# DISCRETE-TIME INTERVAL BIDIRECTIONAL ASSOCIATIVE MEMORIES – NOVEL STABILITY RESULTS

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***Abstract:*** *This paper develops the analysis of a special type of asymptotic stability, called componentwise asymptotic stability* (CWAS)*, for discrete-time Bidirectional Associative Memory* (BAM) *neural networks with interval type parameters. Unlike the standard notion of asymptotic stability, that gives global information on the state-space vector, expressed in terms of arbitrary norms, CWAS allows an individual monitoring of each state-space variable approaching the equilibrium point. At conceptual level, CWAS brings a refinement in the stability theory by revealing the existence of positive invariant time-dependent rectangular sets with respect to the state space trajectories. Our results provide sufficient conditions for testing the CWAS of BAMs with interval type parameters relying on the Schur stability of a matrix which is adequately built from the intervals expressing the parameter uncertainties.*

***Keywords:*** *stability analysis, neural networks, discrete-time systems, nonlinear systems*

## 1. INTRODUCTION

Consider the discrete-time *bidirectional associative memory* (BAM) neural network described by

$$\begin{aligned}
&\boldsymbol{x}^1(t+1) = \boldsymbol{A}^1\boldsymbol{x}^1(t) + \boldsymbol{W}^2\boldsymbol{f}^2\left(\boldsymbol{x}^2(t)\right) + \boldsymbol{I}^1,\\
&\boldsymbol{x}^2(t+1) = \boldsymbol{A}^2\boldsymbol{x}^2(t) + \boldsymbol{W}^1\boldsymbol{f}^1\left(\boldsymbol{x}^1(t)\right) + \boldsymbol{I}^2, \qquad (1)\\
&t \in \mathbb{Z}_+;\ \ \boldsymbol{x}^1(t_0) = \boldsymbol{x}_0^1,\ \ \boldsymbol{x}^2(t_0) = \boldsymbol{x}_0^2,\ \ t_0 \in \mathbb{Z}_+,
\end{aligned}$$

where:

$$\boldsymbol{x}^1 = \left[x_1^1, x_2^1, \ldots, x_m^1\right]^\tau,\ \boldsymbol{x}^2 = \left[x_1^2, x_2^2, \ldots, x_n^2\right]^\tau$$

are the state-vectors ($\tau$ denoting the vector transposition),

$$\boldsymbol{I}^1 = \left[I_1^1, I_2^1, \ldots, I_m^1\right]^\tau,\ \boldsymbol{I}^2 = \left[I_1^2, I_2^2, \ldots, I_n^2\right]^\tau$$

are the input vectors and matrices

$$\boldsymbol{A}^1 = \operatorname{diag}\left\{a_1^1, a_2^1, \ldots, a_m^1\right\},\ \boldsymbol{W}^1 = \left[w_{ji}^1\right],$$

$$\boldsymbol{A}^2 = \operatorname{diag}\left\{a_1^2, a_1^2, \ldots, a_n^2\right\},\ \boldsymbol{W}^2 = \left[w_{ij}^2\right]$$

have appropriate sizes.

All the components of the activation functions

$$\begin{aligned}
&\boldsymbol{f}^1 : \mathbb{R}^m \to \mathbb{R}^m,\\
&\boldsymbol{f}^1(\boldsymbol{x}^1) = \left[f_1^1(x_1^1), f_2^1(x_2^1), \ldots, f_m^1(x_m^1)\right]^\tau,\\
&\boldsymbol{f}^2 : \mathbb{R}^n \to \mathbb{R}^n,\\
&\boldsymbol{f}^2(\boldsymbol{x}^2) = \left[f_1^2(x_1^2), f_2^2(x_2^2), \ldots, f_n^2(x_n^2)\right]^\tau
\end{aligned}$$

are globally Lipschitz continuous, i.e. for all $i = \overline{1,m}$, $j = \overline{1,n}$, there exist $L_i^1, L_j^2 > 0$ so that

$$\begin{aligned}
&0 \le \left|f_i^1(\xi) - f_i^1(\zeta)\right| \le L_i^1 \,|\, \xi - \zeta \,|,\\
&0 \le \left|f_j^2(\xi) - f_j^2(\zeta)\right| \le L_j^2 \,|\, \xi - \zeta \,|,
\end{aligned} \qquad (2)$$

for all $\xi, \zeta \in \mathbb{R}$.

In terms of neural networks, according to our hypotheses, the activation functions are assumed to be neither differentiable nor bounded. If $f$ is

the generic notation for an activation function, our approach covers the following classes of functions: bipolar sigmoid

$$f(s)=\left[1-\exp(-\lambda s)\right]/\left[1+\exp(-\lambda s)\right],\ \lambda>0,$$

piecewise saturation

$$f(s)=\left[|\lambda s+1|-|\lambda s-1|\right]/2,\ \lambda>0,$$

linear $f(s)=\lambda s,\ \lambda>0$, piecewise linear, etc.

Recent papers, such as [2], [9], [12] and [17], provide sufficient conditions, formulated in algebraic terms, for the global (exponential) asymptotic stability of continuous-time BAMs. Work [12] also refers to the discrete-time case.

Taking into consideration the nature of the mathematical problems discussed in this paper, the following notations are used in the remainder of our paper. $\boldsymbol{O}$ and $\boldsymbol{0}$ stand for the null square matrix and the null vector respectively, each of appropriate dimensions. For two matrices having the same sizes $\boldsymbol{\Phi}=[\phi_{rs}]$, $\boldsymbol{\Theta}=[\theta_{rs}]\in\mathbb{R}^{u\times v}$, matrix inequality $\boldsymbol{\Phi}\geq\boldsymbol{\Theta}$ ($\boldsymbol{\Phi}>\boldsymbol{\Theta}$) is understood componentwise, i.e. $\phi_{rs}\geq\theta_{rs}$ ($\phi_{rs}>\theta_{rs}$) for all $r=\overline{1,u}$ and $s=\overline{1,v}$. Given a matrix $\boldsymbol{\Phi}=[\phi_{rs}]\in\mathbb{R}^{u\times v}$, denote by $|\boldsymbol{\Phi}|$ the matrix whose entries are $|\phi_{rs}|$. These conventions apply in the case of vectors or vector functions too.

For many problems encountered in practice it is important to consider that the entries of the matrices $\boldsymbol{A}^k$, $\boldsymbol{W}^k$, $k=\overline{1,2}$, defining the dynamics of BAM (1), are uncertain, in the sense of the matrix componentwise inequalities:

$$\begin{aligned}
&\underline{\boldsymbol{A}}^1=\operatorname{diag}\left\{\underline{a}_1^1,\underline{a}_2^1,...,\underline{a}_m^1\right\}\leq\boldsymbol{A}^1\leq\\
&\qquad\leq\overline{\boldsymbol{A}}^1=\operatorname{diag}\left\{\overline{a}_1^1,\overline{a}_2^1,...,\overline{a}_m^1\right\},\\
&\underline{\boldsymbol{A}}^2=\operatorname{diag}\left\{\underline{a}_1^2,\underline{a}_2^2,...,\underline{a}_n^2\right\}\leq\boldsymbol{A}^2\leq\\
&\qquad\leq\overline{\boldsymbol{A}}^2=\operatorname{diag}\left\{\overline{a}_1^2,\overline{a}_2^2,...,\overline{a}_n^2\right\},
\end{aligned}\tag{3.i}$$

$$\begin{aligned}
&\underline{\boldsymbol{W}}^1=\left[\ \underline{w}_{ji}^1\ \right]_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}}\leq\boldsymbol{W}^1\leq\overline{\boldsymbol{W}}^1=\left[\ \overline{w}_{ji}^1\ \right]_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}},\\
&\underline{\boldsymbol{W}}^2=\left[\ \underline{w}_{ij}^2\ \right]_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}}\leq\boldsymbol{W}^2\leq\overline{\boldsymbol{W}}^2=\left[\ \overline{w}_{ij}^2\ \right]_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}}.
\end{aligned}\tag{3.ii}$$

Consequently, let us introduce the following classes of matrices:

$$\begin{aligned}
\mathsf{A}^1&=\left\{\boldsymbol{A}^1\in\mathbb{R}^{m\times m}\,\middle|\,\underline{\boldsymbol{A}}^1\leq\boldsymbol{A}^1\leq\overline{\boldsymbol{A}}^1\right\},\\
\mathsf{A}^2&=\left\{\boldsymbol{A}^2\in\mathbb{R}^{n\times n}\,\middle|\,\underline{\boldsymbol{A}}^2\leq\boldsymbol{A}^2\leq\overline{\boldsymbol{A}}^2\right\},\\
\mathsf{W}^1&=\left\{\boldsymbol{W}^1\in\mathbb{R}^{n\times m}\,\middle|\,\underline{\boldsymbol{W}}^1\leq\boldsymbol{W}^1\leq\overline{\boldsymbol{W}}^1\right\},\\
\mathsf{W}^2&=\left\{\boldsymbol{W}^2\in\mathbb{R}^{m\times n}\,\middle|\,\underline{\boldsymbol{W}}^2\leq\boldsymbol{W}^2\leq\overline{\boldsymbol{W}}^2\right\}.
\end{aligned}\tag{4}$$

The family of BAMs generated by (1) for all $\boldsymbol{A}^k\in\mathsf{A}^k$, $\boldsymbol{W}^k\in\mathsf{W}^k$, $k=\overline{1,2}$, is called *Interval Bidirectional Associative Memory* neural network, abbreviated as IBAM($\mathsf{A}^1,\mathsf{A}^2,\mathsf{W}^1,\mathsf{W}^2$).

This paper proves that the stability of a single test matrix guarantees a stronger stability property of IBAM($\mathsf{A}^1,\mathsf{A}^2,\mathsf{W}^1,\mathsf{W}^2$), called *componentwise stability*. Unlike the standard concepts of stability, that give global information on the state-space vector, expressed in terms of arbitrary norms, the componentwise stability allows an individual monitoring of each state-space variable. This type of stability was first studied by Voicu in [16] who applied the theory of flow-invariant time-dependent rectangular sets to define and characterize the *componentwise asymptotic stability* (CWAS) and the *componentwise exponential asymptotic stability* (CWEAS) for continuous-time linear systems. Further works extended the analysis of componentwise stability to continuous-time delay linear systems [6], 1-D and 2-D discrete-time linear systems [7], interval matrix systems [13] and a class of Persidskii systems with uncertainties [14]. Despite the existence of these results, the componentwise stability of recurrent neural networks remained almost unexplored, except for a reduced number of recent papers [3]-[5] and [10].

Our paper develops a CWAS/CWEAS analysis of BAM (1) under the hypothesis of parameter uncertainties modeled by $\boldsymbol{A}^k\in\mathsf{A}^k$, $\boldsymbol{W}^k\in\mathsf{W}^k$, $k=\overline{1,2}$. The concepts employed by our work are rigorously defined in Section 2. Section 3 provides the main results, consisting in sufficient criteria for the CWAS/CWEAS of BAMs with uncertainties. Section 4 creates a deeper insight into the particular case of BAMs with fixed parameters. A numerical example is considered in Section 5 to illustrate the validity of the theoretical results. A few final remarks are formulated in Section 6.

## 2. PREREQUISITES ON CWAS / CWEAS

Assume that BAM (1) has a finite number of equilibrium points and let $\boldsymbol{x}_e = \left[ \boldsymbol{x}_e^{1\tau} \; \boldsymbol{x}_e^{2\tau} \right]^\tau$ be one of these, i.e. $\boldsymbol{x}_e^1 = \boldsymbol{A}^1 \boldsymbol{x}_e^1 + \boldsymbol{W}^2 \boldsymbol{f}^2\left(\boldsymbol{x}_e^2\right) + \boldsymbol{I}^1$ and $\boldsymbol{x}_e^2 = \boldsymbol{A}^2 \boldsymbol{x}_e^2 + \boldsymbol{W}^1 \boldsymbol{f}^1\left(\boldsymbol{x}_e^1\right) + \boldsymbol{I}^2$.

*Definition 1.* (**a**) Let $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ be two vector functions $\boldsymbol{p}^1 : \mathbb{Z}_+ \to \mathbb{R}^m$, $\boldsymbol{p}^2 : \mathbb{Z}_+ \to \mathbb{R}^n$, with positive components $p_i^1(t) > 0$, $i = \overline{1,m}$, $p_j^2(t) > 0$, $j = \overline{1,n}$, $t \in \mathbb{Z}_+$, meeting

$$\lim_{t\to\infty} \boldsymbol{p}^k(t) = \boldsymbol{0}, \quad k = \overline{1,2}. \tag{5}$$

If for any $t_0 \in \mathbb{Z}_+$ and any initial condition $\boldsymbol{x}_0 = \left[ \boldsymbol{x}_0^{1\tau} \; \boldsymbol{x}_0^{2\tau} \right]^\tau$, $\boldsymbol{x}_0^1 \in \mathbb{R}^m$, $\boldsymbol{x}_0^2 \in \mathbb{R}^n$, satisfying $\left| \boldsymbol{x}_0^k - \boldsymbol{x}_e^k \right| \le \boldsymbol{p}^k(t_0)$, $k = \overline{1,2}$, the corresponding solution to (1), $\boldsymbol{x}(t) = \left[ \boldsymbol{x}^1(t)^\tau \; \boldsymbol{x}^2(t)^\tau \right]^\tau$, $\boldsymbol{x}^k(t) = \boldsymbol{x}^k(t; t_0, \boldsymbol{x}_0)$, $k = \overline{1,2}$, meets the inequality $\left| \boldsymbol{x}^k(t) - \boldsymbol{x}_e^k \right| \le \boldsymbol{p}^k(t)$, $\forall t \in \mathbb{Z}_+$, $t \ge t_0$, $k = \overline{1,2}$, then we say that the equilibrium point $\boldsymbol{x}_e$ of BAM (1) is *componentwise asymptotically stable* with respect to $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$, abbreviated as CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$).

(**b**) The equilibrium point $\boldsymbol{x}_e$ of BAM (1) is *globally* CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), or CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) *in the large*, abbreviated as GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), if $\boldsymbol{x}_e$ is CWAS($c\boldsymbol{p}^1, c\boldsymbol{p}^2$) for any scalar $c > 0$.

(**c**) BAM (1) is said to be CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if it has an equilibrium point $\boldsymbol{x}_e$ that is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$).

(**d**) IBAM($\mathsf{A}^1, \mathsf{A}^2, \mathsf{W}^1, \mathsf{W}^2$) is said to be CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for all $\boldsymbol{A}^k \in \mathsf{A}^k$, $\boldsymbol{W}^k \in \mathsf{W}^k$, $k = \overline{1,2}$.

*Remark 1.* It can be proved that (**a**) if an equilibrium point $\boldsymbol{x}_e = \left[ \boldsymbol{x}_e^{1\tau} \; \boldsymbol{x}_e^{2\tau} \right]^\tau$ of BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), then it is also *uniformly asymptotically stable* in the sense of the standard definition, e.g. [11], pp. 107; (**b**) if an equilibrium point $\boldsymbol{x}_e = \left[ \boldsymbol{x}_e^{1\tau} \; \boldsymbol{x}_e^{2\tau} \right]^\tau$ of BAM (1) is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), then it is also *uniformly asymptotically stable in the large* in the sense of the standard definition, e.g. [11], pp. 108).

Until this point of our presentation the time-dependence of the vector functions $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, was considered arbitrary. If the CWAS property exists for the particular form of the vector functions

$$\begin{aligned} &\boldsymbol{p}^1(t) = \sigma^t \boldsymbol{\alpha}^1, \; \boldsymbol{p}^2(t) = \sigma^t \boldsymbol{\alpha}^2, \; t \in \mathbb{Z}_+, \\ &\sigma \in (0,1), \boldsymbol{\alpha}^1 \in \mathbb{R}^m, \boldsymbol{\alpha}^1 > \boldsymbol{0}, \boldsymbol{\alpha}^2 \in \mathbb{R}^n, \boldsymbol{\alpha}^2 > \boldsymbol{0}, \end{aligned} \tag{6}$$

then we refer to a special type of stability property called *componentwise exponential asymptotic stability*, abbreviated as CWEAS, and Definition 1 yields the following:

*Definition 2.* If the hypotheses of Definition 1(a-d) are fulfilled with $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, given by (6), then we say that: (**a**) the equilibrium point $\boldsymbol{x}_e$ is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**b**) the equilibrium point $\boldsymbol{x}_e$ is globally CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), abbreviated as GCWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**c**) BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**d**) the interval neural network IBAM($\mathsf{A}^1, \mathsf{A}^2, \mathsf{W}^1, \mathsf{W}^2$) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$).

*Remark 2.* It can be proved that (**a**) if an equilibrium point $\boldsymbol{x}_e = \left[ \boldsymbol{x}_e^{1\tau} \; \boldsymbol{x}_e^{2\tau} \right]^\tau$ of BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then it is also *exponentially asymptotically stable* in the classical sense (e.g. [11], pp. 107); (**b**) if an equilibrium point $\boldsymbol{x}_e = \left[ \boldsymbol{x}_e^{1\tau} \; \boldsymbol{x}_e^{2\tau} \right]^\tau$ of BAM (1) is GCWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then it is also *globally exponentially asymptotically stable* in the classical sense (e.g. [11], pp. 108).

## 3. CWAS / CWEAS RESULTS FOR INTERVAL BAMS

### *3.1. CWAS of an IBAM*

*Theorem 1.* IBAM($\mathsf{A}^1,\mathsf{A}^2,\mathsf{W}^1,\mathsf{W}^2$) is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) if the following inequalities hold

$$\begin{aligned}\boldsymbol{p}^1(t+1) &\geq \hat{\boldsymbol{A}}^1\boldsymbol{p}^1(t)+\hat{\boldsymbol{W}}^2\boldsymbol{L}^2\boldsymbol{p}^2(t),\\ \boldsymbol{p}^2(t+1) &\geq \hat{\boldsymbol{A}}^2\boldsymbol{p}^2(t)+\hat{\boldsymbol{W}}^1\boldsymbol{L}^1\boldsymbol{p}^1(t),\end{aligned} \quad \forall t\in\mathbb{Z}_+, \tag{7}$$

where matrices $\hat{\boldsymbol{A}}^k$, $\hat{\boldsymbol{W}}^k$, $\boldsymbol{L}^k$, $k=\overline{1,2}$, are defined by

$$\begin{aligned}&\hat{\boldsymbol{A}}^1=\operatorname{diag}\left\{\hat{a}_1^1,\hat{a}_2^1,\ldots,\hat{a}_m^1\right\},\\ &\hat{a}_i^1=\max\left\{|\underline{a}_i^1|,|\overline{a}_i^1|\right\},\ i=\overline{1,m},\\ &\hat{\boldsymbol{A}}^2=\operatorname{diag}\left\{\hat{a}_1^2,\hat{a}_2^2,\ldots,\hat{a}_n^2\right\},\\ &\hat{a}_j^2=\max\left\{|\underline{a}_j^2|,|\overline{a}_j^2|\right\},\ j=\overline{1,n},\\ &\hat{\boldsymbol{W}}^1=\left[\hat{w}_{ji}^1\right]\in\mathbb{R}^{n\times m},\\ &\hat{w}_{ji}^1=\max\left\{|\underline{w}_{ji}^1|,|\overline{w}_{ji}^1|\right\},\\ &\hat{\boldsymbol{W}}^2=\left[\hat{w}_{ij}^2\right]\in\mathbb{R}^{m\times n},\\ &\hat{w}_{ij}^2=\max\left\{|\underline{w}_{ij}^2|,|\overline{w}_{ij}^2|\right\},\\ &\boldsymbol{L}^1=\operatorname{diag}\left\{L_1^1,L_2^1,\ldots,L_m^1\right\}\in\mathbb{R}^{m\times m},\\ &\boldsymbol{L}^2=\operatorname{diag}\left\{L_1^2,L_2^2,\ldots,L_n^2\right\}\in\mathbb{R}^{n\times n}.\end{aligned} \tag{8}$$

*Proof*: Given arbitrary $\boldsymbol{A}^k\in\mathsf{A}^k$, $\boldsymbol{W}^k\in\mathsf{W}^k$, $k=\overline{1,2}$, the dynamical behavior of the state-space trajectories of BAM (1) in a vicinity of the equilibrium point $\boldsymbol{x}_e=\left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ may be analyzed by means of the deviations $\boldsymbol{y}^k(t)=\boldsymbol{x}^k(t)-\boldsymbol{x}_e^k$, $k=\overline{1,2}$, that satisfy

$$\begin{aligned}&\boldsymbol{y}^1(t+1)=\boldsymbol{A}^1\boldsymbol{y}^1(t)+\boldsymbol{W}^2\boldsymbol{g}^2\left(\boldsymbol{y}^2(t)\right),\\ &\boldsymbol{y}^2(t+1)=\boldsymbol{A}^2\boldsymbol{y}^2(t)+\boldsymbol{W}^1\boldsymbol{g}^1\left(\boldsymbol{y}^1(t)\right),\\ &\boldsymbol{y}^1(t_0)=\boldsymbol{x}_0^1-\boldsymbol{x}_e^1,\quad \boldsymbol{y}^2(t_0)=\boldsymbol{x}_0^2-\boldsymbol{x}_e^2,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\boldsymbol{g}^1(\boldsymbol{y}^1)&=\boldsymbol{f}^1(\boldsymbol{y}^1+\boldsymbol{x}_e^1)-\boldsymbol{f}^1(\boldsymbol{x}_e^1),\\ \boldsymbol{g}^2(\boldsymbol{y}^2)&=\boldsymbol{f}^2(\boldsymbol{y}^2+\boldsymbol{x}_e^2)-\boldsymbol{f}^2(\boldsymbol{x}_e^2).\end{aligned} \tag{10}$$

Obviously, $\boldsymbol{x}_e$ is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) for (1) if and only if $\boldsymbol{y}_e=\boldsymbol{0}$ is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) for (9). The components $g_i^1$, $i=\overline{1,m}$, and $g_j^2$, $j=\overline{1,n}$, of the activation functions $\boldsymbol{g}^1$ and, respectively, $\boldsymbol{g}^2$, are Lipschitz continuous and satisfy the following conditions derived from (2):

$$0\leq\left|g_i^1(\xi)\right|\leq L_i^1|\xi|,\quad 0\leq\left|g_j^2(\xi)\right|\leq L_j^2|\xi|, \tag{11}$$

for all $\xi\in\mathbb{R}$.

Assuming that for an arbitrary $c>0$, the solution to (9) initiated so that $\left|\boldsymbol{y}^k(t_0)\right|\leq c\boldsymbol{p}^k(t_0)$, $k=\overline{1,2}$, satisfies $\left|\boldsymbol{y}^k(t)\right|\leq c\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, for a certain $t\geq t_0$, and taking (11) into account, we get

$$\begin{aligned}&\left|y_i^1(t+1)\right|\leq\left|a_i^1\right|\left|y_i^1(t)\right|+\sum_{j=1}^n\left|w_{ij}^2\right|\left|g_j^2\left(y_j^2(t)\right)\right|\leq\\ &\leq\max\left\{|\underline{a}_i^1|,|\overline{a}_i^1|\right\}\left|y_i^1(t)\right|+\\ &\quad+\sum_{j=1}^n\max\left\{|\underline{w}_{ij}^2|,|\overline{w}_{ij}^2|\right\}L_j^2\left|y_j^2(t)\right|\leq\\ &\leq c\left[\hat{a}_i^1p_i^1(t)+\sum_{j=1}^n\hat{w}_{ij}^2L_j^2p_j^2(t)\right],\ i=\overline{1,m},\\ &\left|y_j^2(t+1)\right|\leq\left|a_j^2\right|\left|y_j^2(t)\right|+\sum_{i=1}^m\left|w_{ji}^1\right|\left|g_i^1\left(y_i^1(t)\right)\right|\leq\\ &\leq\max\left\{|\underline{a}_j^2|,|\overline{a}_j^2|\right\}\left|y_j^2(t)\right|+\\ &\quad+\sum_{j=1}^n\max\left\{|\underline{w}_{ji}^1|,|\overline{w}_{ji}^1|\right\}L_i^1\left|y_i^1(t)\right|\leq\\ &\leq c\left[\hat{a}_j^2p_j^2(t)+\sum_{j=1}^n\hat{w}_{ji}^1L_i^1p_i^1(t)\right],\ j=\overline{1,n}.\end{aligned} \tag{12}$$

If (7) is satisfied, it follows that $\left|\boldsymbol{y}^k(t+1)\right|\leq c\boldsymbol{p}^k(t+1)$, $k=\overline{1,2}$; through mathematical induction the fulfillment of $\left|\boldsymbol{y}^k(t)\right|\leq c\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, is ensured for all $t\geq t_0$, meaning that $\boldsymbol{y}_e=\boldsymbol{0}$ is CWAS($c\boldsymbol{p}^1,c\boldsymbol{p}^2$) for (9). Since this happens for all $c>0$, the equilibrium point $\boldsymbol{x}_e$ of BAM (1) is GCWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$), showing that BAM (1) is

CWAS( $\boldsymbol{p}^1, \boldsymbol{p}^2$ ). This conclusion can be drawn for all BAMs described by (1) with $\boldsymbol{A}^k \in \mathsf{A}^k$, $\boldsymbol{W}^k \in \mathsf{W}^k$, $k = \overline{1,2}$, which completes the proof.

*Remark 3*. Let us introduce the augmented vector function

$$\boldsymbol{p}: \mathbb{Z}_+ \to \mathbb{R}^{m+n}, \ \boldsymbol{p}(t) = \left[ \boldsymbol{p}^1(t)^\tau \ \ \boldsymbol{p}^2(t)^\tau \right]^\tau, \quad (13)$$

and the matrix $\boldsymbol{\Theta} \in \mathbb{R}^{(m+n)\times(m+n)}$ defined by

$$\boldsymbol{\Theta} = \begin{bmatrix} \hat{\boldsymbol{A}}^1 & \boldsymbol{O} \\ \boldsymbol{O} & \hat{\boldsymbol{A}}^2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} & \hat{\boldsymbol{W}}^2 \\ \hat{\boldsymbol{W}}^1 & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{L}^1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}^2 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{A}}^1 & \hat{\boldsymbol{W}}^2 \boldsymbol{L}^2 \\ \hat{\boldsymbol{W}}^1 \boldsymbol{L}^1 & \hat{\boldsymbol{A}}^2 \end{bmatrix}, \quad (14)$$

where matrices $\hat{\boldsymbol{A}}^k$, $\hat{\boldsymbol{W}}^k$, $\boldsymbol{L}^k$, $k = \overline{1,2}$, given by (8). The sufficient condition (7) stated by Theorem 1 can be equivalently written as

$$\boldsymbol{p}(t+1) \geq \boldsymbol{\Theta} \boldsymbol{p}(t), \ \forall t \in \mathbb{Z}_+, \quad (15)$$

Remark 3 suggests us to explore the role of matrix $\boldsymbol{\Theta}$ (14) in ensuring the CWAS( $\boldsymbol{p}^1, \boldsymbol{p}^2$ ) of IBAM($\mathsf{A}^1, \mathsf{A}^2, \mathsf{W}^1, \mathsf{W}^2$). Let us first notice the special structure of the test matrix $\boldsymbol{\Theta}$, which is *nonnegative* (all its elements are nonnegative). This remark motivates us to present some preparatory results.

*Lemma 1.* Let $\boldsymbol{\Theta} = [\theta_{rs}] \in \mathbb{R}^{q\times q}$ be a nonnegative square matrix and let us denote by $\lambda_r(\boldsymbol{\Theta})$, $r = \overline{1,q}$, its eigenvalues. Then, $\boldsymbol{\Theta}$ has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(\boldsymbol{\Theta})$, which fulfils the dominance condition $|\lambda_r(\boldsymbol{\Theta})| \leq \lambda_{\max}(\boldsymbol{\Theta})$ for all $r = \overline{1,q}$. Moreover, $\theta_{rr} \leq \lambda_{\max}(\boldsymbol{\Theta})$, $r = \overline{1,q}$.

*Proof:* It results from ([13], Lemma 2.1) and from ([8], Corollary 8.1.20).

*Lemma 2:* If $\boldsymbol{\Theta}, \boldsymbol{\Psi} \in \mathbb{R}^{q\times q}$ are nonnegative matrices satisfying $\boldsymbol{\Theta} \leq \boldsymbol{\Psi}$, then $\lambda_{\max}(\boldsymbol{\Theta}) \leq \lambda_{\max}(\boldsymbol{\Psi})$.

*Proof:* It results from ([8], Theorem 8.1.18).

*Lemma 3*: If $\boldsymbol{\Theta} = [\theta_{rs}] \in \mathbb{R}^{q\times q}$ is a nonnegative matrix, then, for any $\sigma \in \mathbb{R}_+$, with $\lambda_{\max}(\boldsymbol{\Theta}) < \sigma$, there exists a positive vector $\gamma \in \mathbb{R}^q$, $\boldsymbol{\gamma} > \boldsymbol{0}$, such that $\boldsymbol{\Theta}\boldsymbol{\gamma} < \sigma\boldsymbol{\gamma}$.

*Proof:* Since $\boldsymbol{\Theta}$ is nonnegative, for any $\sigma \in \mathbb{R}_+$, $\lambda_{\max}(\boldsymbol{\Theta}) < \sigma$, there exists an $\varepsilon = \varepsilon(\sigma) > 0$ such that $\lambda_{\max}(\boldsymbol{\Theta} + \varepsilon \boldsymbol{E}) \leq \sigma$, where $\boldsymbol{E} = [e_{rs}] \in \mathbb{R}^{q\times q}$, with $e_{rs} = 1$, $r,s = \overline{1,q}$. Thus, for the Perron eigenvector $\gamma \in \mathbb{R}^q$, $\boldsymbol{\gamma} > \boldsymbol{0}$, of the positive matrix $\boldsymbol{\Theta} + \varepsilon \boldsymbol{E} > \boldsymbol{O}$ we can write $\boldsymbol{\Theta}\boldsymbol{\gamma} < (\boldsymbol{\Theta} + \varepsilon \boldsymbol{E})\boldsymbol{\gamma} = \lambda_{\max}(\boldsymbol{\Theta} + \varepsilon \boldsymbol{E})\boldsymbol{\gamma} \leq \sigma\boldsymbol{\gamma}$. Note that when $\boldsymbol{\Theta}$ is irreducible, the existence of the Perron-Frobenius eigenvector $\gamma \in \mathbb{R}^q$, $\boldsymbol{\gamma} > \boldsymbol{0}$, ensures the equality $\boldsymbol{\Theta}\boldsymbol{\gamma} = \lambda_{\max}(\boldsymbol{\Theta})\boldsymbol{\gamma}$.

We are now able to establish the following result.

*Theorem 2.* If matrix $\boldsymbol{\Theta}$ defined by (14) is Schur stable, then there exist two vector functions $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, satisfying the conditions from Definition 1 so that IBAM($\mathsf{A}^1, \mathsf{A}^2, \mathsf{W}^1, \mathsf{W}^2$) is CWAS( $\boldsymbol{p}^1, \boldsymbol{p}^2$ ).

*Proof*: Indeed, if $\boldsymbol{\Theta}$ is Schur stable, then the vector function

$$\boldsymbol{p}(t) = \boldsymbol{\Theta}^t \boldsymbol{p}(0) + \sum_{\zeta=0}^{t-1} \boldsymbol{\Theta}^\zeta \boldsymbol{v}(t-1-\zeta) \quad (16)$$

(defined with $\boldsymbol{p}(0) > \boldsymbol{0}$ and adequate $\boldsymbol{v}(\zeta) \geq 0$, $\zeta \in \mathbb{Z}_+$, such that $\lim_{t\to\infty} \sum_{\zeta=0}^{t} \boldsymbol{\Theta}^\zeta \boldsymbol{v}(t-\zeta) = \boldsymbol{0}$ ), satisfies the algebraic inequality (15) and $\boldsymbol{p}(t) > \boldsymbol{0}$, $\forall t \in \mathbb{Z}_+$, $\lim_{t\to\infty} \boldsymbol{p}(t) = \boldsymbol{0}$. The functions ensuring CWAS( $\boldsymbol{p}^1, \boldsymbol{p}^2$ ) result from the appropriate partitioning of $\boldsymbol{p}$, according to (13).

### *3.2. CWEAS of an IBAM*

*Theorem 3.* IBAM($\mathsf{A}^1, \mathsf{A}^2, \mathsf{W}^1, \mathsf{W}^2$) is CWEAS( $\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$ ) if the following algebraic inequalities hold

$$\begin{aligned} \sigma\boldsymbol{\alpha}^1 &\geq \hat{\boldsymbol{A}}^1\boldsymbol{\alpha}^1 + \hat{\boldsymbol{W}}^2\boldsymbol{L}^2\boldsymbol{\alpha}^2, \\ \sigma\boldsymbol{\alpha}^2 &\geq \hat{\boldsymbol{A}}^2\boldsymbol{\alpha}^2 + \hat{\boldsymbol{W}}^1\boldsymbol{L}^1\boldsymbol{\alpha}^1. \end{aligned} \quad (17)$$

*Proof:* It is a direct consequence of Theorem 1 when the time-dependence of vector-functions $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, is given by (6).

*Remark 4*. The same as in Remark 3, let us notice that by introducing the augmented vector

$\boldsymbol{\alpha}=\left[\boldsymbol{\alpha}^{1\tau}\ \ \boldsymbol{\alpha}^{2\tau}\right]^{\tau}\in\mathbb{R}^{m+n}$, inequalities (17) may be written in the equivalent matrix-form

$$\sigma\boldsymbol{\alpha}\geq\boldsymbol{\Theta}\boldsymbol{\alpha}, \tag{18}$$

where $\boldsymbol{\Theta}$ is given by (14).

Similarly to Theorem 2, the following result is available for CWEAS.

*Theorem 4.* If matrix $\boldsymbol{\Theta}$ defined by (14) is Schur stable, then there exist two positive vectors $\boldsymbol{\alpha}^1\in\mathbb{R}^m$, $\boldsymbol{\alpha}^1>\boldsymbol{0}$, $\boldsymbol{\alpha}^2\in\mathbb{R}^n$, $\boldsymbol{\alpha}^2>\boldsymbol{0}$, and a scalar $\sigma\in(0,1)$, so that IBAM($\mathsf{A}^1,\mathsf{A}^2,\mathsf{W}^1,\mathsf{W}^2$) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$).

*Proof*: If matrix $\boldsymbol{\Theta}$ is Schur stable, then Lemma 3 ensures the existence of $\sigma\in\mathbb{R}_+$, $\lambda_{\max}(\boldsymbol{\Theta})<\sigma<1$ and $\boldsymbol{\alpha}\in\mathbb{R}^{m+n}$, $\boldsymbol{\alpha}>0$, satisfying inequality (18). The two positive vectors ensuring CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) result from the appropriate partitioning of $\boldsymbol{\alpha}$. ■

## 4. CWAS / CWEAS RESULTS FOR BAMS WITH FIXED PARAMETERS

The generality of our results on CWAS/CWEAS of IBAMs includes the particular case of a BAM with fixed parameters (i.e. without uncertainties), obtained for $\underline{\boldsymbol{A}}^k=\overline{\boldsymbol{A}}^k=\boldsymbol{A}^k$, $\underline{\boldsymbol{W}}^k=\overline{\boldsymbol{W}}^k=\boldsymbol{W}^k$, $k=\overline{1,2}$, in (3). The CWAS/CWEAS approach relies on the replacement of the test matrix $\boldsymbol{\Theta}\in\mathbb{R}^{(m+n)\times(m+n)}$, built according to (14), by the test matrix $\boldsymbol{\Omega}\in\mathbb{R}^{(m+n)\times(m+n)}$ defined as:

$$\boldsymbol{\Omega}=\begin{bmatrix}|\boldsymbol{A}^1| & \boldsymbol{O}\\ \boldsymbol{O} & |\boldsymbol{A}^2|\end{bmatrix}+\begin{bmatrix}\boldsymbol{O} & |\boldsymbol{W}^2|\\ |\boldsymbol{W}^1| & \boldsymbol{O}\end{bmatrix}\begin{bmatrix}\boldsymbol{L}^1 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{L}^2\end{bmatrix}=\\ =\begin{bmatrix}|\boldsymbol{A}^1| & |\boldsymbol{W}^2|\boldsymbol{L}^2\\ |\boldsymbol{W}^1|\boldsymbol{L}^1 & |\boldsymbol{A}^2|\end{bmatrix}, \tag{19}$$

with matrices $\boldsymbol{L}^k$, $k=\overline{1,2}$, given by (8).

*Theorem 5.* If matrix $\boldsymbol{\Omega}$ defined by (19) is Schur stable, then

(**a**) BAM (1) is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) if the augmented vector function $\boldsymbol{p}(t)$ given by (13) fulfills the difference inequality

$$\boldsymbol{p}(t+1)\geq\boldsymbol{\Omega}\,\boldsymbol{p}(t),\ \forall t\in\mathbb{Z}_+; \tag{20}$$

(**b**) BAM (1) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) if the scalar $\sigma$ and the vector $\boldsymbol{\alpha}=\left[\boldsymbol{\alpha}^{1\tau}\ \ \boldsymbol{\alpha}^{2\tau}\right]^{\tau}\in\mathbb{R}^{m+n}$ fulfill the algebraic inequality

$$\sigma\boldsymbol{\alpha}\geq\boldsymbol{\Omega}\boldsymbol{\alpha}. \tag{21}$$

*Remark 5.* The conditions $|a_i^1|<1$, $i=\overline{1,m}$, $|a_j^2|<1$, $j=\overline{1,n}$, are necessary for matrix $\boldsymbol{\Omega}$ defined by (19) to be Schur stable. It is worth noticing that such conditions are formulated as working hypotheses in most papers dealing with discrete-time BAMs.

*Remark 6.* If BAM (1) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$), then, for its unique equilibrium point $\boldsymbol{x}_e=\left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^{\tau}$ we can write

$$\begin{aligned}&\forall\varepsilon>0,\ \forall t_0\in\mathbb{Z}_+,\ \forall\boldsymbol{x}_0\in\mathbb{R}^{m+n},\\ &\left\|\boldsymbol{x}_0-\boldsymbol{x}_e\right\|_{\infty}^{\mathbf{A}}\leq\varepsilon\ \Rightarrow\\ &\Rightarrow\left\|\boldsymbol{x}(t;t_0,\boldsymbol{x}_0)-\boldsymbol{x}_e\right\|_{\infty}^{\mathbf{A}}\leq\varepsilon\sigma^{t-t_0},\\ &\qquad\forall t\in\mathbb{Z}_+,\ t\geq t_0,\end{aligned} \tag{22}$$

where the vector norm $\|\ \|_{\infty}^{\mathbf{A}}$ is defined by $\left\|\boldsymbol{x}\right\|_{\infty}^{\mathbf{A}}=\left\|\mathbf{A}\boldsymbol{x}\right\|_{\infty}$, $\forall\boldsymbol{x}\in\mathbb{R}^{m+n}$, with

$$\mathbf{A}=\operatorname{diag}\left\{1/\alpha_1^1,\ldots,1/\alpha_m^1,1/\alpha_1^2,\ldots,1/\alpha_n^2\right\}. \tag{23}$$

This shows that for each concrete neural network the definition of exponential stability of $\boldsymbol{x}_e$, e.g. [11], pp.107, with respect to the norm $\|\ \|_{\infty}^{\mathbf{A}}$ is fulfilled in the particular case $\delta(\varepsilon)=\varepsilon$, $\forall\varepsilon>0$. Similarly, the definition of global exponential stability of $\boldsymbol{x}_e$, e.g. [11], pp.108, is satisfied for the particular case $M=1$,

$$\begin{aligned}&\forall t_0\in\mathbb{Z}_+,\forall\boldsymbol{x}_0\in\mathbb{R}^{m+n},\\ &\left\|\boldsymbol{x}(t;t_0,\boldsymbol{x}_0)-\boldsymbol{x}_e\right\|_{\infty}^{\mathbf{A}}\leq M\sigma^{t-t_0}\left\|\boldsymbol{x}_0-\boldsymbol{x}_e\right\|_{\infty}^{\mathbf{A}},\\ &\qquad\forall t\geq t_0.\end{aligned} \tag{24}$$

*Remark 7.* In terms of the matrix norms induced by the vector norms, the sufficient condition for BAM (1) to be CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) can be formulated as

$$\left\|\boldsymbol{\Omega}\right\|_{\infty}^{\mathbf{A}}=\left\|\mathbf{A}\boldsymbol{\Omega}\mathbf{A}^{-1}\right\|_{\infty}\leq\sigma<1, \tag{25}$$

where matrix $\mathbf{A}$ is given by (23). This is a direct consequence of Theorem 5(b).

## 5. ILLUSTRATIVE EXAMPLE

To offer the intuitive support of "visualizing" the CWEAS property, let us consider the BAM with $m=2$ and $n=1$ described by

$$\begin{aligned} x_1^1(t+1) &= -0.1x_1^1(t) - 0.5f_1^2\left(x_1^2(t)\right), \\ x_2^1(t+1) &= 0.2x_2^1(t) + 0.4f_1^2\left(x_1^2(t)\right), \\ x_1^2(t+1) &= 0.15x_1^2(t) + 0.2f_1^1\left(x_1^1(t)\right) - \\ &\quad - 0.3f_2^1\left(x_2^1(t)\right), \quad t \in \mathbb{Z}_+. \end{aligned} \tag{26}$$

The activation functions $f_1^1, f_2^1, f_1^2 : \mathbb{R} \to \mathbb{R}$ are considered of bipolar sigmoid type with $\lambda = 1$, satisfying inequalities (2) with $L_1^1 = L_2^1 = L_1^2 = 1$. The equilibrium point of (26) is $\boldsymbol{x}_e = [0\ 0\ 0]^\tau$.

By using the notations

$$\begin{aligned} \boldsymbol{A}^1 &= \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \boldsymbol{W}^2 = \begin{bmatrix} -0.5 \\ 0.4 \end{bmatrix}, \\ \boldsymbol{A}^2 &= \begin{bmatrix} 0.15 \end{bmatrix}, \quad \boldsymbol{W}^1 = \begin{bmatrix} 0.2 & -0.3 \end{bmatrix}, \\ \boldsymbol{L}^1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{L}^2 = \begin{bmatrix} 1 \end{bmatrix}, \end{aligned} \tag{27}$$

the concrete form of matrix $\boldsymbol{\Omega}$ (19) is

$$\boldsymbol{\Omega} = \begin{bmatrix} |\boldsymbol{A}^1| & |\boldsymbol{W}^2|\boldsymbol{L}^2 \\ |\boldsymbol{W}^1|\boldsymbol{L}^1 & |\boldsymbol{A}^2| \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0.5 \\ 0 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.15 \end{bmatrix}. \tag{28}$$

Matrix $\boldsymbol{\Omega}$ is Schur stable since its dominant eigenvalue (its spectral radius) is $\lambda_{\max}(\boldsymbol{\Omega}) = 0.6239$. Inequality (21) is satisfied, as an equality, for $\sigma = \lambda_{\max}(\boldsymbol{\Omega})$ and $\boldsymbol{\alpha} = \begin{bmatrix} 0.9543 & 0.9436 & 1 \end{bmatrix}^\tau$, the corresponding positive eigenvector with $\|\boldsymbol{\alpha}\|_\infty = 1$. Theorem 5(b) shows that BAM (26) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) with $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ resulting from the appropriate partitioning of vector $\boldsymbol{\alpha}$, namely $\boldsymbol{\alpha}^1 = \begin{bmatrix} 0.9543 & 0.9436 \end{bmatrix}^\tau$ and $\boldsymbol{\alpha}^2 = \begin{bmatrix} 1 \end{bmatrix}$. The exponential type functions $\boldsymbol{p}^1(t) = \sigma^t \boldsymbol{\alpha}^1$, $\boldsymbol{p}^2(t) = \sigma^t \boldsymbol{\alpha}^2$, $t \in \mathbb{Z}_+$, ensure the fulfillment of the CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) condition or, equivalently, of the CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) condition. This property is graphically illustrated by the evolution of the discrete-time state trajectory plotted in Fig. 1, considered as a generic representation showing that the trajectories do not leave the time-dependent set defined by

$$\begin{aligned} \mathcal{I}(t) = [-p_1^1(t), p_1^1(t)] \times [-p_2^1(t), p_2^1(t)] \times \\ \times [-p_1^2(t), p_1^2(t)], \quad t \in \mathbb{Z}_+, \end{aligned} \tag{29}$$

once initialized inside $\mathcal{I}(0)$.

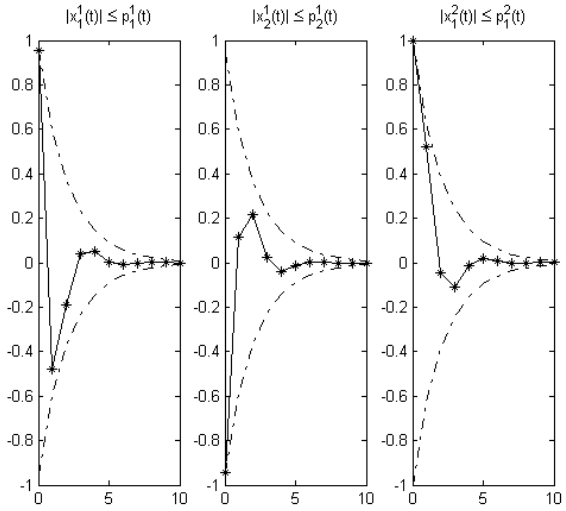


**Fig. 1.** Plots of a generic state trajectory of BAM (26) illustrating the intuitive meaning of the CWEAS property.

In this example we deal with a BAM having fixed parameters in order to construct the intuitive image for the state-space trajectories meeting the CWEAS condition. When an IBAM is considered, such intuitive elements characterize each system belonging to the whole family and, therefore, the overall analysis requires a unique instrument. An instrument of this type is given by the main results of our paper referring to the stability test of the matrix $\boldsymbol{\Theta}$ (14).

## 6. CONCLUSIONS

This paper provides easy-to-apply algebraic criteria for exploring the componentwise (exponential) asymptotic stability of discrete-time BAM neural networks with interval type parameters. These criteria are formulated in terms of Schur stability of a test matrix adequately built from the nonlinear system. Theorems 2 and 4 are the key elements of our

approach that gives a qualitative characterization of the dynamics at the level of the state vector components. This novel point of view refines the classical results in stability theory based on global information about the state vector, expressed in terms of arbitrary norms.

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