Period dependent averaging of a class of pulse modulated systems. Application to the cardiovascular system

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Abstract: In approximating nonlinear systems, averaging theory provides very useful tools, which permit one to obtain simpler mathematical models. The current paper addresses the problem of averaging pulse-frequency modulated systems with constant duty ratios, whose trajectories exhibit a moving average dependent on the modulation period. Because for such systems conventional averaging methods lead to period - independent averaged models, the paper proposes a period-weighted averaging approach, which leads to a period - dependent averaged model (simpler than the original one). The proposed averaging method is further used in a case study involving the cardiovascular system (regarded as a pulse-frequency modulated system). Finally, simulation results illustrate the effectiveness of the proposed averaging method.

Keywords: averaging theory, pulse-modulation, nonlinear systems, biomedical systems, cardiovascular system.

1. INTRODUCTION

Averaging theory has been widely used for approximating nonlinear systems, and thus facilitating future analysis and control design approaches. Especially for the class of periodic (quasi-periodic) systems, averaging methods permit one to obtain a simpler non-periodic system, which approximates to a certain degree the original system. Applications of averaging can be found in power electronics (Pedicini et al., 2012), pneumatic systems (Shen et al., 2006), robotic manipulators (Sira-Ramirez et al., 1993), adaptive control (Sastry et al., 2001), vibrational control (Baillieul et al., 2011), switched controllers (Sedghi, 2003), extremum seeking control (Moura et al., 2013), synchronization of oscillators (Stilwell et al., 2006), multi-agent systems (Porfiri et al., 2007) and congestion control (Marquez et al., 2005).

Over the years, several averaging methods have been proposed, ranging from rather heuristic or application oriented methods to theoretical methods for specific classes of systems. In power electronics, the circuit averaging and state space averaging methods were among the first to be used in applications. The circuit averaging method involves averaging the waveforms of the signals and manipulations of the circuit diagram (different circuit parts are replaced with equivalent ones), which requires a physical insight of the system (Erickson et al., 2001). The state space averaging method provides a more general framework, with a simpler a more straightforward methodology, by averaging directly the equations of the state space model associated to the system ([11]). The results obtained with the state space averaging method are equivalent with those obtained by averaging based on perturbation theory (Khalil, 2000, ch. 10) or through the Krylov-Bogoliubov-Mitropolsky (KBM) averaging method of 1st order (Krein et al., 1990); however both of these methods provide additional theoretical guarantees on the approximation error involved in the averaging process. An increase in the accuracy of the approximation (with the price of increased complexity of the averaged model) is obtained either by using a KBM averaging method of 2nd order (Bass et al., 1998), or through a multifrequency averaging approach (Caliskan et al., 1999; Almér et al., 2012), which implies the use of a generalized average defined based on Fourier series. When the periodic behavior is induced by a relatively high-frequency signal at the input of a static nonlinearity, other approaches are that of the dithering technique – (Iannelli et al., 2006; Iannelli et al., 2008), or the incremental-input describing function (an extension of the describing function method) – (Gelb et al., 1968), which both finally lead to replacing the original nonlinearity by an equivalent (averaged) nonlinearity. Finally, recent studies focus on developing averaging methods for hybrid systems (Pedicini et al., 2011; Wang et al., 2012), and systems with disturbances (Wang et al., 2010).

Pulse modulated systems are widely encountered in both technical control applications (Gelig et al., 2006) and biological control mechanisms (Jones et al., 1961). Although many averaging methods have been proposed in conjunction with pulse modulated systems, most of them actually deal with pulse-width modulation (used especially in power electronics). However, some control applications use pulse-frequency modulation (e.g. Todo et al., 1999), while many biological systems also exhibit pulse-frequency modulation (neural structures) – (Jones et al., 1961). As it will be shown through the case study presented in this paper (referring to the cardiovascular system – regarded as a pulse-frequency modulated system - Codrean et al., 2013 - controlled by the nervous system), there are even some situations when the pulse-frequency modulation is with constant duty ratios and the moving-averages of the system's trajectories are dependent on the modulation frequency (or period). In such a context, conventional averaging approaches usually lead to frequency independent averaged models, which can not properly approximate the original periodic system.

In addressing this issue, the current study proposes a new period-weighted averaging approach, which leads to a period-dependent averaged model, while also maintaining the averaged model as simple as possible. In the first part of the study a theoretical framework is developed for the period-weighted averaging method using perturbation theory, in order to ensure an error bound for the approximation between the original systems and the averaged system, and also to relate the stability of the averaged system with that of the original system. In the second part of the study, a step-by-step description is provided, on how the proposed averaging method can be used in the case study referring to the cardiovascular system, while pointing out ways of copping with the practical issues that emerge.

The remainder of the paper is structured as follows. Section 2 presents the problem formulation for the weighted averaging approach, and shows when the standard averaging approach fails. Section 3 presents the theoretical framework encompassing the proposed averaging method. The case study is presented in Section 4, along with simulation results. The concluding remarks are given in Section 5.

2. PROBLEM FORMULATION

Consider the class of nonlinear systems

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}) \tag{1}$$

characterized by a periodic regime of period T, such that $\mathbf{f}(t,\mathbf{x})=\mathbf{f}(t+T,\mathbf{x})$.

Assumption 1 $\mathbf{f}$ is piecewise continuous in t and locally Lipschitz in $\mathbf{x}$.

Assumption 2 The system (1) can be approximated by a piecewise continuous system, with a finite number of points of discontinuity, and with the left hand side expressed as

$$\mathbf{f}(t,\mathbf{x}) = \begin{cases} \mathbf{f}_1(\mathbf{x}), & nT \le t < (n+d_1)T \\ \mathbf{f}_2(\mathbf{x}), & (n+d_1)T \le t < (n+d_1+d_2)T \\ \vdots & \\ \mathbf{f}_m(\mathbf{x}), & (n+d_1+d_2...+d_{m-1})T \le t < (n+1)T \end{cases}, \tag{2}$$

$$\sum_{i=1}^{m} d_i = 1,\; n \in \mathrm{N},$$

where the duty ratios d_i are considered to be constant.

In order to state the main problem addressed in this paper, the system (1) is rewritten- with the right hand side (2) - as

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}(\omega t,\mathbf{x}) \tag{3}$$

with the angular frequency $\omega=2\pi/T$, and where

$$\tilde{\mathbf{f}}(\omega t,\mathbf{x}) = \begin{cases} \mathbf{f}_1(\mathbf{x}), & 2\pi n \le \omega t < 2\pi(n+d_1) \\ \mathbf{f}_2(\mathbf{x}), & 2\pi(n+d_1) \le \omega t < 2\pi(n+d_1+d_2) \\ \vdots & \\ \mathbf{f}_m(\mathbf{x}), & 2\pi(n+d_1+d_2+...d_{m-1}) \le \omega t < 2\pi(n+1) \end{cases}. \tag{4}$$

Moreover, in accordance with Assumption 2, the system can be recasted as a pulse-modulated switched system, with a switching function q:$[0,\infty)\rightarrow\{1,2,...m\}$:

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}_{q(\omega t)}(\mathbf{x}),\; q(\omega t) = \begin{cases} 1, & 2\pi n \le \omega t < 2\pi(n+d_1) \\ 2, & 2\pi(n+d_1) \le \omega t < 2\pi(n+d_1+d_2) \\ \vdots & \\ m, & 2\pi(n+d_1+d_2+...d_{m-1}) \le \omega t < 2\pi(n+1) \end{cases}. \tag{5}$$

Next, through a time scaling of the form $\tau=\omega t$, system (3) becomes

$$\mathbf{x}' = \frac{1}{\omega}\tilde{\mathbf{f}}(\tau,\mathbf{x}) \tag{6}$$

where $\mathbf{x}' = d\mathbf{x}/d\tau$ and $\tilde{\mathbf{f}}(\tau,\mathbf{x}) = \tilde{\mathbf{f}}(\tau+2\pi,\mathbf{x})$. Finally, by adopting the small positive parameter $\varepsilon=1/\omega$, (6) can be brought to the “standard” form

$$\mathbf{x}' = \varepsilon\,\tilde{\mathbf{f}}(\tau,\mathbf{x}),\quad \mathbf{x}(0) = \mathbf{x}_0\,. \tag{7}$$

The trajectories of (7) describe a periodic orbit Γ, which is dependent on the modulation period T. It will be further considered, as a working hypothesis, that the (geometric) center χ of the periodic orbit Γ is also dependent on the modulation period T (i.e. $\chi \stackrel{\Delta}{=} \chi(T)$); the center corresponds to an average operation point Λ of the periodic system.

In the above presented context, the problem addressed in the current study is as follows:

Problem statement Determine an averaged system which approximates the original periodic system (7) within a certain error bound and which is dependent on the modulation period T.

The novelty of the current problem formulation consists in the fact that the averaged system has to be dependent on the modulation period T. Standard averaging approaches (like the one from (Khalil, 2000, ch. 10.4)) fail in addressing this problem. In particular, for the periodic system (7), the standard averaging method associates the following averaged system

$$\mathbf{x}'_{av} = \varepsilon\,\tilde{\mathbf{f}}_{av}(\mathbf{x}_{av}) \tag{8}$$

with

$$\tilde{\mathbf{f}}_{av}(\mathbf{x}) = \frac{1}{2\pi}\int_{\tau-2\pi}^{\tau}\tilde{\mathbf{f}}(s,\mathbf{x})\,ds \tag{9}$$

Returning to the absolute time ($t=\tau/\omega$), yields

$$\dot{\mathbf{x}}_{av} = \tilde{\mathbf{f}}_{av}(\mathbf{x}_{av}) \tag{10}$$

By taking into account that $\tilde{\mathbf{f}}$ was defined also through (4), one further obtains

$$\tilde{\mathbf{f}}_{av}(\mathbf{x}) = \sum_{i=1}^{m} d_i\,\tilde{\mathbf{f}}_i(\mathbf{x}) \tag{11}$$

Therefore, the averaged system (10) is independent of the modulation period T, and can not properly approximate the original periodic system (3) when the average operation point Λ changes as function of the period T.

Remark 1 Even though the study deals with pulse-frequency modulated systems, in which the period T actually varies in time, it is considered that these variations are relatively slow in respect with the duration of a period T, and as a consequence the averaging approach considers T to be constant ($T=T_{max}$). However, the emphasis is that the resulting averaged model should be T dependent, such that T becomes a new slow-varying input of the averaged system.

3. THEORETICAL FRAMEWORK

In addressing the problem stated in the previous section, a period-weighting averaging approach for system (7) will be considered. First, an additional simplifying assumption is imposed.

Assumption 3 Suppose $\tilde{\mathbf{f}}$ can be decomposed as $\tilde{\mathbf{f}}(\tau,\mathbf{x}) = \tilde{\mathbf{f}}_0(\tau,\mathbf{x}) + \tilde{\mathbf{g}}(\tau)$, where the piecewise continuous functions $\tilde{\mathbf{f}}_0$ and $\tilde{\mathbf{g}}$ are also periodic, and defined in a similar manner as $\mathbf{f}$.

Next, the following time averaged functions are defined:

$$\tilde{\mathbf{f}}_{0,av}(\mathbf{x}) = \frac{1}{2\pi}\int_{\tau-2\pi}^{\tau} \tilde{\mathbf{f}}_0(s,\mathbf{x})\, ds$$
$$\tilde{\mathbf{g}}_{av}(T) = \frac{1}{2\pi}\int_{\tau-2\pi}^{\tau} \tilde{\mathbf{g}}(s)\, m(s,T)\, ds \tag{12}$$

where $m(s,T)$ is a weighting function with the (fixed) parameter T, on the interval $s\in(\tau-2\pi,\tau)$; e.g. $m(s,T)=e^{aT(s-\tau+2\pi)/(2\pi)}$, with a as a tuning parameter. One can note that for $a=0$, the standard averaging approach can be recovered.

Remark 2 Although a weighted average directly for the function $\tilde{\mathbf{f}}$ could have been defined, i.e. without the decomposition given by Assumption 3, the mixed averaging approach given by (12) has been chosen instead because it leads to a simpler averaged model. Moreover, it is expected that the $\tilde{\mathbf{g}}$ component is the main cause for why the average operation point Λ changes as function of the period T.

Let us associate to (7) the following averaged system:

$$\mathbf{x}'_{av} = \varepsilon\, \tilde{\mathbf{f}}_{av}(\mathbf{x}_{av},T), \quad \mathbf{x}_{av}(0) = \mathbf{x}_{av0}, \tag{13}$$

where the left hand side is obtained through (12) as $\tilde{\mathbf{f}}_{av}(\mathbf{x}_{av},T) = \tilde{\mathbf{f}}_{0,av}(\mathbf{x}_{av}) + \tilde{\mathbf{g}}_{av}(T)$.

Finally, by returning to the absolute time, one obtains the averaged system associated to (1):

$$\dot{\mathbf{x}}_{av} = \mathbf{f}_{av}(\mathbf{x}_{av},T) \tag{14}$$

Assumption 4 $\mathbf{f}_{av}$ is locally Lipschitz in respect with x_{av}.

Remark 3 Considering the particular choice of the weighting function $m(s,T)=e^{aT(s-\tau+2\pi)/(2\pi)}$, and that the function $\tilde{\mathbf{g}}$ is defined in a similar piecewise manner as $\mathbf{f}$, it is sometimes useful to approximate de averaged function $\tilde{\mathbf{g}}_{av}$ from (12) as:

$$\begin{aligned}\tilde{\mathbf{g}}_{av}(T) &= \frac{1}{2\pi}\int_{\tau-2\pi}^{\tau} \tilde{\mathbf{g}}(s)\, e^{aT(s-\tau+2\pi)/(2\pi)}\, ds \\ &= \frac{1}{2\pi}\sum_{i=1}^{m}\tilde{\mathbf{g}}_i \int_{\tau-2\pi+2\pi d_1+\ldots 2\pi d_{i-1}}^{\tau-2\pi+2\pi d_1+\ldots 2\pi d_i} e^{aT(s-\tau+2\pi)/(2\pi)}\, ds = \\ &= \sum_{i=1}^{m}\tilde{\mathbf{g}}_i \frac{1}{aT} e^{aT(s-\tau+2\pi)/(2\pi)}\Big|_{\tau-2\pi+2\pi d_1+\ldots 2\pi d_{i-1}}^{\tau-2\pi+2\pi d_1+\ldots 2\pi d_i} \approx \sum_{i=1}^{m}\tilde{\mathbf{g}}_i \underbrace{(\alpha_i+\beta_i T)}_{\theta_i}\end{aligned} \tag{15}$$

where the coefficients of θ_i can be determined either through a first order Taylor series expansion or linear interpolation. Such an approximation further simplifies the resulting averaged model, and holds for a sufficiently small range of T. Moreover, the theoretical results further presented hold even when this approximation is used, instead of the original function $\tilde{\mathbf{g}}_{av}$ from (12).

Next, the following theorem provides a bound on the closeness between the trajectories of the averaged system and original system. The proof is inspired from (Khalil, 2000, ch. 10.4), and adapted for the weighted averaged case defined through (12).

Theorem 1 If the initial conditions for (7) and (13) are such that $\|\mathbf{x}(0)-\mathbf{x}_{av}(0)\| = O(\varepsilon)$, then for a sufficiently small ε, there exists a positive constant b, such that (13) represents an $O(\varepsilon)$ approximation of (7) on the time interval $[0, b/\varepsilon]$, i.e.:

$$\|\mathbf{x}(\tau/\omega)-\mathbf{x}_{av}(\tau/\omega)\| = O(\varepsilon), \quad \tau\in[0, b/\varepsilon]. \tag{16}$$

Proof The following functions are defined:

$$\mathbf{h}(\tau,\mathbf{x}) = \tilde{\mathbf{f}}(\tau,\mathbf{x}) - \tilde{\mathbf{f}}_{av}(\mathbf{x}) - \varphi(T) \tag{17}$$

with the function φ adopted such that $\mathbf{h}$ has zero mean

$$\varphi(T) = \begin{cases} \frac{1}{2\pi}\int_{\tau-2\pi}^{\tau}\tilde{\mathbf{g}}(s)\, ds - \frac{1}{2\pi}\int_{\tau-2\pi}^{\tau}\tilde{\mathbf{g}}(s)\, m(s,T)\, ds, & T\neq 0 \\ 0, & T=0\end{cases} \tag{18}$$

and

$$\mathbf{u}(\tau,\mathbf{x}) = \int_0^{\tau}\mathbf{h}(s,\mathbf{x})\, ds. \tag{19}$$

Assumption 5 The function φ is locally Lipschitz in respect with T.

Remark 4 It can be shown that for the class of weighting functions $m(s,T)=e^{aT(s-\tau+2\pi)/(2\pi)}$, the function φ is of class C^1, and as a result it is also locally Lipschitz.

By taking into account how ε was defined, it can be noticed that φ is actually a function of ε: $\varphi(T)=\varphi(2\pi\varepsilon)$. Thus, φ is also Lipschitz in respect with ε.

The functions **u** and **h** are periodic in τ, and **u** is bounded. It can be shown that the partial derivatives of **u**

$$\frac{\partial \mathbf{u}}{\partial \tau} = \mathbf{h}(\tau, \mathbf{x}), \quad \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \int_0^\tau \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(s, \mathbf{x})\, ds \,, \tag{20}$$

are also periodic in τ and bounded.

Next, consider the system (7), with the following change of variables:

$$\mathbf{x} = \mathbf{y} + \varepsilon \mathbf{u}(\tau, \mathbf{y}) \,. \tag{21}$$

Differentiating both the left hand side and the right hand side in respects with τ leads to

$$\frac{d\mathbf{x}}{d\tau} = \frac{d\mathbf{y}}{d\tau} + \varepsilon \frac{\partial \mathbf{u}}{\partial \tau} + \varepsilon \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \frac{d\mathbf{y}}{d\tau} \,. \tag{22}$$

Further using (7), (17) and (20), expression (22) becomes

$$\begin{aligned} \left[\mathbf{I} + \varepsilon \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \right] \frac{d\mathbf{y}}{d\tau} &= \varepsilon \tilde{\mathbf{f}}(\tau, \mathbf{y} + \varepsilon \mathbf{u}) - \varepsilon \frac{\partial \mathbf{u}}{\partial \tau} \\ &= \varepsilon \tilde{\mathbf{f}}(\tau, \mathbf{y} + \varepsilon \mathbf{u}) - \varepsilon \tilde{\mathbf{f}}(\tau, \mathbf{y}) + \varepsilon \tilde{\mathbf{f}}_{av}(\mathbf{y}) + \varepsilon \varphi(2\pi\varepsilon) \\ &= \varepsilon \tilde{\mathbf{f}}_{av}(\mathbf{y}) + \varepsilon \left[\tilde{\mathbf{f}}(\tau, \mathbf{y} + \varepsilon \mathbf{u}) - \tilde{\mathbf{f}}(\tau, \mathbf{y}) \right] + \varepsilon \left[\varphi(2\pi\varepsilon) - \varphi(0) \right] \end{aligned} \tag{23}$$

Because $\tilde{\mathbf{f}}$ is Lipschitz in $\boldsymbol{y}$ and φ is Lipschitz in ε, the differences of the right hand side can be expressed as $\left\| \tilde{\mathbf{f}}(\tau, \mathbf{y} + \varepsilon \mathbf{u}) - \tilde{\mathbf{f}}(\tau, \mathbf{y}) \right\| = O(\varepsilon)$ and $\left\| \varphi(2\pi\varepsilon) - \varphi(0) \right\| = O(\varepsilon)$. Also, because $\partial \mathbf{u} / \partial \mathbf{y}$ is bounded, the matrix $\mathbf{I} + \varepsilon\, \partial \mathbf{u} / \partial \mathbf{y}$ is nonsingular for sufficiently small ε, and its inverse can be written as

$$\left[\mathbf{I} + \varepsilon \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \right]^{-1} = \mathbf{I} + O(\varepsilon) \tag{24}$$

As a result, (23) becomes

$$\mathbf{y}' = \varepsilon \tilde{\mathbf{f}}_{av}(\mathbf{y}) + \varepsilon^2\, \mathbf{q}(\tau, \mathbf{y}, \varepsilon) \,, \tag{25}$$

where $\boldsymbol{q}$ is a periodic function in τ, bounded, with first partial derivatives with respect to **y** and ε continuous and bounded.

Further on, (25) is compared with the averaged system (13). By integrating both sides and subtracting the resulting equations yields

$$\begin{aligned} &\mathbf{y}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) = \mathbf{y}(0) - \mathbf{x}_{av}(0) + \\ &+ \varepsilon \int_0^\tau \left[\tilde{\mathbf{f}}_{av}(\mathbf{y}(s)) - \tilde{\mathbf{f}}_{av}(\mathbf{x}_{av}(s), T) \right] ds + \varepsilon^2 \int_0^\tau \mathbf{q}(s, \mathbf{y}(s), \varepsilon)\, ds \end{aligned} \,. \tag{26}$$

Taking the norm of the above expression and applying the triangle inequality leads to

$$\begin{aligned} &\left\| \mathbf{y}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) \right\| \le \left\| \mathbf{y}(0) - \mathbf{x}_{av}(0) \right\| + \\ &+ \varepsilon \int_0^\tau \left\| \tilde{\mathbf{f}}_{av}(\mathbf{y}(s)) - \tilde{\mathbf{f}}_{av}(\mathbf{x}_{av}(s), T) \right\| ds + \varepsilon^2 \int_0^\tau \left\| \mathbf{q}(s, \mathbf{y}(s), \varepsilon) \right\| ds \end{aligned} \,. \tag{27}$$

The following notation can be introduced $\left\| \mathbf{y}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) \right\| = \xi(\tau/\omega)$. Using the fact that $\tilde{\mathbf{f}}_{av}$ is Lipschitz (with Lipschitz constant L) and that $\boldsymbol{q}$ is bounded (i.e. the norm is bounded by a positive constant μ), (27) becomes

$$\xi(\tau/\omega) \le \xi(0) + \varepsilon L \int_0^\tau \xi(s)\, ds + \varepsilon^2 \mu\, \tau \,. \tag{28}$$

The further use of Gronwall-Belman's inequality with respect to ξ yields

$$\xi(\tau/\omega) \le \left[\xi(0) + \frac{\varepsilon\mu}{L} \right] e^{\varepsilon L \tau} \tag{29}$$

Therefore, when $\left\| \mathbf{y}(0) - \mathbf{x}_{av}(0) \right\| = O(\varepsilon)$ one has $\left\| \mathbf{y}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) \right\| = O(\varepsilon)$ for $\tau < 1/(\varepsilon L)$. Through the change of variables (21) it result that $\left\| \mathbf{x}(\tau/\omega) - \mathbf{y}(\tau/\omega) \right\| = O(\varepsilon)$. By further using the triangle inequality one obtains

$$\left\| \mathbf{x}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) \right\| \le \left\| \mathbf{x}(\tau/\omega) - \mathbf{y}(\tau/\omega) \right\| + \left\| \mathbf{y}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) \right\| \tag{30}$$

thus finally reaching the result (16), with $b = 1/L$.

In order to extend this result to the infinite time interval, one must add a stability condition.

Theorem 2 If the system (13) is exponentially stable and $\left\| \mathbf{x}(0) - \mathbf{x}_{av}(0) \right\| = O(\varepsilon)$, then for sufficiently small ε, (13) represents an O(ε) approximation of (7):

$$\left\| \mathbf{x}(\tau/\omega) - \mathbf{x}_{av}(\tau/\omega) \right\| = O(\varepsilon) \,, \quad \tau \in [0, \infty] \tag{31}$$

Proof Through the change of variables $v = \varepsilon\tau$, (25) is brought the form of a perturbed system

$$\frac{d\mathbf{y}}{dv} = \tilde{\mathbf{f}}_{av}(\mathbf{y}) + \varepsilon\, \mathbf{q}(v/\varepsilon, \mathbf{y}, \varepsilon) \tag{32}$$

associated to the nominal system

$$\frac{d\mathbf{y}}{dv} = \tilde{\mathbf{f}}_{av}(\mathbf{y}) \,. \tag{33}$$

This is now a standard perturbation problem. Therefore, Theorem 9.1 from (Khalil, 2000, ch. 9) can be used for proving the O(ε) closeness between the solutions of (32) and (33), when the term ε**q** denotes a persistent perturbation, but bounded. The basic idea is to build a system given by the difference between (32) and (33), with the error between the two corresponding trajectories as state variable. Assuming that the corresponding unperturbed system is exponentially stable, and that the perturbation is bounded in a certain sense, the comparison method provides an upper bound for the solution of the perturbed system. Finally, this means that also the solutions in the τ time scale are O(ε) close.

Finally, from the stability of the averaged system, the stability of the original (periodic) system can be inferred.

Theorem 3 If the system (13) is exponentially stable, then for sufficiently small ε, the system (7) is orbitally exponentially stable.

Proof Again, one can make use of the system in perturbed form, given through (32), and its nominal version (33). This is now regarded as a periodic perturbation problem. In this case, Theorem 10.3 from (Khalil, 2000, ch. 10) provides the desired stability result. The basic idea is to regard the term $\varepsilon\mathbf{q}$ as a periodic perturbation. A change of variables of the form $\mathbf{z}=\mathbf{y}-\mathbf{y}^p$ ($\mathbf{y}^p$ is the periodic solution of (32)), and then a linearization at the origin, brings the system to a perturbed form (still linearized) with a vanishing perturbation. The stability is inferred from the stability of the unperturbed system. Moreover, because one deals with exponential stability, the stability of the linearized system finally implies that of the original nonlinear system.

4. CASE STUDY

Consider the averaging problem for the dynamics of the cardiovascular system, modelled as a pulse frequency modulated switched system as in (Codrean et al., 2013) - (the cardiovascular system is regarded as a plant controlled through pulse modulation by the nervous system):

$$\dot{\mathbf{x}} = \mathbf{A}_{q(\omega t)}\,\mathbf{x} + \mathbf{b}_{q(\omega t)} \tag{34a}$$

$$\mathbf{y} = \mathbf{C}_{q(\omega t)}\,\mathbf{x} \tag{34b}$$

with the state and output vectors $\mathbf{x}=[x_0\ x_1]^T$ and $\mathbf{y}=[y_0\ y_1]^T$ (denoting ventricular and arterial stressed blood volume, respectively ventricular and arterial blood pressure), the T periodic switching signal $q:[0,\infty)\to\{0,1\}$ (mechanism of cardiac contraction), expressed as

$$q(\omega t)=\begin{cases}0, & 2\pi n \le \omega t < 2\pi(n+d_0)\\ 1, & 2\pi(n+d_0)\le \omega t < 2\pi(n+1)\end{cases},$$

$$d_0=\frac{\Delta t_0}{T},\, d_1=\frac{\Delta t_1}{T},\, d_0+d_1=1,$$

and where

$$\mathbf{A}_0=\begin{bmatrix}-\dfrac{E_s}{R_0} & \dfrac{1}{R_0C_1}\\ \dfrac{E_s}{R_0}-\dfrac{1}{R_1C_2} & -\dfrac{1}{R_0C_1}-\dfrac{1}{R_1C_1}-\dfrac{1}{R_1C_2}\end{bmatrix},\ \mathbf{b}_0=\begin{bmatrix}0\\ \dfrac{x_T}{R_1C_2}\end{bmatrix},$$

$$\mathbf{A}_1=\begin{bmatrix}-\dfrac{E_d}{R_2}-\dfrac{1}{R_2C_2} & -\dfrac{1}{R_2C_2}\\ -\dfrac{1}{R_1C_2} & -\dfrac{1}{R_1C_1}-\dfrac{1}{R_1C_2}\end{bmatrix},\ \mathbf{b}_1=\begin{bmatrix}\dfrac{x_T}{R_2C_2}\\ \dfrac{x_T}{R_1C_2}\end{bmatrix},$$

$$\mathbf{C}_0=diag\left(E_s,\frac{1}{C_1}\right),\mathbf{C}_1=diag\left(E_d,\frac{1}{C_1}\right).$$

The parameters have the following interpretation: T represents the duration of the heart period (with the two subintervals denoting duration of systole and diastole); R_0, R_1 and R_2 represent hydraulic resistances; C_1, C_2 represent hydraulic capacitance (compliance), while E_s and E_d represent systolic and diastolic elastances (inverse of compliance); x_T represents total stressed blood volume. Finally, as a remark, it should be noted that the model captures only the systemic circulation (large arteries, peripheral circulation, large veins), along with the left heart (left ventricle).

First, the averaged output equation (34b) is defined as

$$\mathbf{y}_{av}\overset{\Delta}{=}\frac{1}{T}\int_{t-T}^{T}\mathbf{C}_{q(t)}\mathbf{x}\,ds=\mathbf{C}_{av}\,\mathbf{x}_{av},\quad \mathbf{C}_{av}=d_0\mathbf{C}_0+d_1\mathbf{C}_1\,. \tag{35}$$

Next, the output equations are temporarily dropped, proceeding with the averaging method for the state equations, and ensuring an error bound for the approximation in respect with the state variables. The stability inferred for the state equations will extend also to the final case when the output equations are reattached (i.e. in this particular case, internal stability implies external stability).

By time scaling the state equation (34a), one obtains

$$\mathbf{x}'=\frac{1}{\omega}\left[\mathbf{A}_{q(\tau)}\,\mathbf{x}+\mathbf{b}_{q(\tau)}\right]. \tag{36}$$

Adopting the small parameter $\varepsilon=1/\omega$ yields

$$\mathbf{x}'=\varepsilon\left[\mathbf{A}_{q(\tau)}\,\mathbf{x}+\mathbf{b}_{q(\tau)}\right]. \tag{37}$$

Obviously, (37) is a particular case of (7), and respects Assumption 3, by considering $\widetilde{\mathbf{f}}_0(\tau,\mathbf{x})=\mathbf{A}_{q(\tau)}\,\mathbf{x}$ and $\widetilde{\mathbf{g}}(\tau)=\mathbf{b}_{q(\tau)}$. The averages of these components, according to (12) are:

$$\begin{aligned}
\widetilde{\mathbf{f}}_{0,av}(x)&\overset{\Delta}{=}\frac{1}{2\pi}\int_{\tau-2\pi}^{\tau}\mathbf{A}_{q(\tau)}\,\mathbf{x}\,ds=\frac{1}{2\pi}\int_{\tau-2\pi}^{\tau-2\pi+2\pi d_0}\mathbf{A}_0\,\mathbf{x}\,ds+\\
&+\frac{1}{2\pi}\int_{\tau-2\pi+2\pi d_0}^{\tau}\mathbf{A}_1\,\mathbf{x}\,ds=\left(d_0\mathbf{A}_0+d_1\mathbf{A}_1\right)\mathbf{x}\\
\widetilde{\mathbf{g}}_{av}(T)&\overset{\Delta}{=}\frac{1}{2\pi}\int_{\tau-2\pi}^{\tau}\mathbf{b}_{q(\tau)}m(s,T)\,ds=\frac{1}{2\pi}\int_{\tau-2\pi}^{\tau-2\pi+2\pi d_0}\mathbf{b}_0\,m(s,T)\,ds+\\
&+\frac{1}{2\pi}\int_{\tau-2\pi+2\pi d_0}^{\tau}\mathbf{b}_1\,m(s,T)\,ds=\theta_0(T)\mathbf{b}_0+\theta_1(T)\mathbf{b}_1
\end{aligned} \tag{38}$$

with $\theta_0(T)=\left(e^{aTd_0}-1\right)/(aT),\ \theta_1=\left(e^{aT}-e^{aTd_0}\right)/(aT)$.

The weighting terms θ_0 and θ_1 can be further approximated through linear interpolation:

$$\theta_0(T)\approx\alpha_0+\beta_0T,\quad \theta_1\approx\alpha_1+\beta_1T\,, \tag{39}$$

which are now affine functions of the period T.

The averaged system is

$$\mathbf{x}'_{av}=\varepsilon\left[\mathbf{A}_{av}\mathbf{x}_{av}+\mathbf{b}_{av}(T)\right], \tag{40}$$

where $\mathbf{A}_{av}=d_0\mathbf{A}_0+d_1\mathbf{A}_1$, $\mathbf{b}_{av}(T)=\theta_0(T)\mathbf{b}_0+\theta_1(T)\mathbf{b}_1$.

Finally, by scaling back, one obtains the weighted averaged system associated to the original system (34a):

$$\dot{\mathbf{x}}_{av} = \mathbf{A}_{av}\mathbf{x}_{av} + \mathbf{b}_{av}(T) . \quad (41)$$

According to Theorem 1, for a constant nominal value of T, the error between the original and the averaged systems is $O(\varepsilon)$ on a finite time interval. To extend this to the infinite time interval, one needs to check the stability of (40).

By making use of aproximation (39), the system (40) can be written as

$$\mathbf{x}'_{av} = \varepsilon\, \mathbf{A}_{av}\mathbf{x}_{av} + \varepsilon[\alpha_0\mathbf{b}_0 + \alpha_1\mathbf{b}_1] + \varepsilon[\beta_0\mathbf{b}_0 + \beta_1\mathbf{b}_1]T , \quad (42)$$

which is now a linear system with two constant input terms. This means that exponential stability follows if and only if the matrix $\varepsilon\, \mathbf{A}_{av}$ is Hurwitz. Additionally, the system would be stable even as T varies (slowly), and thus T could be further regarded as the new input of the averaged system.

For the cardiovascular model parameters given in Table I (Appendix A), a=-0.7 (adopted such that the averaged system's trajectories follow the variations of the moving averages of the original system's trajectories to step changes in T), d_0=1/3, d_1=2/3, T=1 s (nominal value), and $\theta_0(T) \approx 0.33 - 0.03T$, $\theta_1 \approx 0.61 - 0.18T$ (on the physiological domain $T \in [0.3, 2]$ s), it can be easily checked that $\varepsilon\, \mathbf{A}_{av}$ is Hurwitz. Hence, the averaged system is exponentially stable, and as a result of Theorem 2, the error between the original system and the averaged system is $O(\varepsilon)$ on an infinite time interval. Moreover, according to Theorem 3, also the original system is orbitally exponentially stable. Lastly, it should be mentioned that the results are conserved even when taking the maximal value of T (i.e. T=2s), instead of the nominal value.

Remark 5 In this particular case, the stability of the original system (34a) can be alternatively investigated using Floquet Theory (Richards, 1983), adapted to switched linear systems. Thus, by using the results from (Gökçek, 2004), it can be proved that system (34a), rearranged as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{q(\omega t)} & \mathbf{b}_{q(\omega t)} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix}, \begin{bmatrix} \mathbf{x}(0) \\ x_a(0) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ 1 \end{bmatrix}, \quad (43)$$

is in fact orbitally exponentially stable. Here x_a is a support state variable.

In many practical applications an offset steady state error was observed between the signals of the averaged system, and the (real) moving averages of the signals of the original system (see examples in power electronics – Krein et al., 1990, Caliskan et al., 1999, Lehman et al., 1996). This is a generic issue, specific to most averaging methods (including the one presented here), and usually attributed to a large ripple of the signal of interest, for a relatively low frequency range. While in some applications this error can be neglected, in our particular application this is not the case. Moreover, because this error can not be expressed analytically, there are no systematic methods to correct it without substantially increasing the complexity of the averaged model (Codrean et al., 2013). Consequently, in an attempt to minimize the error as much as possible, a multiplicative type correction is considered for the system matrix by introducing the corrector matrix - $\mathbf{M}_c$= diag (ρ_0, ρ_1), with ρ_0>0 and ρ_1>0. Hence, (42) becomes

$$\mathbf{x}'_{av} = \varepsilon\, \mathbf{A}_{av}\mathbf{M}_c\mathbf{x}_{av} + \varepsilon[\alpha_0\mathbf{b}_0 + \alpha_1\mathbf{b}_1] + \varepsilon[\beta_0\mathbf{b}_0 + \beta_1\mathbf{b}_1]T . \quad (44)$$

In the numerical context stated above, these tuning parameters were adopted as ρ_0=0.6 and ρ_1=1.0, so as to ensure that the equilibrium point of (44) is as close as possible to the moving average of (37) in a nominal steady state scenario. For a more detailed discussion on how to determine the set of tuning parameters {a, ρ_0, ρ_1} see (Codrean et al., 2013).

Remark 6 Note that, despite the correction that transforms the system (42) into (44), the stability is preserved: because the main diagonal elements of $\varepsilon\, \mathbf{A}_{av}$ are always negative (from a physiological interpretation) and the elements of $\mathbf{M}_c$ are positive, the Hurwitz determinants of $\varepsilon\, \mathbf{A}_{av}$ and $\varepsilon\, \mathbf{A}_{av}\mathbf{M}_c$ have the same signs. In other words, for this particular application (2nd order system), (42) is stable if and only if (44) is stable. This result can be generalized for higher order systems by further taking into account the fact that we are actually dealing with a positive linear system (most models of the cardiovascular system have as state variables either volumes or pressure, which can not take negative values). By considering the correction matrix $\mathbf{M}_c$ as a known multiplicative perturbation, it can be shown through the D-stability theorem (see Theorem 16 from Farina et al., 2000) that the nominal system – given here by (42) – is stable if and only if the perturbed system – given here by (44) – is stable. Furthermore, one can intuitively expect that the correction would lower the error bound $O(\varepsilon)$ between the trajectory of the averaged system and the original system.

Finally, a scenario for the resulting averaged system is considered - with the output equation (35) reattached

$$\begin{aligned} \dot{\mathbf{x}}_{av} &= \mathbf{A}_{av}\mathbf{M}_c\mathbf{x}_{av} + [\alpha_0\mathbf{b}_0 + \alpha_1\mathbf{b}_1] + [\beta_0\mathbf{b}_0 + \beta_1\mathbf{b}_1]T \\ \mathbf{y}_{av} &= \mathbf{C}_{av}\,\mathbf{x}_{av} \end{aligned} , \quad (45)$$

when the input T varies as in Fig. 1 (note that this variations are considered large from a physiological point of view). The trajectories of the averaged system follow relatively close the real moving averages of the original system as the modulation period changes (the spikes of y_0 are due to numerical errors, and do not influence the averaging process – Codrean et al., 2013). Without a weighted averaging approach as the one presented here, i.e. through a standard averaging approach, the averaged system would have been invariant in respect with the modulation period, and thus the trajectories would remain constant during the entire scenario (see also Codrean et al., 2013).

Remark 7 As shown in (Codrean et al., 2013), the averaged system (45) can be obtained more straightforwardly through a state space weighted averaging approach, but without any theoretical guarantees for the result.

5. CONCLUSIONS

The current paper has presented a novel averaging approach for pulse-frequency modulated systems with constant duty ratios. The approach involves a period-weighting component that makes the resultant averaged system dependent on the modulation period, which is important for situations when the average operating point of the original periodic system is also dependent on the modulation period. In such cases the standard averaging approach fails because it leads to a period-independent averaged model, which can not provide a suitable approximation for the original system.

A theoretical framework was developed for the period-weighted averaging method, which ensures an error bound for the approximation between the original systems and the averaged system, and which relates the stability of the averaged system with that of the original system. Finally, the new averaging method is used for a case study involving a model of the cardiovascular system. Simulation results show that the period-dependent average model of the cardiovascular system represents a good approximation of the original periodic systems. Because the averaged model is simpler than the original periodic one, it could be further used for closed-loop analysis of cardiovascular regulation - the cardiovascular model coupled with a model for the nervous control loop (among which the most important is the baroreflex feedback mechanism). Such a coupling would be straightforward because, as in the case of many technical control systems, the (nervous) feedback control loop actually regulates the time-averages of key state variables of the plant (cardiovascular system), instead of instantaneous values (Heldt et al., 2005).

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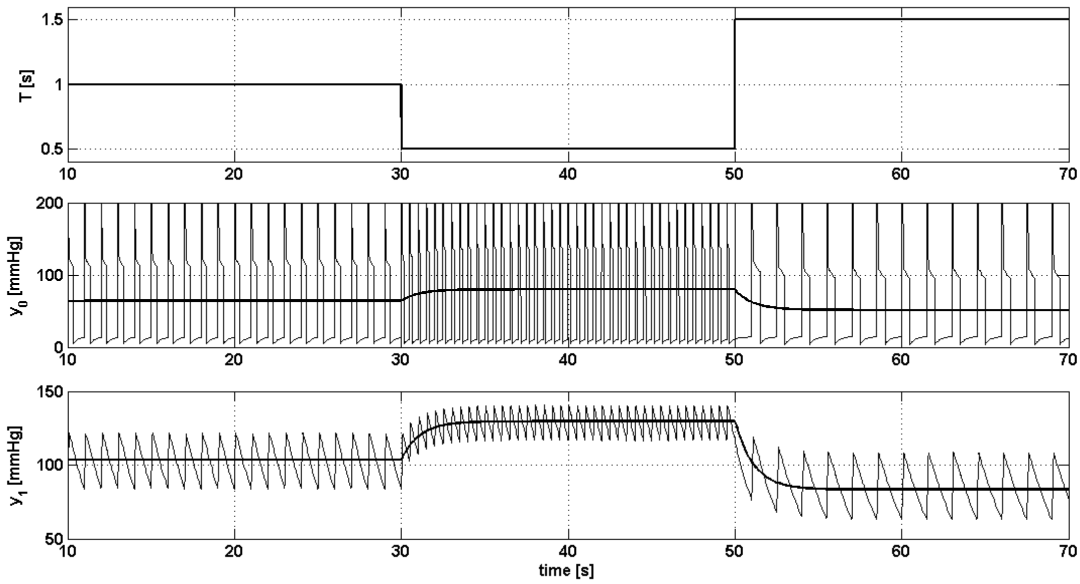

Fig. 1. Trajectories of the averaged system and the original system as T varies from 1 s to 0.5 s, and from 0.5 s to 1.5 s.

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Appendix A. FIRST APPENDIX

Table 1. Parameter values of the cardiovascular model (adapted from Heldt et al., 2005).

Parameter	Value	Measure unit
R_0	0.01	mmHg ml^{-1} s
R_1	1.0	mmHg ml^{-1} s
R_2	0.03	mmHg ml^{-1} s
E_d	0.1	mmHg ml^{-1}
E_s	2.5	mmHg ml^{-1}
C_1	2.0	ml mmHg^{-1}
C_2	100.0	ml mmHg^{-1}
x_T	1734	ml
$x_0(0)$	22.4	ml
$x_1(0)$	112	ml