(k-1)}(t_{0^-}) = 0 \tag{2}$$

that is, $X_1(s) = R(s) + O(s)$ or $x(t) = r(t) + O(t)$. The function $O(t)$ is defined with

$$O(t) = \frac{\sum_{i=2}^{n} c_i \sum_{k=1}^{i-1} s^{i-k-1} e^{(k-1)}(t_{0^-})}{\sum_{i=1}^{n} c_i s^{i-1}} = \frac{\sum_{i=2}^{n} c_i \alpha_i}{\sum_{i=1}^{n} c_i s^{i-1}}, \tag{3}$$

where t_{0^-} is the time instant of the sliding mode occurrence, $e^{(k-1)}(t_{0^-})$ is the value of $(k\text{-}1)^{\text{th}}$ time-derivative of the error signal in $t = t_{0^-}$. Since $G(s)$=0 is a stable polynomial, then $\lim_{t\to\infty} O(t) = 0$ and the perfect tracking is obtained, i.e., $\lim_{t\to\infty} x_1(t) = r(t)$.

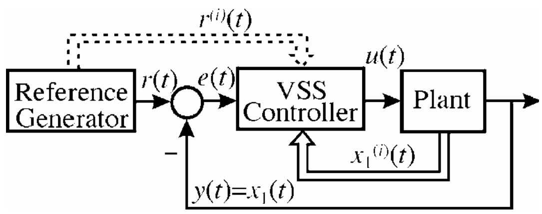

Fig.1. Sliding mode control system structure.

If the switching function is formed as

$$g(t) = \sum_{i=1}^{m} c_i e^{(i-1)}(t) - \sum_{k=m+1}^{n} c_k x_1^{(k-1)}(t) \tag{4}$$

and the stable sliding mode exists in the controlled system, then the tracking error in the complex domain becomes

$$E(s) = \left(R(s) \sum_{i=m+1}^{n} c_i s^{i-1} - \sum_{i=1}^{n-1} \widehat{\alpha}_i s^{i-1} \right) \left(\sum_{i=1}^{n} c_i s^{i-1} \right)^{-1}. \tag{5}$$

In the steady-state, the tracking error may be evaluated as $e(\infty) = \lim_{s\to 0} sE(s)$. If the reference signal is in the form of $r(t) = r_o t^p$, that is, $R(s) = r_o p!/s^{p+1}$, the steady-state tracking error is

$$e(\infty) = \begin{cases} 0 & \text{if } m > p, \\ \dfrac{c_{m+1}}{c_1} r_o m! & \text{if } m = p, \\ \infty & \text{if } m < p. \end{cases} \tag{6}$$

From the relation (6), it may be concluded that the control system of regulator type (p=0) the switching function $g(t)$ may be formed using the error signal $e(t)$ and state coordinates of the plant $x_2,\ldots,x_n$. For the control system which is intended to track ramp reference signals (p=1) without steady-state error, the switching function must be formed using the error signal $e(t)$, its first derivative d$e(t)$/dt and the state coordinates of the plant $x_3,...,x_n$. Hence, in the electromechanical servo-systems designed for tracking parabolic reference signals without tracking error, the second time-derivative of the error must be used. This is not practically possible for servo-systems with reference signal not known in advance. These remarks were made for the continuous-time sliding modes, but they are also valid for the discrete-time sliding modes in the first approximation. The goal in this paper is to design a servo-system for tracking parabolic reference signals with a small constant error that does not require using the second derivative of the error signal.

3. SERVO-SYSTEM MODEL

In the case of positioning servo-system with DC motor, assuming that non-inertial amplifier is used, the state coordinates are: position (θ), velocity (ω) and rotor current (i_r). Mathematical model of the system is

$$\begin{aligned} \frac{d\theta}{dt} &= \omega, \\ \frac{d\omega}{dt} &= -\frac{b}{J}\omega + \frac{c}{J}i_r - \frac{1}{J}M_l, \\ \frac{di_r}{dt} &= -\frac{c}{L_r}\omega - \frac{R_r}{L_r}i_r + \frac{1}{L_r}u_r, \end{aligned} \tag{7}$$

where R_r, L_r, b, J are rotor resistance, inductance, coefficient of viscous friction and moment of inertia, respectively. M_l is a motor shaft load torque and u_r is a rotor voltage, acting as a control signal which has to be synthesized in the controller design procedure.

Assuming that dynamics described by the third relation of the system (7) is much more faster then the first two relations, and using singular perturbations method, the system (7) is reduced to

$$\begin{aligned} \frac{d\theta}{dt} &= \omega, \\ \frac{d\omega}{dt} &= -\frac{1}{T_m}(1+\frac{T_m}{T_v})\omega + \frac{1}{cT_m}Ku - \frac{1}{J}M_l. \end{aligned} \tag{8}$$

The control task of the positioning servo-system is to track a referent signal $\theta_r(t)$. Therefore, it is convenient to represent the system model in tracking error space $e(t) = \theta(t) - \theta_r(t) = e_1$, which gives

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) + \mathbf{b}u(t) + \mathbf{d}f(t);$$
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 0 \\ k_1 \end{bmatrix}; \mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{9}$$

where:

$$f(t) = -\frac{1}{J}M_l - \frac{d^2\theta_r}{dt} - a\frac{d\theta_r}{dt}$$

is an exogenous disturbance;

$$a = \frac{1}{T_m}(1+\frac{T_m}{T_v}); k_1 = \frac{K}{cT_m}; T_m = \frac{R_r J}{c^2};$$

c is a motor constant, $T_v = \dfrac{J}{b}$, $u = u_r/K$, and K is an amplifier gain.

Hence, the model (9) is the simplified model, obtained by neglecting the inertial dynamics – the electrical time constant. The model fulfils the invariance condition:

$$\text{rang}[\mathbf{b} \;\vdots\; \mathbf{d}] = \text{rang}[\mathbf{b}]. \tag{10}$$

4. CONTROL SYNTHESIS

For the synthesis of u(t) it is firstly required to select the control algorithm. There exist a significant number of control algorithms with sliding modes. The attention is only paid to digital algorithms with attribute of "chattering free". These algorithms are presented in [1,2,5,6]. In this paper, the control algorithm [6] is chosen in order to additional performance analysis of this digital control strategy used in tracking systems.

Using δ-transformation, discrete-time model of (9) is obtained as

$$\delta\mathbf{e}(k) = \mathbf{A}_\delta \mathbf{e}(kT) + \mathbf{b}_\delta u(kT) + \mathbf{d}_\delta(kT); \tag{11}$$

$$\delta\mathbf{e}(kT) = \frac{1}{T}[\mathbf{e}((k+1)T) - \mathbf{e}(kT)]; \mathbf{A}_\delta = \frac{1}{T}(\mathbf{A}_d - \mathbf{I});$$
$$\mathbf{b}_\delta = \frac{1}{T}\mathbf{b}_d; \mathbf{A}_d = e^{\mathbf{A}T}; \mathbf{b}_d = \int_0^T e^{\mathbf{A}\tau}d\tau\,\mathbf{b};$$
$$\mathbf{d}_\delta(k) = \frac{1}{T}\int_0^T e^{\mathbf{A}\tau}\mathbf{d}\, f((k+1)T-\tau)d\tau,$$

where T is a sampling period.

In discrete-time domain, the invariance condition (10) is not generally satisfied. However, the introduction of δ-transformation with small T does not substantially disturb the condition, which is effectively proved in [6]. In any case, a disturbance influence can be reduced to a desired value by introduction of the disturbance observer [10]. Further on, the nominal system (11) is considered without presence of the disturbance term. It is shown in [6] that if the control signal is chosen in the form (k stands for kT)

$$u(k) = u_s = -\mathbf{c}_\delta^{\mathrm{T}}\mathbf{A}_\delta\mathbf{e}(k) - \min\left\{\frac{1}{T}|g(k)|, \sigma + q|g(k)|\right\}\mathrm{sgn}(g(k)) \tag{12}$$

system trajectories will softly reach the sliding line $g(k)=0$ from an arbitrary initial state in finite time, and the chattering will not exist. In (12) $\sigma, q > 0$ are the parameters that provide reaching of the sliding hyperplane (line) in finite time and enable system robustness. Vector $\mathbf{c}_\delta^{\mathrm{T}}$ defines sliding hyperplane (line): $g(k) = \mathbf{c}_\delta^{\mathrm{T}}\mathbf{e}(k)$, under the condition $\mathbf{c}_\delta^{\mathrm{T}}\mathbf{b}_\delta = 1$. Term $-\mathbf{c}_\delta^{\mathrm{T}}\mathbf{A}_\delta\mathbf{e}(k)$ is the equivalent control.

The elements of the vector $\mathbf{c}_\delta^{\mathrm{T}}$ are chosen according to the following procedure. First, the real eigenvalues of the system (11) are determined using the relation

$$z_i = \frac{e^{-\alpha_i T} - 1}{T}, \alpha_i > 0, \alpha_i \neq \alpha_j \tag{13}$$

and the elements of the transformed vector $\mathbf{c}^{\mathrm{T}}$ are derived as

$$c_i = \frac{1}{(i-1)!}\frac{d^{i-1}\prod_{i=1}^{n-1}(z = z_i)}{dz^{i-1}}. \tag{14}$$

Afterwards, the elements of $\mathbf{c}_\delta^{\mathrm{T}}$ are found according to

$$\mathbf{c}_\delta^{\mathrm{T}} = \left[\bar{\mathbf{c}}^{\mathrm{T}} \mid 1\right]\mathbf{P}_1^{-1}; \tag{15}$$

where $\mathbf{P}_1$ is a transformation matrix

$$\mathbf{P}_1 = \mathbf{M}_c \begin{bmatrix} a_2 & a_3 & \dots & a_n & 1 \\ a_3 & a_4 & \dots & 1 & \\ . & . & . & . & \\ a_n & 1 & & \mathbf{0} & \\ 1 & & & & \end{bmatrix}, \tag{16}$$

with $\mathbf{M}_c$ being the controllability matrix of the system (11), and a_i being the coefficients of the characteristic polynomial

$$D(z) = \det\left[z\mathbf{I} - \mathbf{A}_\delta\right] = z^n + a_n z^{n-1} + \dots + a_2 z + a_1 \tag{17}$$

5. DISTURBANCE COMPENSATION

As mentioned above, the given control algorithm provides significant robustness to disturbances. An additional improvement can be obtain in two ways: (*i*) by the estimation of the disturbance using an adequate observer; (*ii*) by the introduction of linear PI compensator between the control system and the plant.

5.1. *Disturbance estimator*

This approach represents evaluation of the disturbance. Since the disturbance cannot be directly measured, its estimation is used. Several approaches are recognized. One of them is one step delayed disturbance estimator [10].

Based on (11), a disturbance can be estimated as

$$\mathbf{d}_\delta(k-1) = \mathbf{e}(k) - \mathbf{b}_\delta u(k-1) - \mathbf{A}_\delta\mathbf{e}(k-1). \tag{18}$$

The committed error is [10]:

$$\mathbf{d}_\delta(k) - \mathbf{d}_\delta(k-1) = \int_0^T e^{\mathbf{A}_\delta \tau} \int_{kT=\lambda}^{(k+1)T-\lambda} \mathbf{d}_\delta(\lambda)\, d\tau\, d\lambda = O(T^2). \tag{19}$$

If a disturbance is a slow varying and bounded function, its effect can be eliminated by the enhanced control

$$u(k) = u_s - \mathbf{c}_\delta^{\mathrm{T}}\mathbf{d}_\delta(k-1), \tag{20}$$

where u_s is given by (12).

5.2. *PI compensator*

The paper [8] reveals that the introduction of PI action between the controller and the plant of the designed variable structure system does not violate the sliding mode conditions. On the contrary, it simplifies sliding conditions further. This compensation, from the controller viewpoint, acts like a plant model enhancement with finite zero. The sliding mode control of the plants with finite stable zeros may be efficiently performed in the state subspace [11], which actually is this approach. Taking into account the conventional approach to PI controller design, parameters should be chosen in such manner to compensate unwanted plant dynamics. Since now, the integral action is located in front of disturbance, it will entirely compensate constant load disturbances, and alleviate disturbances of other nature. Besides, if the designed system is of the class 1, it will become now the system of the class 2. Then the system is able to track parabolic signals with constant error. The shortcoming of this approach is a possibility of integrator saturation. However, this problem also exists in the conventional methods, and can be solved by using the appropriate procedure.

This paper does not investigate the problems of the introduction of PI compensator in details. Simulation results will indicate that this approach has its verification and should be explored further.

6. ILUSTRATIVE EXAMPLE

The parameters of dc-motor are: I_n=4 A, T_r=10 ms, T_m=30 ms, c=0.33, amplifier gain K=10 V/V.

The plant with the neglected certain dynamics, used in control system design, is described by transfer function

$$\frac{\theta(s)}{U_r(s)} = \frac{1000}{s(s+33)}.$$

The sampling time is T=0.4 ms, α in (13) is chosen as: 15 or 45.

The discrete-time model (11) of the nominal system is determined as

$$\mathbf{A}_\delta = \begin{bmatrix} 0 & 0.9934 \\ 0 & -32.7832 \end{bmatrix}; \ \mathbf{b}_\delta = \begin{bmatrix} -0.1991 \\ -993.4289 \end{bmatrix}.$$

For α=15 control system parameters are selected as $\sigma = 20, q = 10$.

$$\mathbf{c}_\delta^{\mathrm{T}} = [-0.0151 \ \ -0.0010], \ \mathbf{c}_\delta^{\mathrm{T}}\mathbf{A}_\delta = [0 \ \ 0.0179],$$

The PI control is realized as

$$\begin{aligned} &u_{PI}(k) = u_s(k) - u_I(k); \\ &u_I(k) = k_I g(k) + u_i(k-1), k_I = 100. \end{aligned} \tag{21}$$

First, the simulation of the nominal system (without unmodeled dynamics), subjected to the trapezoidal reference signal of the form

$$\theta_r = 2r(t); r(t) = \begin{cases} t \ \text{ for } 0 < t \le 2, \\ 2 \ \text{ for } 2 < t \le 6, \\ 2 - (t-6) \ \text{ for } 6 < t \le 10, \\ 0 \ \text{ for } 10 < t \le 12; \end{cases} \tag{22}$$

is carried out, and the load is given by

$$M_l = -[6 - 2\sin(6.28t)][h(t-4) - h(t-10)].$$

Instead of the error signal derivative, it is used the derivative of the output signal. The simulation results are presented in Fig. 2. It may be noticed that the system has an error in the tracking of ramp signals, according to the relation (6). Namely, for α=15, $c_{m+1} = c_2$; $c_2 / c_1 = 1/\alpha$; $m = 1, r_o = 2$ and, therefore, for the ramp signal, in the time interval 0 to 2s, and 6 to 8s error is, respectively, 2/15=0.133 and -0.133. The great robustness to the disturbance is evident (curves labeled with $\alpha = 15$, $\alpha = 45$), which is improved by the introduction of the disturbance observer (curve $\alpha = 15, do$). PI compensator almost absolutely eliminates action of the load disturbance (curves labeled with α=15,PI and α=45,PI).

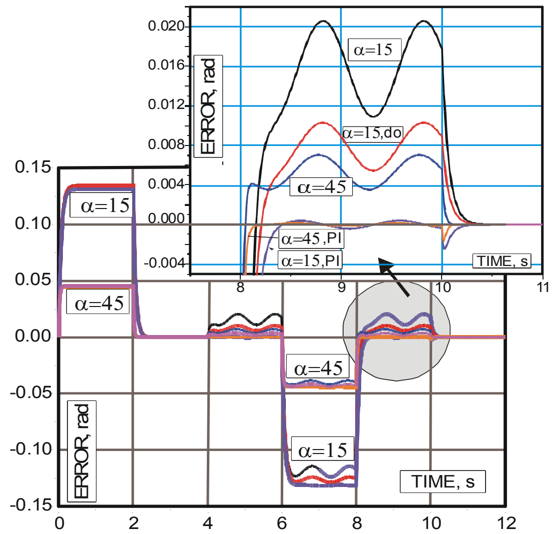

Fig.2. Tracking errors of trapezoidal signal (22) for different conditions, without using error signal derivative.

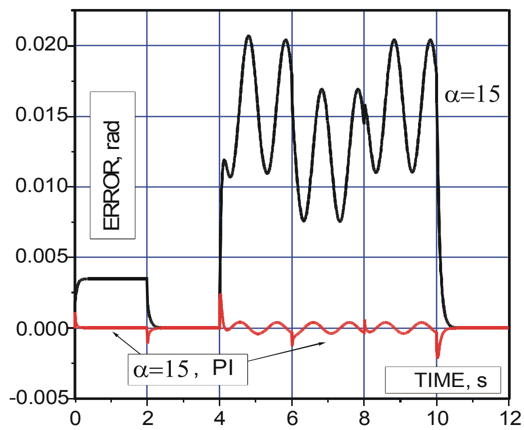

Fig.3. Tracking errors of trapezoidal signal (22) for different conditions, using error signal derivative.

Figure 3. shows the tracking errors of the reference (22), when the error signal derivative is used, without the PI (curve α=15) and with

the PI compensator (line α=15,PI). Notice that the system retains the same features with respect to the load disturbance, but the reference tracking error is significantly reduced (without PI compensator) and is equal to zero when the PI compensator is applied (see the time interval 0-2s, when ramp signal $\theta_r = 2t$ is used and plant is no loaded). This shows that the servo-system behaves as the system of the class 2.

Finally, Fig. 4. shows the tracking error of the reference, obtained by integration of (22), using the error signal derivative (Euler's derivative with T=0.1 ms) and the PI compensator. The tracking error of the parabolic signal (time interval 0-2s) is $\sim 5 \cdot 10^{-5}$ (curve B). The effect of the constant component of load is eliminated whereas the periodic load component is entirely attenuated. It should be emphasized that the system without the PI compensator is unable to track the parabolic signals. The almost identical results are obtained in the case of the presence of unmodeled inertial dynamics as well as Coulomb friction.

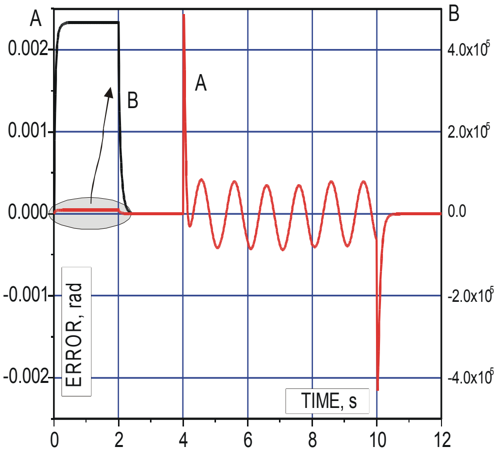

Fig.4. Tracking error of the reference signal which is integral of signal defined by (22) using error signal derivative and PI compensator.

7. CONCLUSION

The dominant effects on the performance improvement of the servo-system, besides the implementation of chattering free digital sliding mode control, have two factors: (*i*) the employment of the error signal derivative in forming the sliding hyperplane (line) and (*ii*) the introduction of the PI compensator. The disturbance observer has not gained the expected performance. The fundamental problem in the realization of the proposed system is obtaining the truthful error signal derivative in real system, where the noises exist. The tracking of the parabolic signals needs to be further investigated, as well as a system behavior in the vicinity of the equilibrium point.

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