Algebraic State Estimation For Linear Time-Varying Systems ⋆

Y. Tian * **H.P. Wang** * **T. Floquet** **

* *Sino-French Joint Laboratory of Automation and Signal Processing and School of Automation, Nanjing University of Science and Technology, Nanjing, China. (e-mail:tianyang@njust.edu.cn, hp.wang@njust.edu.cn)*

** *LAGIS (CNRS, UMR 8146), École Centrale de Lille, 59650 Villeneuve d'Ascq and Équipe Projet Non-A, INRIA Lille-Nord Europe, France.(e-mail:thierry.floquet@ec-lille.fr)*

Abstract: In this paper, we introduce a new algebraic state estimation approach for the linear time-varying (LTV) systems. This approach is based on the following mathematical tools: Leibniz formula, generalized integration by parts, operational calculus and distribution theory. Firstly, we estimate the successive time derivatives of the measured output. Then, a generalized expression of the state variables of the system as a function of the integrals of the output and the input is obtained. Comparisons with the Kalman-type observer and some simulation results are given to illustrate the performance of the proposed approach.

Keywords: Algebraic approach, State estimation, Linear time-varying systems.

1. INTRODUCTION

State estimation is undoubtedly crucial research topic in control systems, and the associated problems are of great interest for engineers. Indeed, the state is not always available by direct measurement (for cost reasons, technological constrains, etc), especially such signals come in a quite large number, including time varying signals characterizing the system and unmeasured external disturbances, etc. Thus, a state observer (a dynamic auxiliary system), which gives a complete estimate based on measurements and inputs, must be designed (Wang et al., 2012, 2014a,b).

In the context of deterministic linear finite-dimensional time-invariant systems, an observer can be designed if the system is observable, i.e. if any initial state $x(t_0)$ at t_0 can be determined from the knowledge of the system output y and the control u on some time interval $[t_0, t_0 + T]$. The observability can be verified by the well-known Kalman rank condition (Kalman, 1960) and an observer leading to the asymptotic estimation of the state was firstly introduced by Luenberger (Luenberger, 1966).

For the observer design problem of LTV systems (i.e. contain some time-dependent parameters), there are also some pioneering studies. According to the Theorem (2.2) in (O'Reilly, 1983), if a system is completely observable (the definition will be recalled later), there exists an asymptotically stable observer. Such a type of observers takes the form of Kalman-Bucy filter (Kalman and Bucy, 1961). In (Hammouri and Morales, 1990), an observer for state-affine systems was constructed and it depends on the input of the the systems. In (Brdiek and Rotella, 1993), a design method of full order and reduced order observer for the linear time-varying systems without stochastic terms is reviewed, but some Riccati equation need to be solved. Furthermore, in the work of J. Trumpf (Trumpf, 2007), the necessary and sufficient existence conditions for tracking and asymptotic observers for linear functions of the state are given, whereas the way to find the right matrices satisfied with those conditions is not so evident.

The purpose of this article is to design a new fast state estimator for LTV systems by using an algebraic approach, which is an extension of M. Fliess and H. S. Ramirez's work originated from the linear time-invariant systems identification (Fliess and Sira-Ramirez, 2003, 2008). Based on this algebraic method, (Mboup et al., 2007) work on the signal time derivative estimation, (Gensior et al., 2007) study the experimental applications of parameter identification following similar ideas, in the linear context see also (Tian et al., 2008), and the switched systems estimation see (Zheng et al., 2009), (Tian et al., 2011). The main idea of this new approach is to apply some algebraic operations to a linear differential equation of the analyzed signals in the complex domain, when come back to the time domain with the inverse Laplace transform, one obtains an expression of the integral of the input and the output signal. As a result, the process of estimation is represented by an exact integral formula, rather than by an auxiliary dynamic system, without any other equations to be solved. In this approach, the successive time derivatives of the output are expressed as a function of the integral of the output $y(t)$ itself and of the input $u(t)$ so that the state

⋆ This work was partially supported by the National Natural Science Foundation of China (61304077, 61203115), by the Natural Science Foundation of Jiangsu Province (BK20130765), by the JiangSu planned projects for postdocatoral research funds (1202018c), and by the Zijin Intelligent Program of Nanjing University of Science and Technology (2013_ZJ_0105)
Corresponding author: H.P Wang (hp.wang@njust.edu.cn)

$x(t)$ can be estimated in terms of the integral of $y(t)$ and $u(t)$ in order to attenuate the influence of measurement noises.

The proposed algebraic method exhibits the following features:

- the state can be efficiently approximated which is independent of the initial values,
- it is non-asymptotic: the estimated value tracks the true value in a finite time,
- state estimation is given by an explicit formula which can be computer-implemented formally and quickly,
- robustness properties with respect to additive noise.

This paper is organized as follows. We begin with the problem formulation. In Section 3, the successive time derivatives of the measured output are estimated by Leibniz formula, operational calculus, integration by parts and distribution theory. Then, we apply the estimated successive time derivatives of the output to achieve the state estimation of the systems. Simulation results are given in Section 4 to highlights the efficiency and the robustness properties of the proposed approach w.r.t noisy measurements, and comparisons with the Kalman-type observer are also included. Finally, some conclusions and perspectives are given in Section 5.

2. PROBLEM STATEMENT

Consider the linear time-varying systems given by:

$$\begin{cases} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{cases} \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the input, $y \in R^d$ represents the output. $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times m}$ and $C(t) \in R^{d\times n}$ are matrices with time varying coefficients.

For the LTV systems, the definition of completely/totally observable is recalled (Kreindler and Sarachik, 1964):

Let $t_f > t_0$. Then, the dynamic system (1) is

- completely observable on $[t_0; t_f]$ if any initial state $x(t_0)$ at t_0 can be determined from the knowledge of the output $y(t)$ and the control $u(t)$ on $[t_0; t_f]$;
- totally observable on $[t_0; t_f]$ if it is completely observable on every subinterval of $[t_0; t_f]$.

In (Silverman and Meadows, 1967), the observability of the system (1) characterized in terms of $A(t)$, $C(t)$ and their appropriate time derivatives is defined as follows:

On the interval $[t_0; t_f]$, the dynamic system (1) is

- completely observable if rank $O(t) = n$ on $[t_0; t_f]$;
- totally observable if and only if rank $O(t) = n$ on every subinterval of $[t_0; t_f]$.

where $O(t)$ is the observability matrix defined by:

$$\begin{aligned} O(t) &= [S_0(t),\ S_1(t), ...,\ S_{n-1}(t)]^T, \qquad (2)\\ S_0(t) &= C^T(t),\\ S_{k+1}(t) &= A^T(t)S_k(t) + \dot{S}_k(t), k = 0, ..., n-2. \end{aligned}$$

In (Fliess and Diop, 1991), module theory notions are used to define the observability [1] of the system (1) as the possibility to express all the variables of the system, (in particular all the state variables) as combinations of the components of the input variable, the output variable and of their time derivatives up to a finite order. In this context, this observability matrix $O(t)$ can be rewritten as follows:

$$\begin{aligned} O(t) &= \left[C^T(t), \Delta(t)C^T(t), ..., \Delta^{n-1}(t)C^T(t)\right]^T,\\ \Delta(t) &= A^T(t) + \frac{d}{dt}. \end{aligned}$$

If the dynamic system (1) is completely observable and $(A(t), C(t))$ are bounded, there exists a Kalman-type observer (Besancon, 2007) of the form :

$$\dot{x}_e(t) = A(t)x_e(t) + B(t)u(t) - K(t)(C(t)x_e(t) - y(t)) \tag{3}$$

$K(t)$ is given by:

$$\begin{aligned} \dot{P}(t) &= P(t)A^T(t) + A(t)P(t) - P(t)C^T(t)W^{-1}(t)C(t)P(t)\\ &\quad + V(t) + \delta P(t),\\ K(t) &= P(t)C^T(t)W^{-1}(t),\\ x_e(0) &= x_{e0}, \quad P(0) = P_0 = P_0^T > 0. \end{aligned}$$

with either $\delta > 2\|A(t)\|$ for all t or $V = V^T > 0$. $W(t)$ and $P(t)$ are symmetric positive definite matrices of adapted dimensions.

Hereafter, we introduce the algebraic state estimation and the comparisons with this Kalman-type observer will be given later.

3. ALGEBRAIC STATE ESTIMATION

From now on, for the sake of the clarity and without loss of generality, only observable mono-variable systems are considered, that is to say: $u \in R$ and $y \in R$ (This approach can be extended to certain Multi-input Multi-output systems with the decoupling technique). It is aimed to estimate the state $x(t)$ in a fast way and on the basis of possibly noisy measurements. For this, exact expressions of the state are derived as a function of the integral of the output and the input. Since the integral operator has a filtering effect, the influence of measurement noise can be reduced.

3.1 Notations

For the sake of convenience, some useful formulas are introduced as follows (see (Yosida, 1984)):

[1] Which is also valid for nonlinear systems (Barbot et al., 2007; Fliess and Sira-Ramirez, 2003).

$$\text{(i)}\mathcal{L}^{-1}\left(\frac{1}{s^l}\frac{d^kY(s)}{ds^k}\right)=\begin{cases}\displaystyle\int_0^t \frac{(-\tau_1)^k(t-\tau)^{l-1}y(\tau)}{(l-1)!}d\tau, & l\geq 1\\ \displaystyle\frac{d^l\left((-t)^k y(t)\right)}{dt^l}, & l\leq 0\end{cases}$$

(ii)Leibniz formula:

$$\frac{d^h(x(s)y(s))}{ds^h}=\sum_{j=0}^{h}\binom{h}{j}\frac{d^{h-j}(x(s))}{ds^{h-j}}\frac{d^j(y(s))}{ds^j}$$

$$\text{(iii)}(f*g)(t)=\int_0^t f(t-\lambda)g(\lambda)d\lambda$$

$$\text{(iv)}\mathcal{L}^{-1}\left(g_1(s)g_2(s)\right)=g_1(t)*g_2(t)\quad \text{Convolution theorem}$$

3.2 Estimation of the successive time derivatives of the measured output

In the following, an algebraic method is developed to obtain a fast and accurate estimate of the output and a finite number of its time derivatives. Since the observable system's observability matrix is invertible, once those variables are known, one can recover the state of the system.

Theorem 1. For the linear time-varying mono-variable systems, the estimates of the successive time derivatives of the measured output y are given by:

$$y_e(t)=\frac{1}{(-t)^n}M_0(t) \tag{4}$$

$$\begin{pmatrix} y_e^{(1)}(t)\\ y_e^{(2)}(t)\\ y_e^{(3)}(t)\\ \vdots\\ y_e^{(n-1)}(t)\end{pmatrix}=\frac{1}{(-t)^n}\left(\begin{pmatrix} M_1(t)\\ M_2(t)\\ M_3(t)\\ \vdots\\ M_{n-1}(t)\end{pmatrix}-N(t)\begin{pmatrix} y_e(t)\\ y_e^{(1)}(t)\\ y_e^{(2)}(t)\\ \vdots\\ y_e^{(n-2)}(t)\end{pmatrix}\right) \tag{5}$$

with

$$N(t)=\begin{pmatrix} N_{1,1}(t) & 0 & 0 & \dots & 0\\ N_{2,1}(t) & N_{2,2}(t) & 0 & \dots & 0\\ N_{3,1}(t) & N_{3,2}(t) & N_{3,3}(t) & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ N_{n-1,1}(t) & N_{n-1,2}(t) & N_{n-1,3}(t) & \dots & N_{n-1,n-1}(t)\end{pmatrix}$$

$$N_{p,l}(t)=\sum_{j=n-p}^{n-1}\gamma_j n^1_{l-1,w}(t)+n^2_{p,l-1}(t),\quad \gamma_j=\frac{n!n!}{j!j!(n-j)!},$$

$$n^1_{g,w}(t)=\binom{w}{g}\frac{j!(-1)^j}{(j-w+g)!}t^{j-w+g},\quad w=p+j-n,$$

$$n^2_{p,k}(t)=\binom{p}{k}\frac{(-1)^n n!}{(n-p+k)!}t^{n-p+k},$$

$$m_{p,f_i,i}=\{(t-\tau)^{n-p-1}(-\tau)^n f_i(\tau)\}^{(i)},$$

$$M_p(t)=\sum_{i=0}^{n-1}\frac{(-1)^i\int_0^t\left(m_{p,b_i,i}u(\tau)-m_{p,a_i,i}y(\tau)\right)d\tau}{(n-p-1)!}$$
$$-\sum_{j=0}^{n-p-1}\gamma_j\int_0^t\frac{(t-\tau)^{-w-1}(-\tau)^j y(\tau)}{(-w-1)!}d\tau.$$

Demonstration

From system (1) which satisfies the observability assumption, one obtains the input-output relation :

$$\sum_{i=0}^{n}a_i(t)\ y^{(i)}(t)=\sum_{i=0}^{n-1}b_i(t)\ u^{(i)}(t), \tag{6}$$

with $a_n=1$.

a) Apply the Laplace transform to this I/O relation

$$s^n y(s)-...-y^{(n-1)}(0)+\sum_{i=0}^{n-1}\mathcal{L}\left(a_i(t)\ y^{(i)}(t)\right)$$
$$=\sum_{i=0}^{n-1}\mathcal{L}\left(b_i(t)\ u^{(i)}(t)\right).$$

b) Algebraic manipulations.

Deriving the preceding expression n times with respect to s, in order to eliminate the initial conditions, using the Leibniz formula and the relation

$$\frac{d^k(s^l)}{ds^k}=\begin{cases}\displaystyle\frac{l!}{(l-k)!}s^{l-k}, & \text{if } 0<k\leq l\\ 0, & \text{if } 0<l<k\\ \displaystyle\frac{(-1)^k(k-l-1)!}{(-l-1)!}s^{l-k}, & \text{if } l<0<k\end{cases} \tag{7}$$

Setting $\gamma_j=\frac{n!n!}{j!j!(n-j)!}$ and then multiply each side of the expression by $s^{-(n-p)}$, one obtains:

$$\sum_{j=0}^{n}\gamma_j\frac{s^j}{s^{n-p}}\frac{d^j(y(s))}{ds^j}+\sum_{i=0}^{n-1}\frac{1}{s^{n-p}}\frac{d^n\mathcal{L}\left(a_i(t)\ y^{(i)}(t)\right)}{ds^n}$$
$$=\sum_{i=0}^{n-1}\frac{1}{s^{n-p}}\frac{d^n\mathcal{L}\left(b_i(t)\ u^{(i)}(t)\right)}{ds^n}. \tag{8}$$

c) Return to time domain.

Applying the Inverse Laplace Transform (ILT) to (8), one gets:

$$\underbrace{\sum_{j=0}^{n}\mathcal{L}^{-1}\left(\frac{\gamma_j}{s^{n-p-j}}\frac{d^j(y(s))}{ds^j}\right)}_{\widetilde{A}_p}+\underbrace{\sum_{i=0}^{n-1}\mathcal{L}^{-1}\left(\frac{1}{s^{n-p}}\frac{d^n\mathcal{L}\left(a_i(t)\ y^{(i)}(t)\right)}{ds^n}\right)}_{\widetilde{B}_p}$$
$$=\underbrace{\sum_{i=0}^{n-1}\mathcal{L}^{-1}\left(\frac{1}{s^{n-p}}\frac{d^n\mathcal{L}\left(b_i(t)\ u^{(i)}(t)\right)}{ds^n}\right)}_{\widetilde{C}_p}. \tag{9}$$

Now, one needs to express $\widetilde{A}_p$, $\widetilde{B}_p$ et $\widetilde{C}_p$ as a function of y, u and their successive derivatives of order less than p.

c1) Using the formula (i), one gets:

$$\mathcal{L}^{-1}\left(\frac{1}{s^{n-p-j}}\frac{d^j(y(s))}{ds^j}\right)=$$
$$\begin{cases}\displaystyle\int_0^t\frac{(t-\tau)^{n-p-j-1}(-\tau)^j y(\tau)}{(n-p-j-1)!}d\tau, & 0\leq j\leq n-p-1\\ \displaystyle\frac{d^{p+j-n}\left((-t)^j y(t)\right)}{dt^{p+j-n}}, & n-p\leq j\leq n.\end{cases} \tag{10}$$

Setting $w = p + j - n$ and applying the Leibniz formula and the relation (7), one has:

$$\frac{d^w\left((-t)^j y(t)\right)}{dt^w} = \sum_{g=0}^{w} \binom{w}{g} \frac{j!(-1)^j t^{j-w+g} y^{(g)}(t)}{(j-w+g)!}. \quad (11)$$

When $j = n$, one has:

$$\frac{d^p\left((-t)^n y(t)\right)}{dt^p} = (-t)^n y^{(p)}(t) + \sum_{k=0}^{p-1} \binom{p}{k} \frac{n!(-1)^n t^{n-p+k} y^{(k)}(t)}{(n-p+k)!}. \quad (12)$$

Using (10), (11) and (12), $\widetilde{A}_p$ can be rewritten as follows:

$$\widetilde{A}_p = \sum_{j=0}^{n-p-1} \widetilde{F}_{p,j} + \sum_{j=n-p}^{n-1} \gamma_j \sum_{g=0}^{w} \binom{w}{g} \frac{j!(-1)^j t^{j-w+g} y^{(g)}(t)}{(j-w+g)!} + (-t)^n y^{(p)}(t) + \sum_{k=0}^{p-1} \binom{p}{k} \frac{n!(-1)^n t^{n-p+k} y^{(k)}(t)}{(n-p+k)!}, \quad (13)$$

where

$$\widetilde{F}_{p,j} = \gamma_j \int_0^t \frac{(t-\tau)^{-w-1}(-\tau)^j y(\tau)}{(-w-1)!} d\tau.$$

c2) In order to express $\widetilde{B}_p$ and $\widetilde{C}_p$, one applies the convolution theorem and gets

$$\widetilde{B}_p = \sum_{i=0}^{n-1} \frac{t^{n-p-1}\epsilon(t)}{(n-p-1)!} * (-t)^n a_i(t) y^{(i)}(t),$$

where $\epsilon(t)$ is the step function.

If g_1 is a C^1–function such that $g_1(0) = 0$ and g_2 is a C^0–function then

$$\int_0^t g_1(t-\tau) g_2(\tau) d\tau = \left[g_1(t-\tau) \int_0^\tau g_2(\mu) d\mu \right]_0^t - \int_0^t \frac{dg_1(t-\tau)}{d\tau} \left(\int_0^\tau g_2(\mu) d\mu \right) d\tau = \int_0^t \frac{dg_1(t-\tau)}{d(t-\tau)} \left(\int_0^\tau g_2(\mu) d\mu \right) d\tau.$$

This result can be extended for two distributions (g_1, g_2) with left hand side limited supports which implies the existence of the convolution product $g_1 * g_2$ and to the following more general result

$$\int_0^t g_1'(t-\lambda) g_2(\lambda) d\lambda = \int_0^t g_1(t-\lambda) g_2'(\lambda) d\lambda$$

which reads as

$$g_1'(t) * g_2(t) = g_1(t) * g_2'(t), \quad (14)$$

where the prime notation denotes the distribution derivation.

Using the formulas which were indicated at the beginning and (14), one has:

$$t^{n-p-1}\epsilon(t) * (-t)^n a_i(t) y^{(i)}(t) \overset{(14)}{=} (n-p-1)!\epsilon(t) * \underbrace{\int_0^t \ldots \int}_{(n-p-1)} (-\tau)^n a_i y^{(i)} d\tau \overset{(iii)}{=} (n-p-1)! \underbrace{\int_0^t \ldots \int}_{(n-p)} \epsilon(t-\tau)(-\tau)^n a_i y^{(i)} d\tau = \int_0^t (t-\tau)^{n-p-1} (-\tau)^n a_i y^{(i)} d\tau$$

So

$$\widetilde{B}_p = \sum_{i=0}^{n-1} \int_0^t \frac{(t-\tau)^{n-p-1}}{(n-p-1)!} (-\tau)^n a_i(\tau) y^{(i)}(\tau) d\tau.$$

Then, applying the integration by parts, which can be generalized for the function of class C^i:

$$\int_a^b f(\tau) g^{(i)}(\tau) d\tau = \left[\sum_{k=0}^{i-1} (-1)^k f^{(k)}(\tau) g^{(i-1-k)}(\tau) \right]_a^b + (-1)^i \int_a^b f^{(i)}(\tau) g(\tau) d\tau$$

one gets:

$$\widetilde{B}_p = \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} \left[\sum_{j=0}^{i-1} (-1)^j m_{p,a_i,j} y^{(i-j-1)}(\tau) \right]_0^t + \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} (-1)^i \int_0^t m_{p,a_i,i} y(\tau) d\tau,$$

where

$$m_{p,a_i,j} = \{(t-\tau)^{n-p-1}(-\tau)^n a_i(\tau)\}^{(j)} = \sum_{f=0}^{j} \binom{j}{f} \frac{d^{j-f}\{(\tau^2 - t\tau)^{n-p-1}\}}{d\tau^{j-f}} \frac{d^f\{(-\tau)^{p+1} a_i(\tau)\}}{d\tau^f} = \sum_{f=0}^{j} \binom{j}{f} \frac{d^{j-f}((\tau^2 - t\tau)^{n-p-1})}{d(\tau^2 - t\tau)^{j-f}} \frac{d(\tau^2 - t\tau)^{j-f}}{d\tau^{j-f}} \frac{d^f\{(-\tau)^{p+1} a_i(\tau)\}}{d\tau^f}.$$

Using the relation (7), one gets

$$\frac{d^{j-f}((\tau^2 - t\tau)^{n-p-1})}{d(\tau^2 - t\tau)^{j-f}} = \begin{cases} \dfrac{(n-p-1)!(\tau^2 - t\tau)^{n-p-1-j+f}}{(n-p-1-j+f)!}, & j-f \leq n-p-1 \\ 0, & n-p-1 < j-f \end{cases}$$

So

$$\sum_{j=0}^{i-1} (-1)^j \left[m_{p,a_i,j} y^{(i-j-1)}(\tau) \right]_0^t = 0,$$

and

$$\widetilde{B}_p = \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} \left((-1)^i \int_0^t m_{p,a_i,i} y(\tau) d\tau \right). \quad (15)$$

Applying the same operation for $\widetilde{C}$, one gets:

$$\widetilde{C}_p = \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} \left((-1)^i \int_0^t m_{p,b_i,i} u(\tau) d\tau \right). \quad (16)$$

Substituting the preceding results (13), (15) and (16) into (9), one obtains the following expressions for the time derivatives of y:

$$y^{(p)}(t) = \frac{1}{(-t)^n}\left(M_p(t) - \Gamma_p(y)\right), \qquad (17)$$

with

$$M_p(t) = \widetilde{C}_p - \widetilde{B}_p - \sum_{j=0}^{n-p-1} \widetilde{F}_{p,j},$$

$$\begin{aligned}
\Gamma_p(y) &= \sum_{j=n-p}^{n-1} \gamma_j \sum_{g=0}^{w} \binom{w}{g} \frac{j!(-1)^j t^{j-w+g}}{(j-w+g)!} y^{(g)}(t) + \\
&\quad \sum_{k=0}^{p-1} \binom{p}{k} \frac{n!(-1)^n t^{n-p+k}}{(n-p+k)!} y^{(k)}(t) \\
&= \left(\gamma_{n-p} n^1_{0,0} + \ldots + \gamma_{n-1} n^1_{0,p-1} + n^2_{p,0}\right) y + \ldots + \\
&\quad \left(\gamma_{n-p} n^1_{p-1,0} + \ldots + \gamma_{n-1} n^1_{p-1,p-1} + n^2_{p,p-1}\right) y^{(p-1)} \\
&= \left(\sum_{j=n-p}^{n-1} \gamma_j n^1_{0,w} + n^2_{p,0}\right) y + \ldots + \\
&\quad \left(\sum_{j=n-p}^{n-1} \gamma_j n^1_{p-1,w} + n^2_{p,p-1}\right) y^{(p-1)} \\
&= \sum_{l=1}^{p} N_{p,l} y^{(l-1)}.
\end{aligned}$$

In the particular case where $p = 0$, applying the result of (13), (15) and (16), one gets:

$$\begin{aligned}
\widetilde{A}_0 &= (-t)^n y(t) + \sum_{j=0}^{n-1} \gamma_j \frac{\int_0^t (t-\tau)^{n-j-1}(-\tau)^j y(\tau)d\tau}{(n-j-1)!}, \\
\widetilde{B}_0 &= \sum_{i=0}^{n-1} \frac{(-1)^i \int_0^t \{(t-\tau)^{n-1}(-\tau)^n a_i(\tau)\}^{(i)} y(\tau)d\tau}{(n-1)!}, \\
\widetilde{C}_0 &= \sum_{i=0}^{n-1} \frac{(-1)^i \int_0^t \{(t-\tau)^{n-1}(-\tau)^n b_i(\tau)\}^{(i)} u(\tau)d\tau}{(n-1)!}.
\end{aligned} \qquad (18)$$

So

$$M_0(t) = \widetilde{B}_0 - \widetilde{C}_0 - \sum_{j=0}^{n-1} \widetilde{F}_{0,j}$$

Clearly, $y_e(t)$ can be rewritten as in (4) as a function of the integral of the output y and the input u. Then, one substitutes $y_e(t)$ in (17) such that one obtains (5) as an expression of the estimate of the successive time derivatives of the measured output y. Due to the triangular structure of the matrix $N(t)$, one gets the estimate of the $p-th$ time derivative of y as a function of the integral of y and the input u only.

3.3 State reconstructor

In the following, with the knowledge of the output and the input and a finite number of their time derivatives, one can reconstruct the state of the system. Let us note (cf the observability matrix defined in (2)):

$$\begin{aligned}
S_0(t) &= C^T(t), \\
S_k(t) &= \left(A^T(t) + \frac{d}{dt}\right)^k C^T(t), \quad 0 < k < n \\
Q_{k0}(t) &= S_k^T(t)B(t), \\
Q_{kj}(t) &= \begin{cases} C(t)B(t), & j = k \\ Q_{(k-1)(j-1)}(t) + \dot{Q}_{(k-1)j}(t), & 1 \le j < k. \end{cases}
\end{aligned}$$

One can show that for all $0 \le k < n$:

$$y^{(k)}(t) = S_k^T(t)x(t) + \sum_{j=1}^{k} Q_{kj}(t)u^{(j-1)}(t). \qquad (19)$$

Indeed, one has:

$$\begin{aligned}
y(t) &= S_0^T(t)x(t) \\
\dot{y}(t) &= S_1^T(t)x(t) + Q_{11}(t)u(t)
\end{aligned}$$

Assume (19) is true for integer $k > 0$, one has:

$$\begin{aligned}
&y^{(k+1)}(t) \\
&= \dot{S}_k^T(t)x + S_k^T(t)\dot{x} + \dot{Q}_{k1}(t)u + Q_{k1}(t)\dot{u} + \ldots \\
&\quad + Q_{k(k-1)}(t)u^{(k-1)} + \dot{Q}_{kk}(t)u^{(k-1)} + Q_{kk}(t)u^{(k)} \\
&= \left(\dot{S}_k^T(t) + S_k^T(t)A(t)\right)x + \left(Q_{k0}(t) + \dot{Q}_{k1}(t)\right)u + \ldots \\
&\quad + \left(Q_{k(k-1)}(t) + \dot{Q}_{kk}(t)\right)u^{(k-1)} + Q_{(k+1)(k+1)}(t)u^{(k)} \\
&= S_{k+1}^T(t)x + \sum_{j=1}^{k+1} Q_{(k+1)j}(t)u^{(j-1)}.
\end{aligned}$$

Then it is true for integer $k+1$. Thus, one can estimate all the state x_e as a function of y_e, u and their time derivatives as follows:

$$x_e(t) = \begin{pmatrix} S_0^T(t) \\ S_1^T(t) \\ S_2^T(t) \\ \vdots \\ S_{n-1}^T(t) \end{pmatrix}^{-1} \left[\begin{pmatrix} y_e \\ \dot{y}_e \\ y_e^{(2)} \\ \vdots \\ y_e^{(n-1)} \end{pmatrix} - Q(t)\begin{pmatrix} u \\ \dot{u} \\ u^{(2)} \\ \vdots \\ u^{(n-2)} \end{pmatrix}\right].$$

with

$$Q(t) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ Q_{11}(t) & 0 & \ldots & 0 \\ Q_{21}(t) & Q_{22}(t) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ Q_{(n-1)1}(t) & Q_{(n-1)2}(t) & \ldots & Q_{(n-1)(n-1)}(t) \end{pmatrix}.$$

For the input u, it is assumed to be sufficiently differentiable and its derivatives are known. If it's not the case, an estimate technique of the numerical derivation developed by the similar algebraic method (Mboup et al., 2007) can solve out it. Thus, one can obtain an expression that no longer involves the derivatives of the input.

The observability matrix $O(t) = [S_0(t), S_1(t), ..., S_{n-1}(t)]^T$ is invertible since the system is assumed observable, but it should be pointed out that in some cases, for the numerical problem, there are some singular points needed to be treated particularly.

Note that all these computations are singular at $t = 0$ but becomes valid for any arbitrary small instant. Therefore one must to evaluate the formula not at $t = 0$ but after a small time ϵ.

4. NUMERICAL EXAMPLE AND SIMULATIONS

Consider a DC motor system whose electric part is neglected, and its equations are given by:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{1}{\tau(t)}x_2(t) + \dfrac{k}{\tau(t)}u(t) \end{cases}$$

with $y = x_1$ as measured output; x_1 is the angular position of the rotor, x_2 is the angular velocity of the rotor and u is the control input voltage. k is strictly positive constant and $\tau(t)$ is time-varying strictly positive parameter.

4.1 Application of developed algebraic approach

Write the input-output relationship:

$$y^{(2)}(t) + \frac{1}{\tau(t)}\dot{y} = \frac{k}{\tau(t)}u(t) \tag{20}$$

Step 1: Express y_e as a function of the integral of y.

a) Apply the Laplace transform to the relation (20).

$$s^2y(s) - sy(0) - \dot{y}(0) + \mathcal{L}\left(\frac{\dot{y}}{\tau(t)}\right) = k\mathcal{L}\left(\frac{u(t)}{\tau(t)}\right) \tag{21}$$

b) Derive (21) twice to eliminate the initial conditions:

$$2y(s) + 4s\frac{dy(s)}{ds} + s^2\frac{d^2y(s)}{ds^2} + \frac{d^2\mathcal{L}\left(\frac{\dot{y}}{\tau(t)}\right)}{ds^2} = k\frac{d^2\mathcal{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2} \tag{22}$$

Multiply each side of (22) by s^{-2}:

$$\frac{2}{s^2}y(s) + \frac{4}{s}\frac{dy(s)}{ds} + \frac{d^2y(s)}{ds^2} + \frac{1}{s^2}\frac{d^2\mathcal{L}\left(\frac{\dot{y}}{\tau(t)}\right)}{ds^2} = \frac{k}{s^2}\frac{d^2\mathcal{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2} \tag{23}$$

c) Apply the inverse Laplace transform to (23) using the expressions of (4) or (18) in order to return to time domain, one obtains the estimation of the output:

- $\tau(t) = a_0t + a_1$

$$y_e(t) = \frac{\int_0^t y(\lambda)\left(\frac{-2a_0\lambda^3 - 3a_1\lambda^2 + t(a_0\lambda^2 + 2a_1\lambda)}{(a_0\lambda + a_1)^2} + 6\lambda - 2t\right)d\lambda}{t^2} + k\frac{\int_0^t \frac{(t-\lambda)\lambda^2u(\lambda)}{a_0\lambda + a_1}d\lambda}{t^2} \tag{24}$$

- $\tau(t) = b_0 \sin b_1t + b_2$

$$y_e(t) = \frac{\int_0^t y(\lambda)\left(\frac{-(3\lambda^2 - 2t\lambda)}{b_0\sin(b_1\lambda) + b_2} + \frac{(\lambda^3 - t\lambda^2)b_0b_1\cos(b_1\lambda)}{(b_0\sin(b_1\lambda) + b_2)^2}\right)d\lambda}{t^2} + \frac{\int_0^t y(\lambda)(6\lambda - 2t)d\lambda + \int_0^t \frac{k(t-\lambda)\lambda^2u(\lambda)}{b_0\sin(b_1\lambda) + b_2}d\lambda}{t^2} \tag{25}$$

Step 2: Express $y^{(1)}$ as a function of y, u and their integral.

Multiply each side of (22) by s^{-1}:

$$\frac{2}{s}y(s) + 4\frac{dy(s)}{ds} + s\frac{d^2y(s)}{ds^2} + \frac{1}{s}\frac{d^2\mathcal{L}\left(\frac{\dot{y}}{\tau(t)}\right)}{ds^2} = \frac{k}{s}\frac{d^2\mathcal{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2} \tag{26}$$

Apply the ILT to (26) using the expressions of (5) or (13, 15, 16), one obtains:

- $\tau(t) = a_0t + a_1$

$$y^{(1)} = \frac{\int_0^t \left(y(\lambda)\left(\frac{a_0\lambda^2 + 2a_1\lambda}{(a_0\lambda + a_1)^2} - 2\right) + \frac{k\lambda^2u(\lambda)}{a_0\lambda + a_1}\right)d\lambda}{t^2} + \frac{2ty(t) - \frac{t^2y(t)}{\tau(t)}}{t^2} \tag{27}$$

- $\tau(t) = b_0 \sin b_1t + b_2$

$$y^{(1)} = \frac{\int_0^t y(\lambda)\left(\frac{2\lambda}{b_0\sin(b_1\lambda) + b_2} - \frac{\lambda^2b_0b_1\cos(b_1\lambda)}{(b_0\sin(b_1\lambda) + b_2)^2} - 2\right)d\lambda}{t^2} + \frac{\int_0^t \frac{k\lambda^2u(\lambda)}{b_0\sin(b_1\lambda) + b_2}d\lambda + 2ty(t) - \frac{t^2y(t)}{\tau(t)}}{t^2} \tag{28}$$

Then, one substitutes $y_e(t)$ (24) (respectively (25)) in the expression of $y^{(1)}(t)$ (27) (respectively (28)) such that one obtains $y_e^{(1)}(t)$ as an expression of the estimate of the successive time derivatives of the measured output y.

Step 3: Reconstruction of state.

3a) $\tau(t) = a_0t + a_1$, one obtains:

$$\begin{cases} \widehat{x}_1 = \dfrac{\int_0^t y(\lambda)\left(\frac{t(a_0\lambda^2 + 2a_1\lambda) - 2a_0\lambda^3 - 3a_1\lambda^2}{\tau^2(\lambda)} + 6\lambda - 2t\right)d\lambda}{t^2} + \dfrac{\int_0^t u(\lambda)\frac{k(t-\lambda)\lambda^2}{\tau(\lambda)}d\lambda}{t^2} \\ \widehat{x}_2 = \dfrac{\int_0^t \left(y(\lambda)\left(\frac{a_0\lambda^2 + 2a_1\lambda}{\tau^2(\lambda)} - 2\right) + u(\lambda)\frac{k\lambda^2}{\tau(\lambda)}\right)d\lambda}{t^2} + \dfrac{2ty_e(t) - \frac{t^2y_e(t)}{\tau(t)}}{t^2} \end{cases}$$

3b) $\tau(t) = b_0(\sin b_1t) + b_2$, one obtains:

$$\begin{cases} \widehat{x}_1 = \dfrac{\int_0^t y(\lambda)\left(\frac{2t\lambda - 3\lambda^2}{\tau(\lambda)} + \frac{(\lambda^3 - t\lambda^2)b_0b_1\cos(b_1\lambda)}{\tau^2(\lambda)} + 6\lambda - 2t\right)d\lambda}{t^2} + \dfrac{\int_0^t u(\lambda)\frac{k(t-\lambda)\lambda^2}{\tau(\lambda)}d\lambda}{t^2} \\ \widehat{x}_2 = \dfrac{\int_0^t \left(y(\lambda)\left(\frac{2\lambda\tau(\lambda) - \lambda^2b_0b_1\cos(b_1\lambda)}{\tau^2(\lambda)} - 2\right) + u(\lambda)\frac{k\lambda^2}{\tau(\lambda)}\right)d\lambda}{t^2} + \dfrac{2ty_e(t) - \frac{t^2y_e(t)}{\tau(t)}}{t^2} \end{cases}$$

4.2 Simulation

Hereafter, robust estimation with respect to noise and comparative study with Kalman-type observer are depicted. Simulations are given for a polynomial parameter (Fig. 1, Fig. 2): $\tau = a_0t + a_1$ ($a_0 = 0.001$, $a_1 = 1$) and a sinusoidal one (Fig. 3, Fig. 4): $\tau = b_0\sin(b_1t) + b_2$ ($b_0 = 2$, $b_1 = 0.2 * \pi$, $b_2 = 3$), $\delta = 0$, $R = 1$ and $V = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The initial conditions are: $x_2(0) = 5(rad/s)$, $x_1(0) = 1(rad)$, $k = 1$ and the input voltage is chosen as $u(t) = 12\sin(t)$.

For the Kalman-type observer (3), it is assumed that:

- $w(t)$ is a Gaussian white noise of the covariance matrix $W(t)$ and $v(t)$ is a Gaussian white noise of the covariance matrix $V(t)$ with the average values 0, that is to say:

$$E\{w(t)\} = 0,$$
$$E\{v(t)\} = 0,$$
$$E\{w(t)w^T(t)\} = W \times \delta(t - \tau),$$
$$E\{v(t)v^T(t)\} = V \times \delta(t - \tau).$$

- The noise $w(t)$ and $v(t)$ are independent Gaussian random variables, that is to say:

$$E\{w(t)v^T(t)\} = 0.$$

- The noise $w(t)$, $v(t)$ and the initial condition $x(0)$ are mutually independent,that is to say:

$$E\{x(0)w^T(t)\} = 0,$$
$$E\{x(0)v^T(t)\} = 0.$$

- $x(t)$ and $y(t)$ are independent variables.

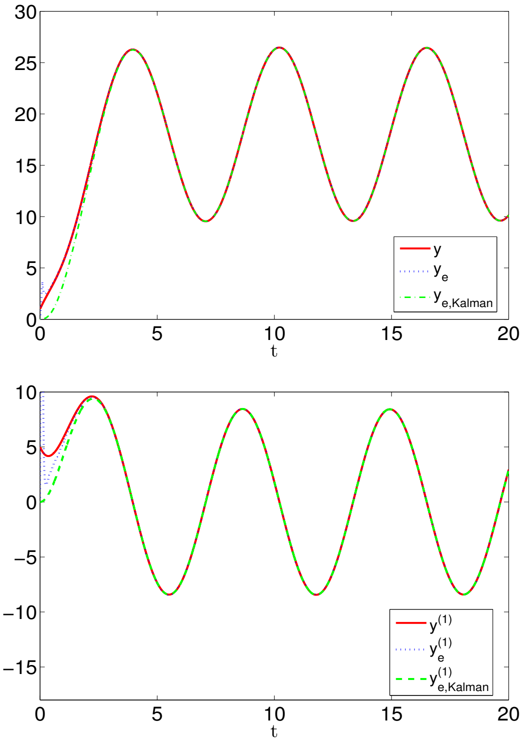

Fig. 1. States and its estimates (without noise) ($\tau = a_0 t + a_1$).

In the figures (Fig. 1 and Fig. 3), there is no measurement noise. It can be seen that the estimated value tracks quasi-instantaneously exactly the real value.

In the figures (Fig. 2 and Fig. 4), the measured signal $y(t)$ was perturbed by a white noise normally distributed in the interval $[-4, 4]$ (standard deviation $4/\sqrt{3}$). It can be seen that this estimator is quite robust w.r.t white noise.

In the following, we conclude the differences between these two methods.

Kalman-type observer:

(1) The observer is an auxiliary system;

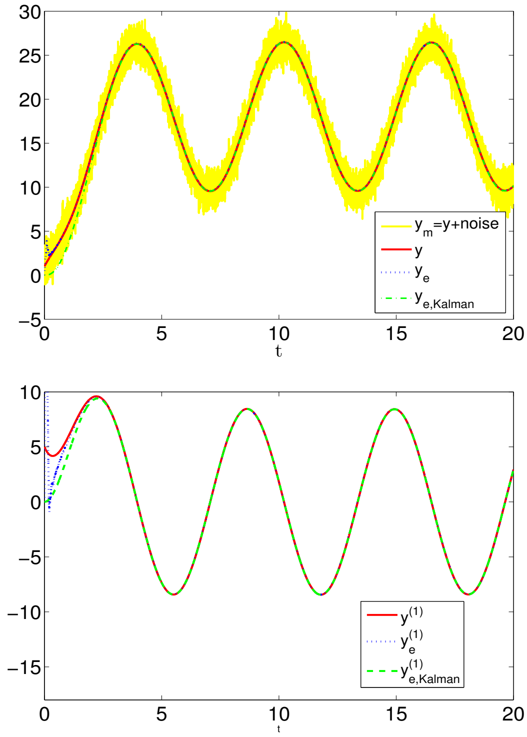

Fig. 2. States and its estimates (affected by white noise) ($\tau = a_0 t + a_1$).

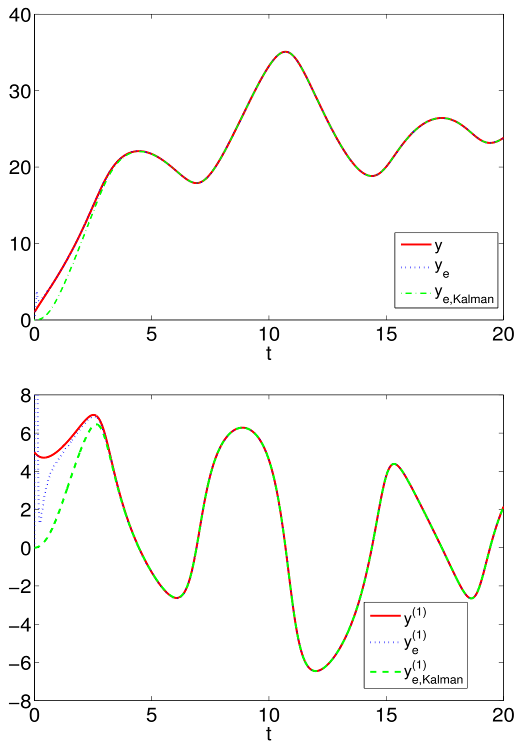

Fig. 3. States and its estimates (without noise) ($\tau = b_0 \sin(b_1 t) + b_2$).

(2) The convergence of the observer is asymptotic, and the convergence speed can be selected by the value of δ or V;

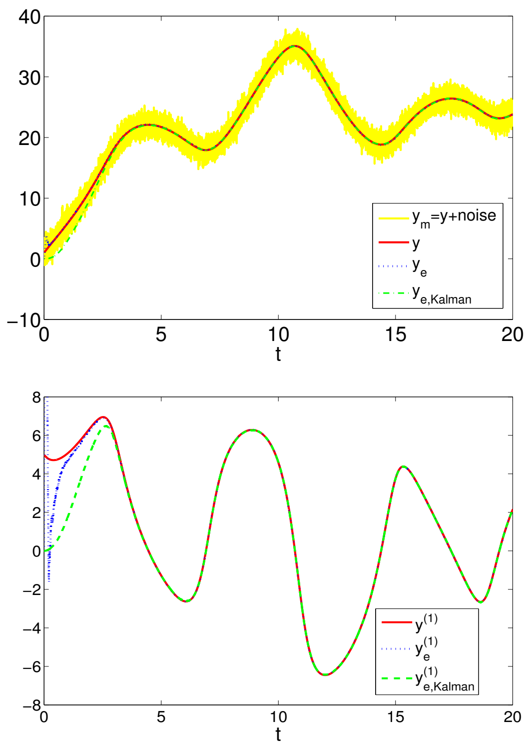

Fig. 4. States and its estimates (affected by white noise) ($\tau = b_0 \sin(b_1 t) + b_2$).

(3) The statistical properties of the noise and disturbance should be known.

Algebraic state estimation:

(1) It is based on algebraic operational calculus (Laplace here) and state estimation is given by an explicit formula;
(2) The estimated value tracks the true value in a finite time. This approach requires no convergence parameter adjustment (it only needs the system parameters $A(t)$ and $C(t)$);
(3) It is deterministic: no knowledge of the statistical property of the noise is required.

5. CONCLUSION AND PERSPECTIVES

In this paper, an algebraic state estimation approach for the linear time-varying systems has been introduced. The estimation is given by an explicit integral formula which can be carried out by the computer formally and quickly, without using a dynamic auxiliary system. The only required condition is that the time-varying parameters should be continuously derivable. Note that such a new technique could be a powerful tool to solve the state estimation problem and the parametric identification. In the near future, the above techniques and results can be generalized to state estimation of the switched system with LTV subsystems, or state estimation of linear time delay systems.

REFERENCES

Barbot, J.P., Fliess, M., and Floquet, T. (2007). An algebraic framework for the design of nonlinear observers with unknown inputs. *46th IEEE Conf. Decis. Contr.*

Besancon, G. (2007). An overview on observer tools for nonlinear systems. *Nonlinear Observers and Applications. Lecture Notes in Control and Information Scinences (LNCIS)*, 363, 1–33. ISBN: 978-3-540-73502-1.

Brdiek, E.B. and Rotella, F. (1993). Design of observers for non linear time variant systems. *IEEE Int. Conf. Syst. Man Cybern.*, 4, 216 – 220.

Fliess, M. and Diop, S. (1991). Nonlinear observability, identifiability and persistent trajectories. *36th IEEE Conf. Decis. Contr.*, 714–719.

Fliess, M. and Sira-Ramirez, H. (2003). An algebraic framework for linear identification. *ESAIM Contr. Optim.*, 9, 151–168.

Fliess, M. and Sira-Ramirez, H. (2008). Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques. In *Continuous-Time Model Identification from Sampled Data*. Springer.

Gensior, A., Weber, J., Guldner, H., and Rudolph, J. (2007). An algebraic parameter identification algorithm and asymptotic observers for estimation of the load of a boost converte. *IEEE Int. Symposium on Industrial Electronics.*

Hammouri, H. and Morales, J.D.L. (1990). Observer synthesis for state-affine systems. *29th IEEE Conf. Decis. Contr.*

Kalman, R.E. (1960). A new approach to linear filtering and prediction problems. *Trans. ASME - Journal of Basic Engineering*, 82-D, 35–45.

Kalman, R.E. and Bucy, R.S. (1961). New results in linear filtering and prediction theory. *Trans. ASME - Journal of Basic Engineering*, 83, 95–107.

Kreindler, E. and Sarachik, P.E. (1964). On the concepts of controllability and observability of linear systems. *IEEE Trans. Autom. Contr.*, 9, 129–136.

Luenberger, D.G. (1966). Observers for multivariable systems. *IEEE Trans. Autom. Contr.*, 11(2), 190–197.

Mboup, M., Join, C., and Fliess, M. (2007). A revised look at numerical differentiation with an application to nonlinear feedback control. *15th Medit. Conf. Contr. Automat.*

O'Reilly, J. (1983). Observers for linear systems. *Academic Press.*

Silverman, L.M. and Meadows, H.E. (1967). Controllability and observability in time-variable linear systems. *SIAM J. Contr.*, 5, 64–73.

Tian, Y., Floquet, T., Belkoura, L., and Perruquetti, W. (2011). Algebraic switching time identification for a class of linear hybrid systems. In *J. Nonlinear Analysis: Hybrid systems*, volume 5, 233–241.

Tian, Y., Floquet, T., and Perruquetti, W. (2008). Fast state estimation in linear time-invariant systems: an algebraic approach. In *16th Medit. Conf. Contr. Automat.* Ajaccio, France.

Trumpf, J. (2007). Observers for linear time-varying systems. *Linear Algebra Appl*, 425(2-3), 303–312.

Wang, H., Tian, Y., and Christov, N. (2014a). Event-triggered observer based control of networked visual servoing control systems. *Contr. Eng. Appl. Inform.*, 16(1), 22–30.

Wang, H., Tian, Y., and Christov, N. (2014b). Piecewise-continuous observers for linear systems with sampled and delayed output. *Int. J. of Syst. Sci.*, DOI:10.1080/00207721.2014.953798.

Wang, H., Vasseur, C., Christov, N., and Koncar, V. (2012). Vision servoing of robot systems using piecewise continuous controllers and observers. *Mech. Syst. Sign.Proc.*, 33, 132–141.

Yosida, K. (1984). *Operational Calculus.* Springer, New York.

Zheng, G., Yu, L., Boutat, D., and Barbot, J. (2009). Algebraic observer for a class of switched systems with zeno phenomenon. *48th IEEE Conf. Decis. Contr.*