STABILITY AND SET-INVARIANCE TESTING FOR INTERVAL SYSTEMS

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Abstract: *Many works dealing with the stability analysis of interval systems developed criteria based on matrices that majorize (in a certain sense) the interval matrices describing the system dynamics. Besides this already classical employment, we prove that the majorant matrices also contain valuable information for the study of the exponentially decreasing sets, invariant with respect to the trajectories of the interval systems. The interval systems are considered with both discrete- and continuous-time dynamics. The invariant sets are characterized by arbitrary shapes, defined in terms of Hőlder vector p-norms,* $1 \le p \le \infty$. *Our results cover two types of interval systems, namely described by interval matrices of general form and by some particular classes of interval matrices. For the general case, we formulate necessary and sufficient conditions, when the shape of the invariant sets is defined by the norms* $p = 1, \infty$, *and sufficient conditions, when the shape is defined by the norms* $1 < p < \infty$. *For the particular cases, we provide necessary and sufficient conditions for all norms* $1 \le p \le \infty$.

Keywords: *interval dynamical systems, discrete-time systems, continuous-time systems, invariant sets, diagonally invariant exponential stability, Hőlder norms.*

1. INTRODUCTION

Our paper considers *interval systems* (ISs) described by the equation:

$$x'(t) = Ax(t)\,,\ x(t_0) = x_0 \in \mathbb{R}^n\,,\quad t, t_0 \in \mathbb{T}, t \ge t_0,\ A \in \mathsf{A}^I, \tag{1}$$

where:

(i) t is the independent variable with *discrete-time* (DT) ($\mathbb{T} = \mathbb{Z}_+$), or *continuous-time* (CT) ($\mathbb{T} = \mathbb{R}_+$) meaning;

(ii) the operator $(\ )'$ acts according to

DT case: $\boldsymbol{x}'(t) = \boldsymbol{x}(t+1)\,,\ t \in \mathbb{T} = \mathbb{Z}_+\,,$ (2-DT)

CT case: $\boldsymbol{x}'(t) = \dot{\boldsymbol{x}}(t)\,,\ t \in \mathbb{T} = \mathbb{R}_+\,;$ (2-CT)

(iii) A^I is an interval matrix:

$$\mathsf{A}^I = \{A \in \mathbb{R}^{n \times n} \mid A^- \le A \le A^+\} \quad (3)$$

defined by the componentwise inequalities $a_{ij}^- \le a_{ij} \le a_{ij}^+$, $i,j=1,\ldots,n$, for a_{ij}^-, a_{ij}, a_{ij}^+ representing the generic elements of matrices A^-, A and A^+ respectively.

During the past decades a large body of work has been developed for testing the stability of interval matrices [1]-[3], [5], [7]-[10], [16]-[18]. An interval matrix A^I is said to be stable if $\forall A \in \mathsf{A}^I$, A is Schur stable (DT case) or Hurwitz stable (CT case). An IS uses an interval matrix for describing parameter uncertainties, in the sense that the dynamics of a real plant is modeled by a time-invariant system belonging to the system family (1) with a fixed but unknown matrix $A \in \mathsf{A}^I$. According to [9], an IS is said to be stable if the associated interval matrix is stable.

The qualitative analysis of IS (1) developed by many works [1]-[3], [7], [9], [16], [18] relies on the properties of a majorant matrix $U = [u_{ij}]_{i,j=1,\ldots,n}$, built for A^I as follows:

DT case:

$$u_{ij} = \max\{|a_{ij}^-|, |a_{ij}^+|\},\ i,j=1,\ldots,n; \quad \text{(4-DT)}$$

CT case:

$$\begin{cases} u_{ii} = a_{ii}^+,\ i=1,\ldots,n, \\ u_{ij} = \max\{|a_{ij}^-|,|a_{ij}^+|\}, i \ne j, i,j=1,\ldots,n. \end{cases} \quad \text{(4-CT)}$$

The stability (in the sense of Schur or Hurwitz) of this majorant matrix provides a sufficient condition for the stability of IS (1), as shown in ([1], Theorem 3), [2], ([7], Theorem 3.4.17), ([16], Corollary 1.2) (*DT case*) and ([3], Theorem 3.1), [9], ([16], Corollary 2.2) (*CT case*). Moreover, papers ([1], Corollary 1 - time-invariant case), ([7], Lemma 3.4.18), ([16], Corollary 1.3) (*DT case*) and ([16], Corollary 2.3), ([18], Theorem 1) (*CT case*) prove that "U stable" is also a necessary condition for the stability of some classes of interval systems.

On the other hand, our researches [12] and [13] show that that "U stable" is a necessary and sufficient condition for the existence of exponentially decreasing (contractive) invariant sets of rectangular form (property called *componentwise exponential asymptotic stability*). These results demonstrate that the role played by the majorant matrix U in the qualitative analysis of IS (1) is not limited to the study of stability and deserves further investigation.

The current paper reveals new connections between the majorant matrix U and the invariance properties exhibited by IS (1). By the help of the majorant matrix U, we explore exponentially decreasing invariant sets with general forms defined by Hölder p-norms ($1 \le p \le \infty$). We show that the existence of such sets represents a stronger property for IS (1) than the traditional concept of stability; therefore this property has been called *diagonally invariant exponential stability* (abbreviated DIES). While the papers cited above [[1]-[3], [7], [9], [16], [18] use U for testing the standard stability of IS (1), our work uses U for testing the stronger property of DIES. Moreover, this scenario developed for DIES of IS accommodates our previous results on componentwise exponential asymptotic stability [12], [13] as a particular case corresponding to $p = \infty$.

Throughout the paper we use the following notations. For a vector $x \in \mathbb{R}^n$: $\| x \|_p$ is the Hölder vector p-norm defined for $1 \le p \le \infty$; $\| x \|_p^D = \| D^{-1} x \|_p$, where D is a positive definite diagonal matrix. For a square matrix $M \in \mathbb{R}^{n \times n}$: $\| M \|_p$ is the matrix norm induced by the vector norm $\| \ \|_p$; $\| M \|_p^D = \| D^{-1} M D \|_p$ is the matrix norm induced by the vector norm $\| \bullet \|_p^D$; $m_p^D(M) = \lim_{\theta \downarrow 0} (\| I + \theta M \|_p^D - 1)/\theta$ is a matrix measure ([4], pp. 29) associated with the matrix norm $\| \ \|_p^D$; $\sigma(M) = \{z \in \boldsymbol{C} \mid \det(zI - M) = 0\}$ is the spectrum of M, and $\lambda_i(M) \in \sigma(M)$, $i = 1,\ldots,n$, denote its eigenvalues.

The remainder of the text is organized as follows. Section 2 defines the DIES of IS (1) and provides its characterization in terms of (i) the transition matrix (the continuous semigroup of linear operators), (ii) Lyapunov functions and (iii) the matrix defining the difference / differential system (the generator of the semigroup). Section 3 formulates results based on the majorant matrix U that allow testing the DIES of IS (1). Section 4 comments on the contributions of our work to the qualitative analysis of interval systems. To support the fluent readability of the text, the proofs with higher complexity are given in the Appendix.

2. DIES OF INTERVAL SYSTEMS

This section introduces and characterizes the concept of diagonally invariant exponential stability relative to the norm $\| x \|_p^D = \| D^{-1}x \|_p$, where D is a positive definite diagonal matrix $D = \mathrm{diag}\{d_1,..., d_n\}$, $d_i > 0$, $i = 1,...,n$

Definition 1. IS (1) is called *diagonally invariant exponentially stable* relative to the norm $\| \; \|_p^D$ (abbreviated as DIES$_{p,D}$) if

DT case: there exists a constant $0 < r < 1$ such that

$$\forall \varepsilon > 0, \quad \forall t, t_0 \in \mathbb{Z}_+, t \ge t_0, \ \forall x_0 \in \mathbb{R}^n,$$
$$\|x_0\|_p^D \le \varepsilon \Rightarrow \|x(t;t_0,x_0)\|_p^D \le \varepsilon r^{t-t_0}; \qquad \text{(5-DT)}$$

CT case*:* there exists a constant $r < 0$ such that

$$\forall \varepsilon > 0, \quad \forall t, t_0 \in \mathbb{R}_+, t \ge t_0, \ \forall x_0 \in \mathbb{R}^n,$$
$$\|x_0\|_p^D \le \varepsilon \Rightarrow \|x(t;t_0,x_0)\|_p^D \le \varepsilon e^{r(t-t_0)}. \qquad \text{(5-CT)}$$

The constant $0 < r < 1$ (DT case), $r < 0$ (CT case) is called the *decreasing rate* for DIES$_{p,D}$. ■

Remark 1. The nomenclature DIES$_{p,D}$ introduced by Definition 1 is motivated by the following two facts:

Fact 1. DIES$_{p,D}$ is equivalent with the existence of a constant $0 < r < 1$ in the DT case, respectively $r < 0$ in the CT case, ensuring that the time-dependent sets

DT case:

$$X_{DT}^{\varepsilon}(t;t_0) = \left\{ x \in \mathbb{R}^n \,\middle|\, \| x \|_p^D \le \varepsilon \, r^{(t-t_0)} \right\},$$
$$t, t_0 \in \mathbb{Z}_+, \ t \ge t_0, \ \varepsilon > 0; \quad \text{(6-DT)}$$

CT case:

$$X_{CT}^{\varepsilon}(t;t_0) = \left\{ x \in \mathbb{R}^n \,\middle|\, \| x \|_p^D \le \varepsilon \, e^{r\,(t-t_0)} \right\},$$
$$t, t_0 \in \mathbb{R}_+, \ t \ge t_0, \ \varepsilon > 0, \quad \text{(6-CT)}$$

are invariant (e.g. [11], pp. 100) with respect to the trajectories of IS (1). The general case of invariant sets with arbitrary time-dependence in the dynamics of interval systems is studied in our recent paper [15]. Using the terminology in [6] the sets $X_{DT}^{\varepsilon}(t;t_0)$ and $X_{CT}^{\varepsilon}(t;t_0)$ are *exponentially contractive* with the coefficient $0 < r < 1$ (respectively $r < 0$). For the usual values $p = 1, 2, \infty$, these sets have well-known geometric shapes (i.e. hyper-diamonds, ellipses and rectangles, respectively), scaled in accordance with the diagonal entries of the matrix D.

Fact 2. DIES$_{p,D}$ represents a sufficient condition for the *exponential stability* (abbreviated as ES) of IS (1), defined as (e.g. [11], pp. 107)

DT case:

$$\exists 0 < r < 1 : \forall \alpha > 0, \exists \delta(\alpha) > 0 : \forall t, t_0 \in \mathbb{Z}_+, t \ge t_0,$$
$$\forall x_0 \in \mathbb{R}^n : \ \| x_0 \| < \delta(\alpha) \Rightarrow$$
$$\Rightarrow \| x(t;t_0,x_0) \| \le \alpha r^{(t-t_0)}; \qquad \text{(7-DT)}$$

CT case:

$$\exists r < 0 : \forall \alpha > 0, \exists \delta(\alpha) > 0 : \forall t, t_0 \in \mathbb{R}_+, t \ge t_0,$$
$$\forall x_0 \in \mathbb{R}^n : \ \| x_0 \| < \delta(\alpha) \Rightarrow$$
$$\Rightarrow \| x(t;t_0,x_0) \| \le \alpha e^{r(t-t_0)}, \qquad \text{(7-CT)}$$

where $\| \; \|$ denotes an arbitrary vector norm. Indeed, if the DIES$_{p,D}$ condition (5) holds, then the ES condition (7) is true for $\| \; \| = \| \; \|_p^D$ and $\delta(\alpha) = \alpha$. ■

The connection discussed above between DIES$_{p,D}$ and ES draws the attention on the possible role played by the standard tools of stability analysis in exploring the DIES$_{p,D}$ of IS (1). The following three propositions provide characterizations of the DIES$_{p,D}$ in terms of the transition matrix (the continuous semigroup of linear operators), Lyapunov functions and the matrix defining the difference / differential system (the generator of the continuous semigroup).

Proposition 1. Consider the transition matrix (continuous semigroup of linear operators) that defines the trajectories of IS (1) for an arbitrary $A \in \mathcal{A}^I$ and denote it by $\phi_A(\tau)$, $\tau \in \mathbb{T}$, where

DT case: $\phi_A(\tau) = A^{\tau}$, $\tau \in \mathbb{Z}_+$; (8-DT)

CT case: $\phi_A(\tau) = e^{A\tau}$, $\tau \in \mathbb{R}_+$. (8-CT)

IS (1) is DIES$_{p,D}$ with the decreasing rate $0 < r < 1$ (DT case), $r < 0$ (CT case), if and only

DT case:

$\forall A \in \mathcal{A}^I$, $\forall \tau \in \mathbb{Z}_+$, $\| \phi_A(\tau) \|_p^D \le r^\tau$; (9-DT)

CT case:

$\forall A \in \mathcal{A}^I$, $\forall \tau \in \mathbb{R}_+$, $\| \phi_A(\tau) \|_p^D \le e^{r\tau}$. (9-CT)

Proof: See the Appendix. ■

Proposition 2. IS (1) is DIES$_{p,D}$ with the decreasing rate $0 < r < 1$ (DT case), $r < 0$ (CT case), if and only if the function

$$V : \mathbb{R}^n \to \mathbb{R}_+, \; V(x) = \| x \|_p^D, \quad (10)$$

is a common Lyapunov function for all linear systems belonging to the family defined by (1) that fulfils

DT case: $\forall t \ge 0$, $\forall x \in \mathbb{R}^n$, $x = x(t)$,

$$V(x(t+1)) \le rV(x(t)) \quad (11\text{-DT})$$

CT case: $\forall t \ge 0$, $\forall x \in \mathbb{R}^n$, $x = x(t)$,

$$D_t^+ V(x(t)) = \lim_{\theta \downarrow 0} \frac{V(x(t+\theta)) - V(x(t))}{\theta} \le \le rV(x(t)). \quad (11\text{-CT})$$

Proof: See the Appendix. ■

Proposition 3. IS (1) is DIES$_{p,D}$ with the decreasing rate $0 < r < 1$ (DT case), $r < 0$ (CT case), if and only if the matrix defining the difference / differential system (generator of the semigroup) fulfils the condition

DT case: $\forall A \in \mathcal{A}^I$, $\| A \|_p^D \le r$; (12-DT)

CT case: $\forall A \in \mathcal{A}^I$, $m_p^D(A) \le r$. (12-CT)

Proof: See the Appendix. ■

Remark 2. Proposition 3 confirms that DIES$_{p,D}$ is a stronger property than the standard stability of IS (1) considered by papers [1]-[3], [7], [9], [16], [18], which is characterized by the eigenvalues $\lambda_i(A)$, $i = 1,\ldots,n$, of the matrices $A \in \mathcal{A}^I$. Indeed, in the DT case $|\lambda_i(A)| \le \| A \|_p^D \le r < 1$, $i = 1,\ldots,n$, $\forall A \in \mathcal{A}^I$, while, in the CT case, $\mathrm{Re}\,\lambda_i(A) \le m_p^D(A) \le r < 0$, $i = 1,\ldots,n$, $\forall A \in \mathcal{A}^I$. ■

3. MAJORANT MATRIX U IN DIES TESTING

The propositions presented in the previous section allow the characterization of DIES$_{p,D}$, but do not offer tractable instruments for practical testing, since they are employing the whole family of matrices $A \in \mathcal{A}^I$. In the current section, we derive results that allow testing the DIES$_{p,D}$ of IS (1) by using the majorant matrix U defined by (4).

Theorem 1.

a) Let $p = 1, \infty$.

DT case: IS (1) is DIES$_{p,D}$ with the decreasing rate $0 < r < 1$ if and only if $\| U \|_p^D \le r$.

CT case: IS (1) is DIES$_{p,D}$ with the decreasing rate $r < 0$ if and only if $m_p^D(U) \le r$.

b) Let $1 < p < \infty$.

DT case: IS (1) is DIES$_{p,D}$ with the decreasing rate $0 < r < 1$ if $\| U \|_p^D \le r$.

CT case: IS (1) is DIES$_{p,D}$ with the decreasing rate $r < 0$ if $m_p^D(U) \le r$.

Proof: See the Appendix. ■

Corollary 1. Let $1 \le p \le \infty$.

DT case: Assume that $U \in \mathcal{A}^I$ or $-U \in \mathcal{A}^I$. IS (1) is DIES$_{p,D}$ with the decreasing rate $0 < r < 1$ if and only if $\| U \|_p^D \le r$.

CT case: Assume that $U \in \mathcal{A}^I$. IS (1) is DIES$_{p,D}$ with the decreasing rate $r < 0$ if and only if $m_p^D(U) \le r$.

Proof: Sufficiency: It is ensured by Theorem 1. *Necessity. DT case:* If $U \in \mathcal{A}^I$ then $\| U \|_p^D \le r$, by Proposition 3 (DT case). If $-U \in \mathcal{A}^I$ then $\| -U \|_p^D \le r$, i.e. $\| U \|_p^D \le r$. *CT case:* If $U \in \mathcal{A}^I$ then $m_p^D(U) \le r$, by Proposition 3 (CT case). ■

Remark 3. The hypothesis on the majorant matrix U used in Corollary 1 is satisfied by any interval matrix for which the mean matrix $A_m = \frac{1}{2}(A^- + A^+)$ is nonnegative or nonpositive in the *DT case* and essentially nonnegative in the *CT case*. Obviously, an important class of interval systems fulfilling the above condition is defined by the nonnegative interval systems (with CT and DT dynamics). ■

Remark 4. For the necessary and sufficient condition in Corollary 1, a more general hypothesis can be considered for the majorant matrix *U*, namely: $\exists A^* \in \mathcal{A}^I$ such that $\| A^* \|_p^D = \| U \|_p^D$ in the DT case and $m_p^D(A^*) = m_p^D(U)$ in the CT case. However the proof of the existence for such a matrix $A^* \in \mathcal{A}^I$ may be a rather cumbersome task for practice. Therefore we preferred to formulate our result under a hypothesis which is easier to handle. ■

Remark 5. The necessary and sufficient condition provided by Theorem 1 for the particular case of $p = \infty$ can be equivalently written as the algebraic inequality $Ud \le rd$, where *d* is a positive vector defined by the diagonal entries $d_i > 0$, $i = 1, \cdots, n$, of matrix *D*. The results of our previous papers [12] and [13] (considering the invariance problem only for rectangular boxes) are presented in terms of the algebraic inequality $Ud \le rd$. ■

The majorant matrix U defined by (4-DT) and (4-CT) is nonnegative and, respectively, essentially nonnegative. For this type of matrices, our recent work [14] proves the following properties:

Lemma 1. Let $1 \le p \le \infty$.

A nonnegative matrix U is Schur stable if and only if there exist $0 < r < 1$ and $d_i > 0$, $i = 1, \ldots, n$, such that $\| D^{-1} U D \|_p \le r$, where $D = \mathrm{diag}\{d_1, \cdots, d_n\}$.

An essentially nonnegative matrix U is Hurwitz stable if and only if there exist $r < 0$ and $d_i > 0$, $i = 1, \ldots, n$, such that $m_p(D^{-1} U D) \le r$, where $D = \mathrm{diag}\{d_1, \cdots, d_n\}$.

Proof: See Definition1 and Corollary 1 in [14]. ■

Relying on these properties we offer a broader interpretation of the role played by the majorant matrix *U* in the qualitative analysis of IS (1).

Theorem 2.

a) Let $p = 1, \infty$.

DT case: There exist positive definite diagonal matrices *D* and constants $0 < r < 1$ such that IS (1) is DIES$_{p,D}$ with the decreasing rate *r*, if and only if matrix *U* is Schur stable.

CT case: There exist positive definite diagonal matrices *D* and constants $r < 0$ such that IS (1) is DIES$_{p,D}$ with the decreasing rate *r*, if and only if matrix *U* is Hurwitz stable.

b) Let $1 < p < \infty$.

DT case: There exist positive definite diagonal matrices *D* and constants $0 < r < 1$ such that IS (1) is DIES$_{p,D}$ with the decreasing rate *r*, if matrix *U* is Schur stable.

CT case: There exist positive definite diagonal matrices *D* and constants $r < 0$ such that IS (1) is DIES$_{p,D}$ with the decreasing rate *r*, if matrix *U* is Hurwitz stable.

Proof: It results from Theorems 1 and Lemma 1.■

Corollary 2. Let $1 \le p \le \infty$.

DT case: Assume that $U \in \mathcal{A}^I$ or $-U \in \mathcal{A}^I$. There exist positive definite diagonal matrices *D* and constants $0 < r < 1$ such that IS (1) is DIES$_{p,D}$ with the decreasing rate *r*, if and only if matrix *U* is Schur stable.

CT case: Assume that $U \in \mathcal{A}^I$. There exist positive definite diagonal matrices *D* and constants $r < 0$ such that IS (1) is DIES$_{p,D}$ with the decreasing rate *r*, if matrix *U* is Hurwitz stable.

Proof: It results from Corollary 1 and Lemma 1.■

Remark 6. Theorem 2 and Corollary 2 show that whenever the usage of the majorant matrix *U* allows concluding the stability of IS (1) as in [1]-[3], [7], [9], [16], [18], one concomitantly gets information on the existence of various families of invariant sets with exponential decrease, defined relative to any *p*-norm ($1 \le p \le \infty$). ■

Remark 7. Theorem 2 and Corollary 2 include, as particular cases, previous results bringing refinements to the stability analysis of interval systems. Thus, [7] points out the difference between the Schur / Hurwitz stability of all matrices $A \in \mathcal{A}^I$ and the exponential stability of IS (1). For some classes of DT interval systems, Lemma 3.4.18 in [7] proves the equivalence between the Schur stability of *U* and the exponential stability of IS (1), but it does not discuss the invariance properties ensured together with the exponential stability. Our papers [12] and [13] using U as a test matrix, established the connection between the exponential stability of IS (1) and the invariant sets, but only for $p = \infty$, yielding the concept of *componentwise exponential asymptotic stability*. Therefore the current paper can be regarded as generalizing this concept to arbitrary Hőlder norms. ■

Remark 8. For the concrete identification of the invariant sets $X_{DT}^{\varepsilon}(t;t_0)$ and $X_{CT}^{\varepsilon}(t;t_0)$, one can use Lemma 3 and Remark 3 in [14] applied to the majorant matrix U. Denote by $\lambda_{\max}(U)$ the Perron-Frobenius eigenvalue of U, i.e. the spectral radius in the DT case (when U is nonnegative) and the spectral abscissa in the CT case (when U is essentially nonnegative). For all $1 \le p \le \infty$, there exist invariant sets of IS (1) with the decreasing rate $r \ge \lambda_{\max}(U)$ (if U is irreducible) and $r > \lambda_{\max}(U)$ (if U is reducible). If U is irreducible, the invariant sets with the decreasing rate $r = \lambda_{\max}(U)$ may be constructed by using $D = \operatorname{diag}\{v_1^{1/q} / w_1^{1/p}, \ldots, v_n^{1/q} / w_n^{1/p}\}$, the diagonal matrix built with the right and left Perron-Frobenius eigenvectors of matrix U, $v = [v_1 \ldots v_n]^T > 0$ and $w = [w_1 \cdots w_n]^T > 0$, for $1/p + 1/q = 1$, where the particular cases of norms $p = 1$ and $p = \infty$ mean $1/p = 1, 1/q = 0$, and $1/p = 0, 1/q = 1$, respectively. ■

Remark 9. The Perron-Frobenius eigenvalue $\lambda_{\max}(U)$ represents the fastest decreasing rate of the invariant sets $X_{DT}^{\varepsilon}(t;t_0)$ and $X_{CT}^{\varepsilon}(t;t_0)$ of IS (1) only when the (Schur or Hurwitz) stability of U is a necessary condition for the existence of the invariant sets, i.e. the hypotheses of Theorem 2 (a) and of Corollary 2. Otherwise, IS (1) may have invariant sets with a decreasing rate $r < \lambda_{\max}(U)$. However this does not reduce the importance of the majorant matrix U for the qualitative analysis of IS (1). Once U is stable, Theorem 2 (b) guarantees the existence of invariant sets defined by all p-norms, $1 \le p \le \infty$, which decrease exponentially with the rate $\lambda_{\max}(U)$. ■

Remark 10. Papers ([1], Corollary 7) (DT case) and ([9], Corollary 2), ([2], Theorems 3.1, 3.4, 3.5) (CT case) formulate sufficient conditions for the asymptotic stability of IS (1) that rely on the placement in the complex plane of the Gershgorin disks associated with the majorant matrix U. For $p = 1, \infty$, our approach reveals an one-to-one correspondence between the invariant sets $X_{DT}^{\varepsilon}(t;t_0)$, $X_{CT}^{\varepsilon}(t;t_0)$ of IS (1) and the families of generalized Gershgorin's disks of U, defined for columns ($p = 1$) or rows ($p = \infty$), placed inside the unit circle (DT case) and in the left half-plane (CT case), respectively. The proof follows from Corollary 1 in [14] applied to the majorant matrix U. ■

4. CONCLUDING REMARKS

Our results expand the knowledge on the role played by the majorant matrices in the qualitative analysis of interval systems with DT or CT dynamics. The stability of the majorant matrix ensures more than the asymptotic stability of the system; it also guarantees the existence of exponentially decreasing invariant sets defined by any p-norm ($1 \le p \le \infty$). The stability of the majorant matrix represents only a sufficient condition for the asymptotic stability of the system, whereas it is necessary and sufficient for the existence of exponentially decreasing invariant sets defined by $p = 1, \infty$ norms. This remark can also be formulated in terms of the generalized Gershgorin disks associated with the majorant matrix, and, thus, it reveals the complete meaning of the disks' usage in the qualitative analysis, as an instrument stronger than the matrix stability. For some classes of interval matrices, we emphasize the equivalence between asymptotic stability of the system dynamics and the existence of exponentially decreasing invariant sets defined by any p-norm ($1 \le p \le \infty$). The construction of the majorant matrix is a simple operation for any interval system, and, therefore, the above results are not limited, as importance, to the theoretical level.

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APPENDIX

Proof of Proposition 1. We give a common proof for both DT and CT cases by using a generic function, defined as:

DT case: $h:\mathbb{Z}_+ \to \mathbb{R}$, $h(\tau)=r^{\tau}$; (A1-DT)

CT case: $h:\mathbb{R}_+ \to \mathbb{R}$, $h(\tau)=e^{r\tau}$. (A1-CT)

Sufficiency: If (9) is true, then, for $\forall A \in \mathcal{A}^I$, $\forall\, t_0 \in \mathbb{T}$ and $\forall\, x_0 \in \mathbb{R}^n$ satisfying $\| x_0 \|_p^D \le \varepsilon$, the corresponding solution to (1) fulfills $\|x(t;t_0,x_0)\|_p^D = \|\phi_A(t-t_0)x_0\|_p^D \le$ $\|\phi_A(t-t_0)\|_p^D \|x_0\|_p^D \le \varepsilon h(t-t_0)$, $\forall t \in \mathbb{T}, t \ge t_0$.
This shows that condition (5) is fulfilled.

Necessity: If (5) is true, then, for $\forall t_0, t \in \mathbb{T}$, $t \ge t_0$, and $\forall\, x_0 \in \mathbb{R}^n$, $\| x_0 \|_p^D \le \varepsilon$, we can write for $\forall A \in \mathcal{A}^I$,

$$\| \phi_A(t-t_0)\|_p^D = \sup_{\|z_0\|_p^D=1} \| \phi_A(t-t_0)z_0\|_p^D =$$

$$\sup_{\|\varepsilon^{-1}x_0\|_p^D=1} \left\| \phi_A(t-t_0)\varepsilon^{-1}x_0 \right\|_p^D =$$

$$\varepsilon^{-1} \sup_{\|\varepsilon^{-1}x_0\|_p^D\neq 0} \|x(t,t_0,x_0)\|_p^D \le$$

$\varepsilon^{-1}\varepsilon h(t-t_0) = h(t-t_0)$. This shows that condition (9) is fulfilled. ■

Proof of Proposition 2. We give a common proof for both DT and CT cases by using the generic function $h:\mathbb{T} \to \mathbb{R}$ defined by (A1). Thus, (11-DT) and (11-CT) can be equivalently written in a common form, namely

$$\begin{aligned}&\forall t_0, t \in \mathbb{T},\, t_0 \le t,\, \forall x_0 = x(t_0) \in \mathbb{R}^n, \\ &V\left(x(t;t_0,x_0)\right) \le h(t-t_0) V\left(x(t_0)\right). \end{aligned} \tag{A2}$$

Necessity: If (5) is true, then $\forall A \in \mathcal{A}^I$, $\forall t_0 \in \mathbb{T}$ and $\forall x_0 = x(t_0) \in \mathbb{R}^n$, we have $V\left(x(t;t_0,x_0)\right) = \| \phi_A(t-t_0)x(t_0)\|_p^D \le$ $\| \phi_A(t-t_0)\|_p^D \|x(t_0)\|_p^D \le h(t-t_0)V\left(x(t_0)\right)$, for $\forall t \in \mathbb{T}$, $t \ge t_0$, showing that condition (A2), equivalent to (11), is met.

Sufficiency: Consider that (11), or equivalently (A2), is fulfilled. We give a proof by contradiction and assume that (5) is not true. This means the set $X^{\varepsilon}(t;t_0)$ defined by (6) is not flow-invariant with respect to the trajectories of IS (1), i.e. $\exists t_0, t \in \mathbb{T}$, $t_0 < t$, and $\exists x_0 = x(t_0) \in \mathbb{R}^n$, so that $\|x(t_0)\|_p^D \le \varepsilon h(t_0)$ and $\varepsilon h(t) < \|x(t;t_0,x_0)\|_p^D$, or equivalently,

$$h^{-1}(t_0)\|x(t_0)\|_p^D \le \varepsilon < h^{-1}(t)\|x(t;t_0,x_0)\|_p^D .$$

Thus, we contradict (A2) which says that $h^{-1}(t)\|x(t;t_0,x_0)\|_p^D \le h^{-1}(t_0)\|x(t_0)\|_p^D$. ■

Proof of Proposition 3. *DT case.*
Necessity: If (5) is true, then inequality (8-DT) in Proposition 1 is true. Inequality (12-DT) results from (8-DT) with $\tau=1$.

Sufficiency: If (12-DT) is true, then $\forall A \in \mathcal{A}^I$, $\forall t \in \mathbb{Z}_+$, $\forall x \in \mathbb{R}^n$, $x(t)=x$, we have

$$\|x(t+1)\|_p^D = \|Ax(t)\|_p^D \le \|A\|_p^D \|x(t)\|_p^D \le r\|x(t)\|_p^D .$$

Thus, inequality (11-DT) in Proposition 2 is satisfied, meaning that (5) is fulfilled.

CT case.
Necessity: If (5) is true, then inequality (8-CT) in Proposition 1 is true. Inequality (12-CT) results from (8-CT) since

$$m_p^D(A) = \lim_{\theta\downarrow 0}\left(\|I+\theta A\|_p^D - 1\right)\Big/\theta =$$

$$\lim_{\theta\downarrow 0}\left(\|\phi_A(\theta)\|_p^D - 1\right)\Big/\theta \le \lim_{\theta\downarrow 0}\left(e^{r\theta}-1\right)\Big/\theta = r .$$

Sufficiency: If (12-CT) is true, then $\forall A \in \mathcal{A}^I$, $\forall t \in \mathbb{R}_+, \forall x = x(t) \in \mathbb{R}^n$, we have $D_t^+\|x(t)\| =$

$$\lim_{\theta\downarrow 0}\left(\|x(t+\theta)\|_p^D - \|x(t)\|_p^D\right)\Big/\theta = \lim_{\theta\downarrow 0}\left(\|\phi_A(\theta)x(t)\|_p^D - \right.$$

$$\left.\|x(t)\|_p^D\right)\Big/\theta \le \left[\lim_{\theta\downarrow 0}\left(\|\phi_A(\theta)\|_p^D - 1\right)\Big/\theta\right]\|x(t)\|_p^D =$$

$m_p^D(A)\|x(t)\|_p^D \le r\|x(t)\|_p^D$, $\forall A \in \mathcal{A}^I$. Thus, inequality (11-CT) in Proposition 2 is satisfied, meaning that (5) is fulfilled. ■

Proof of Theorem 1. *DT case*: a) *Sufficiency* and b): Let $1 \le p \le \infty$. We first prove that

$$\| A \|_p^D \le \| U \|_p^D,\ \forall A \in \mathcal{A}^I . \tag{A3-DT}$$

Given $Q=[q_{ij}]$, $i,j=1,\dots,n$, a real square matrix, denote by $\overline{Q}=[\overline{q}_{ij}]$, $i,j=1,\dots,n$, the nonnegative matrix built from Q as follows

$$\overline{q}_{ij}=|q_{ij}|,\ i,j=1,\dots,n. \qquad \text{(A4-DT)}$$

According to Lemma 4 in [14], we have $\|Q\|_p\le\|\overline{Q}\|_p$ for all $1\le p\le\infty$. If P is a real square matrix fulfilling the componentwise inequality $\overline{Q}\le P$, by the same technique as in the proof of Lemma 4 in [14], we can show that $\|\overline{Q}\|_p\le\|P\|_p$ for all $1\le p\le\infty$.

On the other hand, for any matrix $A\in\mathcal{A}^I$ we can write $A\le\overline{A}\le U$. For $D\succ 0$ diagonal, we get the componentwise matrix inequality $D^{-1}AD\le D^{-1}\overline{A}D\le D^{-1}UD$.

Therefore, $\|D^{-1}AD\|_p\le\|D^{-1}UD\|_p$ for all $1\le p\le\infty$, which completes the proof of (A3-DT).

The inequality $\|U\|_p^D\le r$, $0<r<1$, together with (A3-DT) ensure the fulfilment of condition (12-DT) in Proposition 3, meaning that IS (1) is DIES$_{p,D}$ with the decreasing rate r.

a) *Necessity*: There exits a matrix $A^*\in\mathcal{A}^I$ such that $\overline{A^*}=U$, where the operator $\overline{\circ}$ acts according to (A4-DT). Hence, for $p=1,\infty$, $\|U\|_p^D=\|A^*\|_p^D$. Since IS (1) is DIES$_{p,D}$ with the decreasing rate r, Proposition 3 (DT case) ensures $\|A^*\|_p^D\le r$, and, consequently $\|U\|_p^D\le r$.

CT case: a) *Sufficiency* and b): Let $1\le p\le\infty$. We first prove that

$$m_p^D(A)\le m_p^D(U),\ \forall A\in\mathcal{A}^I. \qquad \text{(A3-CT)}$$

Given $Q=[q_{ij}]$, $i,j=1,\dots,n$, a real square matrix, denote by $\overline{Q}=[\overline{q}_{ij}]$, $i,j=1,\dots,n$, the matrix with nonnegative off-diagonal elements built from Q as follows

$$\overline{q}_{ii}=q_{ii},\ i=1,\dots,n;$$
$$\overline{q}_{ij}=|q_{ij}|,\ i\ne j,\ i,j=1,\dots,n. \qquad \text{(A4-CT)}$$

According to Lemma 4 in [14], we have $m_p(Q)\le m_p(\overline{Q})$ for all $1\le p\le\infty$. If P is a real square matrix fulfilling the componentwise inequality $\overline{Q}\le P$, by the same technique as in the proof of Lemma 4 in [14], we can show that $m_p(\overline{Q})\le m_p(P)$ for all $1\le p\le\infty$.

On the other hand, for any matrix $A\in\mathcal{A}^I$ we can write $A\le\overline{A}\le U$. For $D\succ 0$ diagonal, we get the componentwise matrix inequality $D^{-1}AD\le D^{-1}\overline{A}D\le D^{-1}UD$.

Therefore, $m_p(D^{-1}AD)\le m_p(D^{-1}UD)$ for all $1\le p\le\infty$, which completes the proof of (A3-CT).

The inequality $m_p^D(U)\le r$, $r<0$, together with (A3-CT) ensure the fulfilment of condition (12-CT) in Proposition 3, meaning that IS (1) is DIES$_{p,D}$ with the decreasing rate r.

a) *Necessity*: There exits a matrix $A^*\in\mathcal{A}^I$ such that $\overline{A^*}=U$, where the operator $\overline{\circ}$ acts according to (A4-CT). Hence, for $p=1,\infty$, $m_p^D(U)=m_p^D(A^*)$. Since IS (1) is DIES$_{p,D}$ with the decreasing rate r, Proposition 3 (CT case) ensures $m_p^D(A^*)\le r$, and, consequently $m_p^D(U)\le r$. ∎

REFERENCES

[1] Bauer, P.H. and K. Premaratne (1993). Time-invariant versus time-variant stability of interval matrix systems. In: *Fundamentals of Discrete-Time Systems: A Tribute to Professor E. I. Jury* (M. Jamshidi, M. Mansour and B.D.O. Anderson (Eds.)), pp. 181–188, TSI Press, Albuquerque.

[2] Chen, J. (1992). Sufficient conditions on stability of interval matrices: connections and new results. *IEEE Trans. Automatic Control*, vol. **37**, pp. 541–544.

[3] Chen, J. (1993). Comments on ‘A necessary and sufficient condition for the stability of nonnegative interval discrete systems’ by B. Shafai et al. *IEEE Trans. Autom. Control*, vol. **38**, pp. 189.

[4] Desoer, C.A. and M. Vidyasagar (1975). *Feedback Systems: Input-Output Properties.* Academic Press, New York.

[5] Grman, L., D. Rosinova, V. Vesely and A.K. Kova (2005). Robust stability conditions for polytopic systems, *Int. Jrnl. Systems Science*, vol. **36**, no. *15*, pp. 961-973.

[6] Gruyitch, L., J.-P. Richard, P. Borne and J.-C. Gentina (2004). *Stability Domains,* Chapman & Hall / CRC, Boca Raton.

[7] Kaszkurewicz, E. and A. Bhaya (2000). *Matrix Diagonal Stability in Systems and Computation*. Birkhäuser, Boston.

[8] Kolev, L. and S. Petrakieva (2005). Assessing the stability of linear time-invariant continuous interval dynamic systems, *IEEE Trans. Automatic Control*, vol. **50**, pp. 393 – 397.

[9] Lin, S.H., Y.T. Juang, I.K. Fong, C.F. Hsu and T.S. Kuo (1988). Dynamic interval systems analysis and design. *Int. Jrnl. Control*, vol. **48**, pp. 1807-1818.

[10] Mao, W.-J. and Jian Chu (2003). Quadratic stability and stabilization of dynamic interval systems, *IEEE Trans. Automatic Control*, vol. **48**, pp. 1007- 1012.

[11] Michel, A. and K. Wang (1995). *Qualitative Theory of Dynamical Systems*, Marcel Dekker, Inc., New York-Bassel-Hong Kong.

[12] Pastravanu, O. and M. Voicu (2002). Interval matrix systems - Flow invariance and componentwise asymptotic stability", *Differential and Integral Equations*, vol. **15**, no. *11*, pp. 1377–1394.

[13] Pastravanu, O. and M. Voicu (2004). Necessary and sufficient conditions for componentwise stability of interval matrix systems, *IEEE Trans. Aut. Control*, vol. **49**, no. *5*, pp. 1016-1021.

[14] Pastravanu, O. and M. Voicu (2006). Generalized matrix diagonal stability and linear dynamical systems, *Linear Algebra and its Applications*, vol. **419**, issues *2-3*, pp. 299-310.

[15] Matcovschi, M.H., O. Pastravanu, and M. Voicu (2007). Invariant sets with arbitrary time-dependence in the dynamics of linear systems with interval-type uncertainties, *Control Eng. and Appl. Inf.*, vol. **9**, no. *3-4*, pp. 36-42.

[16] Sezer, M. E. and D. D. Šiljak (1994). On stability of interval matrices, *IEEE Trans. Autom. Control*, vol. **39**, pp. 368–371.

[17] Xiao, Y. and R. Unbehauen (2000). Robust Hurwitz and Schur stability test for interval matrices, *Proc. 39th Conf. Decision and Control*, vol. **5**, pp. 4209 – 4214.

[18] Xin, L.X. (1987). Necessary and sufficient conditions for stability of a class of interval matrices. *Int. J. Control*, vol. **45**, pp. 211-214.