A Robust Sensor Fault Reconstruction Based on a New Sliding Mode Observer Design Method for Uncertain Delayed Systems: A Satellite System Application

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Abstract: In this paper, we present a robust sensor fault reconstruction scheme for uncertain time-delay satellite system using a new Sliding Mode Observer (SMO). Indeed, taking into account the uncertainty and the time-delay, obtaining a robust sensor fault reconstruction is a difficult task. Therefore, to overcome this difficulty, a virtual augmented system is established in which the sensor fault in the real system is treated as actuator fault. For this virtual augmented system, a new SMO design method is proposed. To obtain this observer, the SMO concept, a Lyapunov-Krasovskii approach and the Linear Matrix Inequality (LMI) optimization are derived to guarantee the stability of the estimation error dynamics and compute the SMO gains. Consequently, applying the developed SMO, the Bounded Real Lemma (BRL), the H_∞ concept and a Lyapunov-Krasovskii functional, a robust sensor fault reconstruction is obtained. A satellite system is included to show the efficiency of the proposed methods.

Keywords: Sliding Mode Observer, Satellite System, Robust Fault Reconstruction, Uncertain Time-Delay Systems, Linear Matrix Inequalities.

1. INTRODUCTION

Time delay is frequently encountered in a variety of dynamic systems, such as communication, network, nuclear reactors, biology, population, physics, chemistry etc. Also the time-delay systems belong to the class of differential-difference equations which are infinite dimensional and, in usually cases, it is a source of instability, oscillation and degradation of the system performances (Niculescu, 2002; Gu et al., 2003; Jafarov, 2009). Therefore, the study of this systems is pertinent and valuable both from theoretical and practical perspectives and has some open problems (Richard, 2003). One of these problems is the state estimation. Consequently, this has motivated the current emphasis on the study of robust states estimation of time-delay systems. In fact, the observer based methods are the most widely used (Fattouh et al., 1999; Cacace et al., 2010), but, sometimes the observer cannot reflect the system sufficiently enough. However, it's known that the SMO is an effective estimation approach thanks to its excellent advantage of strong robustness against model uncertainties, parameter variations, and external disturbances (Utkin, 1992; Edwards and Spurgeon, 1994; Andreescu, 2003). Relatively some of researchers have investigated this observer on the estimation area of the uncertain time-delay systems (Jafarov, 2005; Niu and Ho, 2006; Koshkouei and Burnham, 2009). This observer has subsequently been employed in other situations including the Fault Detection and Isolation (FDI) problem of uncertain systems (Tan and Edwards, 2003; Chen et al., 2008; Wu and Saif, 2010; Yan and Edwards, 2007, 2008; Raoufi et al., 2010; Sharma and Aldeen, 2011; Dhahri et al., 2012; Akram et al., 2014). Note that most of these works are focused on the FDI for uncertain systems and little attempt has been made on the FDI problem for uncertain time-delay systems.

Thus, only few SMO approaches are used in the FDI problem of time-delay systems. To solve this problem, a robust high gain observer for state and unknown inputs/faults estimations for a class of Lipschitz nonlinear systems is presented by (Veluvolu et al., 2011), in which they de-coupled the fault signals to obtain the fault and the state estimations. (Zong et al., 2012) presented a sliding mode observer-based fault detection method for two-level distributed networked control systems and two different situations are considered, when all the states of the system are available for measurement or not, respectively. (Liu et al., 2011) presented a proportional and derivative sliding mode observer of sensor fault estimation and fault-tolerant control (FTC) for Markovian jump systems with time-delay and Lipschitz non-linearities. In the presence of sampled output information, a SMO and its application to robust fault reconstruction for uncertain system with delayed output, is presented in (Han et al., 2013).

However, compared with the rich results in FDI based SMO of uncertain systems, few research results are addressed to the FDI for uncertain time-delay systems, and this motivates our work. In this paper we present a new SMO design method for robust sensor fault reconstruction for a class of linear uncertain time-delay systems. First, the sensors work in very harsh environments and are considered the least reliable components of the system and potentially faulty ones. Thus, the sensor fault reconstruction is very delicate. For this reason, a virtual augmented system is established by designing a stable filter to process the outputs where the sensor fault of the real system takes the appearance of actuator fault of the virtual augmented system. For this augmented system, the stability conditions of estimation error dynamic are derived by using Lyapunov-Krasovskii functional method and Linear Matrix Inequality (LMI) optimization. Taking into account for these conditions, the SMO parameters are obtained such that the estimation error is always driven to a pre-defined sliding surface, in finite time, and maintain a sliding motion thereafter. In the existing works, this method differs from other methods in class of systems investigated and particularly, the uncertain time-delay systems. Based on this designed SMO and using the Bounded Real Lemma (BRL) to minimize the H_∞ norm of a transfer matrix between the uncertainty and the fault reconstructed signal, a robust sensor fault reconstruction can be achieved. This problem of minimization is cast as a well-defined convex optimization problem and it's efficiently solved using the LMI approach. Moreover, our robust reconstruction approach can give some information concerning faults such as the magnitude, the shape and the dynamic.

The rest of the paper is organized as follows: Section 2 gives a short description of the linear uncertain delayed system, the SMO and some preliminaries. In section 3, we present a new SMO design method. In section 4, we verify the dynamic properties of the designed observer. A method of robust sensor fault reconstruction is developed in section 5. Simulation results are accessible in section 6 to demonstrate the effectiveness of the proposed schemes and the section 7 will state some conclusions.

Notation. Throughout this paper, the notation ||.|| will be used to represent the Euclidean norm or its induced norm. I_n and 0_n represent an nth order identity and zeros matrix dimensions, respectively. $P>0$ ($P<0$), means that P is a symmetric and positive (negative) definite matrix. λ_{max} (λ_{min}) is the largest (smallest) eigenvalue.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we will introduce some necessary preliminaries and formulate a sensor fault reconstruction problem based on SMO. Considering the following linear uncertain delayed system

$$\dot{x}_o(t) = A_o x_o(t) + A_{oh} x_o(t-h) + B_o u(t) + M_o \xi(t, x(t)), \tag{1}$$

$$y_o(t) = C_o x_o(t) + N_o f_o(t), \tag{2}$$

where $x_o(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input vector and $y_o(t) \in \Re^p$ is the output vector. The signal $f_o(t) \in \Re^r$ represents an additive sensor fault which affects the plant dynamics, it is unknown but bounded so that $\| f_o(t) \| \le \alpha$ where α is a known positive scalar. The uncertainty $\xi(t, x(t)) \in \Re^k$ is unknown and bounded by a positive scalar β subject to $\|\xi(t, x(t))\| \le \beta$. h is a known positive number denoting the state time-delay. The matrices $A_{o,}\, A_{oh}$, B_o, M_o, C_o and N_o are constants and with appropriate dimensions where $q < p \le n$. Assume that C_o is full row rank and N_o is full column rank.

Remark 1. Our problem is to reconstruct sensor fault using a robust SMO. To solve this problem, we will use an effective method developed by (Tan and Edwards, 2002) which consists in overtaking the output of the uncertain system through a stable law passes filter $-A_f \in \Re^{p\times p}$, where both the state equation system and the state equation filter form an augmented system. Then, the original sensor fault will be considered as actuator fault of this augmented system.

This FDI problem can be formulated and summarized into the figure 1.

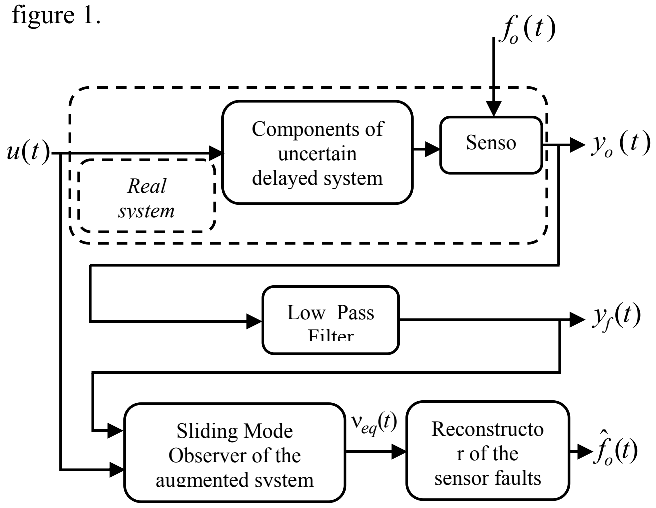

Fig. 1. Robust sensor fault reconstruction based on SMO.

Scaling the output $y_o(t)$ (2) by $-A_f$, so the filtered version of $y_o(t)$ is:

$$\dot{y}_f(t) = -A_f y_f(t) + A_f C_o x_o(t) + A_f N_o f_o(t). \tag{3}$$

Yet, if we combine the equations (1) and (3), we get an augmented state space system of order $n+p$, defined by

$$\begin{aligned}\underbrace{\begin{bmatrix}\dot{x}_o(t)\\ \dot{y}_f(t)\end{bmatrix}}_{\dot{x}(t)} &= \underbrace{\begin{bmatrix}A_o & 0_{n\times p}\\ A_f C_o & -A_f\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_o(t)\\ y_f(t)\end{bmatrix}}_{x(t)} \\ &+ \underbrace{\begin{bmatrix}A_{oh} & 0_{n\times p}\\ 0_{p\times n} & 0_{p\times p}\end{bmatrix}}_{A_h}\underbrace{\begin{bmatrix}x_o(t-h)\\ y_f(t-h)\end{bmatrix}}_{x(t-h)} + \underbrace{\begin{bmatrix}B_o\\ 0_{p\times m}\end{bmatrix}}_{B}u(t) \\ &+ \underbrace{\begin{bmatrix}0_{n\times q}\\ A_f N_o\end{bmatrix}}_{N}f_o(t) + \underbrace{\begin{bmatrix}M_o\\ 0_{p\times k}\end{bmatrix}}_{M}\xi(t, x(t)),\end{aligned} \tag{4}$$

$$y(t) = \underbrace{\begin{bmatrix}0_{p\times n} & I_p\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}x_o(t)\\ y_f(t)\end{bmatrix}}_{x(t)}. \tag{5}$$

Now, it's clear that the sensor fault in (1)-(2) will be actuator fault in (4)-(5).

Our new SMO for the system (4)-(5) is:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + A_h\hat{x}(t-h) + Bu(t) - Ke_y(t) + Gv(t),$$
$$\hat{y}(t) = C\hat{x}(t), \tag{6}$$

where $K \in \Re^{(n+p)\times p}$ and $G \in \Re^{(n+p)\times p}$ are the SMO gains, $e_y(t) = \hat{y}(t) - y(t)$ is the output error.

The discontinuous vector $\nu(t)$ is given by (Utkin, 1992):

$$\nu(t) = \begin{cases} -\rho(t,u,y)\dfrac{\bar{P}_o e_y(t)}{\left\|\bar{P}_o e_y(t)\right\|} & if \quad e_y(t) \neq 0 \\ 0 & otherwise \end{cases}, \tag{7}$$

where $\bar{P}_o \in \Re^{p\times p}$ is a symmetric positive definite matrix will be determined later and $\rho(t,u,y)$ is a scalar gain function. To design the SMO (6), the following assumptions must be verified :

- Assumption A1. Rank(CN)=rank(N)=q.
- Assumption A2. The system (4)-(5) is minimum phase.

The assumption A1 is verified directly through the calculation of the rank and the proof of the assumption A2 is given in the following:

Proof 1. From (Pearson and Fiagbedzi, 1989), for all complex *s* with *Re(s)≥0*, the system $(A + A_h e^{-hs}, N, C)$ is minimum phase if and only if

$$rank\left(\begin{bmatrix} sI - A - A_h\mathrm{e}^{-hs} & 0_{n\times p} \\ C & N \end{bmatrix}\right) = rank\left(\begin{bmatrix} sI - A - A_h\mathrm{e}^{-hs} \\ C \end{bmatrix}\right) + q, \tag{8}$$

however the matrix *N* is full column rank then

$$rank\left(\begin{bmatrix} sI - A - A_h e^{-hs} \\ C \end{bmatrix}\right) = n + p. \tag{9}$$

So, from (8) and (9) the system $(A + A_h e^{-hs}, N, C)$ is minimum phase if and only if $(A + A_h e^{-hs}, C)$ is detectable.

Furthermore, if the two assumptions A1 and A2 are verified, then, a change of coordinates *To* exists, which can transform the state-space system (4)-(5) into tow non-faulty and potentially faulty sub-systems, then the system matrices appropriately would yield (Jafarov, 2009):

$$\bar{A} = T_o A T_o^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{211} & \bar{A}_{22} \\ \bar{A}_{212} & \end{bmatrix},$$

$$\bar{A}_h = T_o A_h T_o^{-1} = \begin{bmatrix} \bar{A}_{h11} & \bar{A}_{h12} \\ \bar{A}_{h21} & \bar{A}_{h22} \end{bmatrix} = \begin{bmatrix} \bar{A}_{h11} & \bar{A}_{h12} \\ \bar{A}_{h211} & \bar{A}_{h22} \\ \bar{A}_{h212} & \end{bmatrix},$$

$$\bar{M} = T_o M = \begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \end{bmatrix},\ \bar{N} = T_o N = \begin{bmatrix} 0_{n\times q} \\ \bar{N}_2 = \begin{bmatrix} 0_{p\times q} \\ \bar{N}_q \end{bmatrix} \end{bmatrix}$$

and $\quad \bar{C} = CT_o^{-1} = \begin{bmatrix} 0_{p\times n} & \bar{C}_2 \end{bmatrix}, \tag{10}$

where $\bar{A}_{11} \in \Re^{n\times n}$, $\bar{A}_{12} \in \Re^{n\times p}$, $\bar{A}_{21} \in \Re^{p\times n}$, $\bar{A}_{211} \in \Re^{(p-q)\times n}$, $\bar{A}_{22} \in \Re^{p\times p}$, $\bar{A}_{h11} \in \Re^{n\times n}$, $\bar{A}_{h12} \in \Re^{n\times p}$, $\bar{A}_{h21} \in \Re^{p\times n}$, $\bar{A}_{h211} \in \Re^{(p-q)\times n}$ and $\bar{A}_{h22} \in \Re^{p\times p}$. For the uncertainty matrix $\bar{M}_1 \in \Re^{n\times k}$ and $\bar{M}_2 \in \Re^{p\times k}$, and for the fault matrix $\bar{N}_2 \in \Re^{p\times q}$, $\bar{N}_q \in \Re^{q\times q}$ is a non-singular matrix and $\bar{C}_2 \in \Re^{p\times p}$ is an orthogonal matrix $\left(\bar{C}_2\bar{C}_2^{-1} = I_p\right)$.

The system (4)-(5) is minimum phase, then the pair $(\bar{A}_{11} + \bar{A}_{h11}e^{-hs}, \bar{A}_{211} + \bar{A}_{h211}\,\mathrm{e}^{-hs})$ is detectable, so a matrix L_q can always be found to make $(\bar{A}_{11} + \bar{A}_{h11}e^{-hs} + L_q(\bar{A}_{211} + \bar{A}_{h211}\,\mathrm{e}^{-hs}))$ is stable.

Also, in the new coordinates, the SMO gains *K* and *G* are given by

$$\bar{K} = \begin{bmatrix} \bar{K}_1 \\ \bar{K}_2 \end{bmatrix}, \tag{11}$$

$$\bar{G} = \begin{bmatrix} -L \\ I_p \end{bmatrix}\bar{C}_2^T, \tag{12}$$

where

$$L = \begin{bmatrix} L_q & 0_{n\times q} \end{bmatrix} \in \Re^{n\times p}, \tag{13}$$

with $L_q \in \Re^{n\times(p-q)}$ and $\bar{C}_2$ is defined in (10).

Corollary 1. The sliding motion is governed by $(\bar{A}_{11} + \bar{A}_{h11}e^{-hs} + L_q(\bar{A}_{211} + \bar{A}_{h211}\,\mathrm{e}^{-hs}))$ which stable, then, the sliding surface $S_g = \{e : Ce(t) = 0\}$ is taken in finite time.

Proof 2. This proof is an extension to Corollary 6.1 in Edwards and Spurgeon (1998) to show that the sliding motion is governed by

$$\left(I_{n+p} - \bar{G}(\bar{C}\bar{G})^{-1}\bar{C}\right)\left(\bar{A} - \bar{K}\bar{C} + A_h\,\mathrm{e}^{-hs}\right) = \begin{bmatrix} \Theta_1 & \Theta_2 \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}, \tag{14}$$

where

$$\Theta_1 = (\bar{A}_{11} + \bar{A}_{h11}e^{-hs}) + L_q(\bar{A}_{211} + \bar{A}_{h211}\,\mathrm{e}^{-hs}),$$

and

$$\Theta_2 = (\bar{A}_{12} + \bar{A}_{h12}e^{-hs}) + L_q(\bar{A}_{22} + \bar{A}_{h22}\,\mathrm{e}^{-hs}).$$

Consequently, it is clear that the sliding dynamic is governed by the radii matrix Θ_1 which is stable. Hence, sliding motion takes place on S_g in finite time.

3. NEW SLIDING MODE OBSERVER DESIGN

This section focuses on new SMO (6) design method. The problem of design will be posed in such a way that Lyapunov-Krasovskii functional and LMIs can be used to obtain the SMO gains $\bar{K}$ and $\bar{G}$, and the Lyapunov matrix $\bar{P}$. After coordinates change, the state estimation error will be $\bar{e} = T_o.e$, i.e.

$$\dot{\bar{e}}(t) = (\bar{A} - \bar{K}\bar{C})\bar{e}(t) + \bar{A}_h\bar{e}(t-h) + \bar{G}\nu(t) - \bar{M}\xi(t,x(t)) - \bar{N}f_o(t). \quad (15)$$

Theorem 1. If there exist positive symmetric definite matrices $\bar{P} \in \Re^{(n+p)\times(n+p)}$, $\bar{Y} \in \Re^{(n+p)\times p}$ and $\bar{S} \in \Re^{(n+p)\times(n+p)}$ such that the following LMI conditions are satisfied:

$$\begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T + \bar{S} & \bar{P}\bar{A}_h \\ \bar{A}_h^T\bar{P} & -\bar{S} \end{bmatrix} < 0 \quad (16)$$

and

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^T & \bar{P}_3 \end{bmatrix} > 0, \quad (17)$$

where $\bar{P}_1 \in \Re^{n\times n}$, $\bar{P}_2 \in \Re^{n\times p}$, $\bar{P}_3 \in \Re^{p\times p}$, then the state estimation error (15) is bounded and belongs to the following set

$$\Psi = \{e : \|\bar{e}\| < \mu_1\beta/\mu_o + \omega\}, \quad (18)$$

where ω is a positive scalar,

$$\mu_1 = \sqrt{\lambda_{max}(\bar{M}^T\bar{P}^2\bar{M})}, \quad (19)$$

$$\mu_o = -\lambda_{\max}\left(\begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T + \bar{S} & \bar{P}\bar{A}_h \\ \bar{A}_h^T\bar{P} & -\bar{S} \end{bmatrix}\right) \quad (20)$$

and $\lambda_{max}(\,.\,)$ is the maximal eigenvalue.

Proof 3. Consider the Lyapunov-Krasovskii functional:

$$V_1(t) = \bar{e}^T(t)\bar{P}\bar{e}(t) + \int_{-h}^{0} \bar{e}^T(t+\tau)\bar{S}\,\bar{e}(t+\tau)d\tau. \quad (21)$$

The derivative of $V_1(t)$ along the trajectory of $\bar{e}(t)$ governed by (15) is

$$\dot{V}_1(t) \le \varphi^T \Xi\, \varphi - 2\bar{e}^T(t)\bar{P}\bar{M}\xi(t,x(t)) + 2\bar{e}^T(t)\bar{P}\bar{G}\nu(t) - 2\bar{e}^T(t)\bar{P}\bar{N}f_o(t), \quad (22)$$

with $\Xi = \begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{P}\bar{K}\bar{C} - \bar{C}^T\bar{K}^T\bar{P} + \bar{S} & \bar{P}\bar{A}_h \\ \bar{A}_h^T\bar{P} & -\bar{S} \end{bmatrix}$ and

$\varphi = \begin{bmatrix} \bar{e}^T(t) & \bar{e}^T(t-h) \end{bmatrix}^T$.

Then if we define from (Tan and Edwards, 2001)

$$\bar{P}_o = \bar{C}_2(\bar{P}_3 - L^T\bar{P}_1L)\bar{C}_2^T \quad (23)$$

and

$$\bar{P}_2 = \bar{P}_1L, \quad (24)$$

with $\bar{P}_2 = \begin{bmatrix} \bar{P}_{21} & 0_{p\times q} \end{bmatrix}$ and $\bar{P}_{21} \in \Re^{n\times(p-q)}$ and from (10), (12), (13), (17) and (23) we can obtain $\bar{P}\bar{N} = \bar{C}^T\bar{P}_o\bar{C}\bar{N}$ and $\bar{P}\bar{G} = \bar{C}^T\bar{P}_o$, hence

$$\dot{V}_1(t) \le \varphi^T \Xi\, \varphi - 2\bar{e}^T(t)\bar{P}\bar{M}\xi(t,x(t)) + 2\bar{e}^T(t)\bar{C}^T\bar{P}_o\nu(t) - 2\bar{e}^T(t)\bar{C}^T\bar{P}_o\bar{C}\bar{N}f_o(t). \quad (25)$$

Since $\bar{e}_y(t) = \bar{C}\bar{e}(t)$ and using (7), the equation (25) leads to

$$\dot{V}_1(t) \le \varphi^T \Xi\, \varphi + 2\|\bar{e}\|\mu_1\beta - 2\|\bar{P}_o\bar{e}_y\|(\rho(t,u,y) - \|\bar{C}\bar{N}\|\alpha) \quad (26)$$

and $\rho(t,u,y)$ in (7) is chosen to satisfy

$$\rho(t,u,y) \ge \|\bar{C}\bar{N}\|\alpha + \varepsilon, \quad (27)$$

where ε is a positive scalar. So it's clear that for $t \to \infty$ we obtain $\|\bar{e}(t)\| \simeq \|\bar{e}(t-h)\|$ and using (27) we get

$$\dot{V}_1(t) \le 2\|\bar{e}\|\left(-\|\bar{e}\|\mu_o + \mu_1\beta\right), \quad (28)$$

then the state estimation error (15) is bound and belongs to Ψ. Using the LMI technique to resolve $\Xi<0$ since this inequality isn't affine in the variable matrices $\bar{P}$ and $\bar{K}$, for this reason, we propose a matrix

$$\bar{K} = \bar{P}^{-1}\bar{Y}. \quad (29)$$

Then using the equation (29), $\Xi<0$ will be equivalent to (16). Consequently, if the condition (27) is verified then the positive definite matrices $\bar{P}, \bar{Y}$ and $\bar{S}$, are the solutions of this LMI. Then the LMI condition of theorem 1 appears. Thus, if (16) and (17) are verified, the state estimation error is bound and belongs to Ψ.

Now, we can find the gain $\bar{K}$ using the equation (29), $\bar{G}$ using (12), $\bar{P}_o$ from (23) and the matrix $L = \bar{P}_1^{-1}\bar{P}_2$ is given by (24), as to find numerical value matrices $\bar{P}, \bar{Y}$ and $\bar{C}_2$ we used the software MATLAB's LMI Control Toolbox Gahinet et al. (1995).

4. DYNAMIC PROPERTIES OF THE OBSERVER

After designing the SMO and verifying the stability of the estimation error, it is necessary to verify if this SMO can drive and maintain this error on the sliding surface S_g in finite time, which famous by the reachability condition. For this reason, a second change of coordinates (Jafarov, 2009) is applied

$$\tilde{T} : \bar{e} \mapsto \tilde{e} \qquad \tilde{T} = \begin{bmatrix} I_n & L \\ 0_{p\times n} & \bar{C}_2 \end{bmatrix}. \quad (30)$$

In this new coordinates, the system matrices will be:

$$\tilde{A} = \tilde{T}\bar{A}\tilde{T}^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

$$\tilde{A}_h = \tilde{T}\bar{A}_h\tilde{T}^{-1} = \begin{bmatrix} \tilde{A}_{h11} & \tilde{A}_{h12} \\ \tilde{A}_{h21} & \tilde{A}_{h22} \end{bmatrix},$$

$$\tilde{N} = \tilde{T}\bar{N} = \begin{bmatrix} 0_{n\times q} \\ \tilde{N}_2 \end{bmatrix} = \begin{bmatrix} 0_{n\times q} \\ \bar{C}_2\bar{N}_2 \end{bmatrix}, \tag{31}$$

$$\tilde{M} = \tilde{T}\bar{M} = \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix} = \begin{bmatrix} \bar{M}_1 + L\bar{M}_2 \\ \bar{C}_2\bar{M}_2 \end{bmatrix},$$

$$\tilde{C} = \bar{C}\tilde{T}^{-1} = \begin{bmatrix} 0_{p\times n} & I_p \end{bmatrix}$$

where

$$\begin{aligned}
\tilde{A}_{11} &= \bar{A}_{11} + L\bar{A}_{21} = \bar{A}_{11} + L_q\bar{A}_{211}, \\
\tilde{A}_{12} &= -(\bar{A}_{11} + L\bar{A}_{21})L\bar{C}_2^T + (\bar{A}_{12} + L\bar{A}_{22})\bar{C}_2^T, \\
\tilde{A}_{21} &= \bar{C}_2\bar{A}_{21}, \\
\tilde{A}_{22} &= -\bar{C}_2\bar{A}_{21}L\bar{C}_2^T + \bar{C}_2\bar{A}_{22}\bar{C}_2^T, \\
\tilde{A}_{h11} &= \bar{A}_{h11} + L\bar{A}_{h21} = \bar{A}_{h11} + L_q\bar{A}_{h211}, \\
\tilde{A}_{h12} &= -(\bar{A}_{h11} + L\bar{A}_{h21})L\bar{C}_2^T + (\bar{A}_{h12} + L\bar{A}_{h22})\bar{C}_2^T, \\
\tilde{A}_{h21} &= \bar{C}_2\bar{A}_{h21}, \\
\tilde{A}_{h22} &= -\bar{C}_2\bar{A}_{h21}L\bar{C}_2^T + \bar{C}_2\bar{A}_{h22}\bar{C}_2^T.
\end{aligned}$$

Thus, the SMO gains become:

$$\tilde{G} = \tilde{T}\,\bar{G} = \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix} \tag{32}$$

and

$$\tilde{K} = \tilde{T}\bar{K} = \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} = \begin{bmatrix} \bar{K}_1 + L\bar{K}_2 \\ \bar{C}_2\bar{K}_2 \end{bmatrix}. \tag{33}$$

The Lyapunov matrix $\bar{P}$ will be:

$$\tilde{P} = (\tilde{T}^{-1})^T\,\bar{P}(\tilde{T}^{-1}) = \begin{bmatrix} \bar{P}_1 & 0_{n\times p} \\ 0_{p\times n} & \bar{P}_o \end{bmatrix}, \tag{34}$$

then, the new estimation error becomes

$$\begin{aligned}
\dot{\tilde{e}}(t) &= (\tilde{A} - \tilde{K}\tilde{C})\tilde{e}(t) + \tilde{A}_h\tilde{e}(t-h) + \tilde{G}\nu(t) \\
&\quad - \tilde{M}\xi(t,x(t)) - \tilde{N}f_o(t).
\end{aligned} \tag{35}$$

Partitioning this error according to the dimensions of (31), it is easy to check that:

$$\begin{aligned}
\dot{\tilde{e}}_1(t) &= \tilde{A}_{11}\tilde{e}_1(t) + (\tilde{A}_{12} - \tilde{K}_1)\tilde{e}_y(t) + \tilde{A}_{h11}\tilde{e}_1(t-h) \\
&\quad + \tilde{A}_{h12}\tilde{e}_y(t-h) - \tilde{M}_1\,\xi(t,x(t)),
\end{aligned} \tag{36}$$

$$\begin{aligned}
\dot{\tilde{e}}_y(t) &= \tilde{A}_{21}\tilde{e}_1(t) + (\tilde{A}_{22} - \tilde{K}_2)\tilde{e}_y(t) + \tilde{A}_{h21}\tilde{e}_1(t-h) \\
&\quad + \tilde{A}_{h22}\tilde{e}_y(t-h) + \nu(t) - \tilde{M}_2\xi(t,x(t)) + \nu(t) \\
&\quad - \tilde{N}_2 f_o(t).
\end{aligned} \tag{37}$$

Lemma 1. Under assumptions A1 and A2, if the scalar gain function $\rho(t,u,y)$ in (7) satisfies

$$\begin{aligned}
\rho(t,u,y) &\ge \|\tilde{N}_2\|\alpha + \|\tilde{M}_2\|\beta + \|\tilde{A}_{21}\|\|\tilde{e}_1(t)\| \\
&\quad + \|\tilde{A}_{h21}\|\|\tilde{e}_1(t-h)\| + \|\tilde{A}_{h22}\|\|\tilde{e}_y(t-h)\| + \varepsilon,
\end{aligned} \tag{38}$$

where ε is a positive scalar, then the system (36)–(37) is driven to the sliding surface S_g in finite time and maintains a sliding motion on it.

Proof 4. Consider a Lyapunov functional:

$$V_2(t) = \tilde{e}_y^T(t)\bar{P}_o\,\tilde{e}_y(t). \tag{39}$$

The derivative of $V_2(t)$ along the trajectory of (37) leads to

$$\dot{V}_2(t) = \dot{\tilde{e}}_y^T(t)\bar{P}_o\tilde{e}_y(t) + \tilde{e}_y^T(t)\bar{P}_o\dot{\tilde{e}}_y(t), \tag{40}$$

then

$$\begin{aligned}
\dot{V}_2(t) &= \tilde{e}_y^T\left[\bar{P}_o(\tilde{A}_{22} - \tilde{K}_2) + (\tilde{A}_{22} - \tilde{K}_2)^T\bar{P}_o\right]\tilde{e}_y \\
&\quad + 2\tilde{e}_y^T(t)\bar{P}_o\tilde{A}_{21}\tilde{e}_1(t) + 2\tilde{e}_y^T(t)\bar{P}_o\tilde{A}_{h21}\tilde{e}_1(t-h) \\
&\quad + 2\tilde{e}_y^T(t)\bar{P}_o\tilde{A}_{h22}\tilde{e}_y(t-h) + 2\tilde{e}_y^T(t)\bar{P}_o\nu \\
&\quad - 2\tilde{e}_y^T(t)\bar{P}_o\tilde{M}_2\xi(t,x(t)) - 2\tilde{e}_y^T(t)\bar{P}_o\tilde{N}_2 f_o(t).
\end{aligned} \tag{41}$$

Since, by design, $\tilde{P}$ is a block diagonal Lyapunov matrix for $(\tilde{A} - \tilde{K}\tilde{C})$ then, $\bar{P}_o(\tilde{A}_{22} - \tilde{K}_2) + (\tilde{A}_{22} - \tilde{K}_2)^T\bar{P}_o < 0$. Also, from the Theorem 1, the estimation error is quadratically stable and ultimately bounded and belongs to the set Ψ, thus for some small $a = 2\mu_1\beta\big/\left(\|\varphi\|^2\mu_o\right) + \omega > 0$, we have $\sup\limits_{0\le t<\infty}\|\tilde{e}(t)\| < a$, therefore,

$$\begin{aligned}
\dot{V}_2(t) &\le -2\bar{P}_o\tilde{e}_y^T\left[\rho(t,u,y) - \|\tilde{M}_2\|\|\xi(t,x(t))\|\right. \\
&\quad - \|\tilde{N}_2\|\|f_o(t)\| - \|\tilde{A}_{21}\|\|\tilde{e}_1(t)\| - \|\tilde{A}_{h21}\|\|\tilde{e}_1(t-h)\| \\
&\quad \left. - \|\tilde{A}_{h22}\|\|\tilde{e}_y(t-h)\| + \varepsilon\right].
\end{aligned} \tag{42}$$

and using (38), it follows that the equation (42) leads to

$$\dot{V}_2(t) \le -2\varepsilon\|\bar{P}_o\tilde{e}_y\| \le -2\varepsilon\sqrt{\lambda_{\min}(\bar{P}_o)}\sqrt{V_2(t)}. \tag{43}$$

This shows that the error system is driven to the sliding surface S_g in finite time and maintained on it, so the reachability condition is satisfied.

5. ROBUST SENSOR FAULT RECONSTRUCTION

In this part, we will present a robust sensor fault reconstruction method using the SMO (6). During the sliding motion $e_y = \dot{e}_y = 0$, therefore the equations (36) and (37) became (44) and (45), respectively:

$$\dot{\tilde{e}}_1(t) = \tilde{A}_{11}\tilde{e}_1(t) + \tilde{A}_{h11}\tilde{e}_1(t-h) - \tilde{M}_1\xi(t,x(t)), \tag{44}$$

$$\begin{aligned} 0 = \tilde{A}_{21}\tilde{e}_1(t) + \tilde{A}_{h21}\tilde{e}_1(t-h) + \nu_{eq}(t) - \tilde{M}_2\,\xi(t,x(t)) \\ - \tilde{N}_2 f_o(t), \end{aligned} \tag{45}$$

where $\nu_{eq}(t)$ is the equivalent output error injection (Utkin (1992)),which is a version of $\nu(t)$ during the sliding and can be approximated to any accuracy by

$$\nu_{eq}(t) = -\rho(t,u,y)\frac{\bar{P}_o\tilde{e}_y(t)}{\left\|\bar{P}_o\tilde{e}_y(t)\right\| + \delta}, \tag{46}$$

where δ is a small positive constant representing the term of smoothing. Since, $\nu_{eq}(t)$ is the responsible to maintain the sliding motion in presence of the fault and the uncertainty, then, the analysis of this term permits us to find the signal of the estimate fault $\hat{f}_o(t)$. Then, from (45) we get

$$\begin{aligned} \nu_{eq}(t) = -\tilde{A}_{21}\tilde{e}_1(t) - \tilde{A}_{h21}\tilde{e}_1(t-h) + \tilde{M}_2\,\xi(t,x(t)) \\ + \tilde{N}_2 f_o(t) \end{aligned} \tag{47}$$

and rewriting (31) in terms of the co-ordinates in (10), we obtain

$$\begin{aligned} \nu_{eq}(t) = -\bar{C}_2\bar{A}_{21}\tilde{e}_1(t) - \bar{C}_2\bar{A}_{h21}\tilde{e}_1(t-h) \\ + \bar{C}_2\bar{M}_2\,\xi(t,x(t)) + \bar{C}_2\bar{N}_2 f_o(t). \end{aligned} \tag{48}$$

The idea now is to extract $\hat{f}_o(t)$ from the equation (48), for this reason, we will use a numerical development in Tan and Edwards (2003). From (10) and (31), it's clear that

$$\tilde{N}_2 = \bar{C}_2\bar{N}_2 = \bar{C}_2\begin{bmatrix} 0_{(p-q)\times q} \\ \bar{N}_q \end{bmatrix}, \tag{49}$$

where $\bar{N}_q$ is a non singular matrix. Then for an arbitrary matrix $W_1 \in \Re^{q\times(p-q)}$

$$\begin{bmatrix} W_1 & \bar{N}_q^{-1} \end{bmatrix}\bar{C}_2^T.\bar{C}_2\begin{bmatrix} 0_{(p-q)\times q} \\ \bar{N}_q \end{bmatrix} = I_q. \tag{50}$$

Now, assume that

$$W = \begin{bmatrix} W_1 & \bar{N}_q^{-1} \end{bmatrix} \in \Re^{q\times p}, \tag{51}$$

then from this special form of W we can obtain

$$\hat{f}_o(t) = W\,\bar{C}_2^T\,\nu_{eq}(t), \tag{52}$$

which implies

$$\begin{aligned} \hat{f}_o(t) = -W\,\bar{A}_{21}\tilde{e}_1(t) - W\bar{A}_{h21}\tilde{e}_1(t-h) + W\bar{M}_2\xi(t,x(t)) \\ + f_o(t). \end{aligned} \tag{53}$$

Using (44) and (53), the last equation can be

$$\hat{f}_o(t) = H(s)\,\xi(t,x(t)) + f_o(t), \tag{54}$$

where

$$\begin{aligned} H(s) = W(\bar{A}_{21} + \bar{A}_{h21}e^{-hs}) \times (sI_n - \tilde{A}_{11} - \bar{A}_{h11}e^{-hs})^{-1} \\ \times(\bar{M}_1 + L\bar{M}_2) + W\bar{M}_2, \end{aligned} \tag{55}$$

where s is the Laplace variable, $W\bar{C}_2^T\tilde{A}_{21} = W\bar{A}_{21}$, $W\bar{C}_2^T\tilde{A}_{h21} = W\bar{A}_{h21}$, $\tilde{M}_1 = \bar{M}_1 + L\bar{M}_2$ and $W\bar{C}_2^T\tilde{M}_2 = W\bar{M}_2$. So, in this case, the transfer matrix *H(s)* joins the exogenous input signal $\xi(t,x(t))$ and the estimate fault signal $\hat{f}_o(t)$.

Remark 2. To obtain $\hat{f}_o(t) \cong f_o(t)$ is sufficient to minimize the H_∞ norm of *H(s)*. To solve this problem, we will use the LMI technique, the BRL and we extend a numerical development in (Tan and Edwards, 2003) which will be, after this extension, applicable in the linear uncertain delayed systems. Then, we end up with an optimization problem, which will be solved by the following theorem.

Theorem 2. Obtaining $\hat{f}_o(t) \cong f_o(t)$ is equivalent to minimize a positive scalar γ subject to $\|H\|_\infty < \gamma$ and for any non-zero $\xi \in L_2[0,\infty)$ under the zero initial condition, if and only if there exist variable matrices $\bar{P}_1, \bar{P}_2, \tilde{S}_1$ and W_1 such that the following inequality holds:

$$\begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 & -(W\bar{A}_{21})^T \\ \Upsilon_2^T & -\tilde{S}_1 & 0_{n\times k} & -(W\bar{A}_{h21})^T \\ \Upsilon_3^T & 0_{k\times n} & -\gamma^2 I_k & (W\bar{M}_2)^T \\ -W\bar{A}_{21} & -W\bar{A}_{h21} & W\bar{M}_2 & -I_q \end{bmatrix} < 0, \tag{56}$$

where

$$\Upsilon_1 = \bar{P}_1\bar{A}_{11} + \bar{P}_2\bar{A}_{21} + (\bar{P}_1\bar{A}_{11} + \bar{P}_2\bar{A}_{21})^T + \tilde{S}_1,$$

$$\Upsilon_2 = \bar{P}_1\bar{A}_{h11} + \bar{P}_2\bar{A}_{h21},$$

$$\Upsilon_3 = \bar{P}_1\bar{M}_1 + \bar{P}_2\bar{M}_2.$$

Proof 5. Consider the Lyapunov-Krasovskii functional

$$V_3(t) = \tilde{e}_1^T(t)\bar{P}_1\tilde{e}_1(t) + \int_{-h}^{0}\tilde{e}_1^T(t+\tau)\tilde{S}_1\tilde{e}_1(t+\tau)d\tau. \tag{57}$$

The derivative of (53) along the trajectory of (40) is equivalent to

$$\begin{aligned} V_3(t) = \tilde{e}_1^T(t)\left[\tilde{A}_{11}^T\bar{P}_1 + \bar{P}_1\tilde{A}_{11}\right]\tilde{e}_1(t) \\ + 2\tilde{e}_1^T(t)\bar{P}_1\tilde{A}_{h11}\tilde{e}_1(t-h) - 2\tilde{e}_1^T(t)\bar{P}_1\tilde{M}_1\xi(t,x(t)) \\ + \tilde{e}_1^T(t)\tilde{S}_1\tilde{e}_1(t) - \tilde{e}_1^T(t-h)\tilde{S}_1\tilde{e}_1(t-h), \end{aligned} \tag{58}$$

in terms of the coordinates in (8), (54) re-arranges yields

$$V_3(t) = \tilde{e}_1^T(t)\Upsilon_1\tilde{e}_1(t) + 2\tilde{e}_1^T(t)\Upsilon_2\tilde{e}_1(t-h) - 2\tilde{e}_1^T(t)\Upsilon_3\xi(t,x(t)) + \tilde{e}_1^T(t)\tilde{S}_1\tilde{e}_1(t) - \tilde{e}_1^T(t-h)\tilde{S}_1\tilde{e}_1(t-h), \quad (59)$$

which is equivalent to

$$V_3(t) = \vartheta^T\Upsilon_1\vartheta, \quad (60)$$

where

$$\Upsilon = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Upsilon_2^T & -\tilde{S}_1 & 0_{n\times k} \\ \Upsilon_3^T & 0_{k\times n} & 0_{k\times k} \end{bmatrix} \text{ and } \vartheta = \begin{bmatrix} \tilde{e}_1(t) \\ \tilde{e}_1(t-h) \\ \xi(t,x(t)) \end{bmatrix}.$$

Assuming there exist symmetric positive definite matrices $\bar{P}_1$ and $\tilde{S}_1$ which can guarantee $\Upsilon < 0$, and implies $\dot{V}_3(t) < 0$.

Also, for

$$\left\|\hat{f}_o(t)\right\|_2 \le \gamma^2 \left\|\xi(t,x(\mathrm{t}))\right\|_2, \quad (61)$$

so,

$$\hat{f}_o^T(t)\hat{f}_o(t) - \gamma^2\xi^T(t,x(t))\xi(t,x(\mathrm{t})) = \vartheta^T\Delta\vartheta, \quad (62)$$

where

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \Delta_2^T & \Delta_4 & \Delta_5 \\ \Delta_3^T & \Delta_5^T & \Delta_6 \end{bmatrix},$$

$$\Delta_1 = \bar{A}_{21}^T W^T W \bar{A}_{21},$$

$$\Delta_2 = \bar{A}_{21}^T W^T W \bar{A}_{h21},$$

$$\Delta_3 = -\bar{A}_{21}^T W^T W \bar{M}_2,$$

$$\Delta_4 = \bar{A}_{h21}^T W^T W \bar{A}_{h21},$$

$$\Delta_5 = -\bar{A}_{h21}^T W^T W \bar{M}_2,$$

and

$$\Delta_6 = \bar{M}_2^T W^T W \bar{M}_2 - \gamma^2 I_q.$$

In the next, we shall investigate the H_∞ performance and the BRL, thus, under the zero initial condition, for $t > 0$ and for any nonzero $\xi \in L_2[0,\infty)$, it can be shown

$$\dot{V}_3(t) - \gamma^2\xi^T(t,x(t))\xi(t,x(\mathrm{t})) + \hat{f}_o^T(t)\hat{f}_o(t) < 0, \quad (63)$$

which is equivalent to $\Upsilon + \Delta < 0$ and to

$$\begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Upsilon_2^T & -\tilde{S}_1 & 0_{n\times k} \\ \Upsilon_3^T & 0_{k\times n} & -\gamma^2 I_k \end{bmatrix} - \begin{bmatrix} -(W\bar{A}_{21})^T \\ -(W\bar{A}_{h21})^T \\ (W\bar{M}_2)^T \end{bmatrix}(-I_q)\begin{bmatrix} -(W\bar{A}_{21})^T \\ -(W\bar{A}_{h21})^T \\ (W\bar{M}_2)^T \end{bmatrix}^T < 0, \quad (64)$$

applying the Schur Lemma (Boyd et al., 1994), the last inequality will be equivalent to Eq. (56).

Consequently, this method consists to use the BRL to minimize the H_∞ norm of the transfer matrix *H(s)* by minimizing γ with respect to the variables matrices $\bar{P}$ and W_1 subject to (16), (17) and (56). To solve this convex optimization problem, software like MATLAB's LMI Control Toolbox (Gahinet et al., 1995) is able to find γ, $\bar{P}$ and W_1.

6. NUMERICAL EXAMPLE AND SIMULATIONS

Now, we will demonstrate the validity of the theoretical approaches presented in this paper. Consider a satellite system in Li et al. (2009) described by the following dynamic equations:

$$u(t) = J_1\ddot{\theta}_1(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k(\theta_1(t) - \theta_2(t)), \quad (65)$$

$$0.1\xi(t,x(t)) = J_2\ddot{\theta}_2(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k(\theta_1(t) - \theta_2(t)). \quad (66)$$

This satellite system consists of two rigid bodies joined by a flexible link. Where *u(t)* is the control torque, *k* is the torque constant and *f* represents the viscous damping. The yaw angles for the two bodies are θ_1 and θ_2. The moments of inertia of the two bodies are J_1 and J_2 and $\xi(t,x(t))$ includes all uncertainties and disturbances. The equations (65) and (66) allow us to obtain the following state space model:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}\xi(t,x(t)), \quad (67)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}x(t),$$

where $x(t) = \begin{bmatrix} \theta_1^T(t) & \theta_2^T(t) & \dot{\theta}_1^T(t) & \dot{\theta}_2^T(t) \end{bmatrix}^T$.

The parameters are chosen as in (Li et al., 2009): $J_1 = J_2 = 1$, $k = 0.09$, $f = 0.04$. To show the results which have been developed and is not a feature of (Li et al., 2009), affecting this system by sensor fault $f_o(t)$ (Benso and Carlo, 2011) and assuming that

$$u(t)=[-1.1789\ -1.3096\ -1.6629\ -7.3974]x(t-h) + 1.25u(t).$$

Then, the system is described in the form of (67) is equivalent to

$$
\frac{d}{dt}\underbrace{\begin{bmatrix}\theta_1^T(t)\\ \theta_2^T(t)\\ \dot{\theta}_1^T(t)\\ \dot{\theta}_2^T(t)\end{bmatrix}}_{x(t)}=\underbrace{\begin{bmatrix}0&0&1&0\\0&0&0&1\\-0.09&0.09&-0.04&0.04\\0.09&-0.09&0.04&-0.04\end{bmatrix}}_{A_O}x(t)
+\underbrace{\begin{bmatrix}0&0&0&0\\0&0&0&0\\-1.1789&-1.3096&-1.6629&-7.3974\\0&0&0&0\end{bmatrix}}_{A_{ho}}x(t-h)
+\underbrace{\begin{bmatrix}0\\0\\1.25\\0\end{bmatrix}}_{B_O}u(t)+\underbrace{\begin{bmatrix}0\\0\\0\\0.1\end{bmatrix}}_{M_O}\xi(t,x(t)),
\tag{68}
$$

$$
y(t)=\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}}_{C_O}x(t)+\underbrace{\begin{bmatrix}12\\0\\0\end{bmatrix}}_{N_O}f_o(t).
$$

In order to make the proposed system better conditioned, an additional feedback has been added to assign closed-loop poles of the matrix A at $\{-0.8378+0.5388i,\ -0.8378-0.5388i,\ -0.1411+0.3292i,\ -0.1411-0.3292i\}$. That, it is easy to check that Assumptions A1 and A2 hold. The coordinate transformation matrices are

$$
\overline{T}=\begin{bmatrix}
0&0&0&-1.0000&0&0&0\\
-1.0000&0&0&0&0&0&0\\
0&-1.0000&0&0&0&0&0\\
0&0&-1.0000&0&0&0&0\\
0&0&0&0&0&0&1.0000\\
0&0&0&0&0&1.0000&0\\
0&0&0&0&-1.0000&0&0
\end{bmatrix},
$$

$$
\tilde{T}=\begin{bmatrix}
1.0000&0&0&0&1.5782&-34.1898&0\\
0&1.0000&0&0&-1.3155&-43.2251&0\\
0&0&1.0000&0&1.2619&-29.3459&0\\
0&0&0&1.0000&0&0&0\\
0&0&0&0&0&0&-1.0000\\
0&0&0&0&0&1.0000&0\\
0&0&0&0&1.0000&0&0
\end{bmatrix}.
$$

Using Theorems 1 and 2, the following parameters are obtained:

$$
\overline{P}=\left[\begin{array}{c|c}\overline{P}_1&\overline{P}_2\\ \hline \overline{P}_2^T&\overline{P}_3\end{array}\right]
=\left[\begin{array}{cccc|ccc}
1.3288&-0.0890&-1.4284&0.0221&-0.4951&0.8947&0\\
-0.0890&0.1035&-0.0315&0.0136&-0.8726&-0.1625&0\\
-1.4284&-0.0315&2.5343&-0.0088&1.3465&-24.3987&0\\
0.0221&0.0136&-0.0088&0.0342&-1.3967&-0.2178&0\\ \hline
-0.4951&-0.8726&1.3465&-1.3967&171.0898&-20.0165&0.0000\\
0.8947&-0.1625&-24.3987&-0.2178&-20.0165&796.2985&-0.0000\\
0&0&0&0&0.0000&-0.0000&111.8173
\end{array}\right],
$$

$$
L=\overline{P}_1^{-1}\overline{P}_2=\left[L_q \;\middle|\; 0_{n\times q}\right]=\left[\begin{array}{cc|c}
1.5782&-34.1898&0\\
-1.3155&-43.2251&0\\
1.2619&-29.3459&0\\
-41.0067&25.3190&0
\end{array}\right],
$$

$$
\overline{P}_o=\overline{C}_2(\overline{P}_3-L^T\overline{P}_1L)\overline{C}_2^T=\begin{bmatrix}111.8173&0.0000&0.0000\\0.0000&109.3755&0.2164\\0.0000&0.2164&111.7493\end{bmatrix},
$$

$$
W=\left[W_1 \;\middle|\; \overline{N}_q^{-1}\right]=\left[0.2862\quad 0.8759\,|\,0.8333\right],
$$

$\delta = 7,2.10^{-5}$ and $\gamma = 0.0178$.

The SMO gains of the augmented system are

$$K = T_o^{-1}.\tilde{T}^{-1}.\tilde{K} = \begin{bmatrix} 344.8094 & 522.6985 & -188.6697 \\ 150.2371 & 362.1675 & -94.9140 \\ -184.3148 & -433.0516 & 773.9188 \\ 184.7383 & 406.1020 & -121.9188 \\ -1.1000 & -0.0000 & -0.0000 \\ -4.4188 & -11.0928 & 2.3961 \\ -0.1231 & 1.1635 & -7.2659 \end{bmatrix},$$

$$G = T_o^{-1}.\tilde{T}^{-1}.\tilde{G} = \begin{bmatrix} 0 & -43.2251 & -1.3155 \\ 0 & -29.3459 & 1.2619 \\ 0 & -25.3190 & -41.0067 \\ 0 & -34.1898 & 1.5782 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

Finally, choose $\rho(t,u,y) = 110$ to satisfy (38). For simulation, the time delay is chosen as $h=0.7s$, the uncertainty $\xi(t,x(t)) = 0.52x_1(t) + 0.21x_3(t)$ and with initial conditions $x(0) = \begin{bmatrix} 0.003 & -0.006 & 0.005 & -0.002 \end{bmatrix}^T$ and $\hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

From the figures 2, 3 and 4 we notice that the signal of the sensor faults and its estimations (respectively) are almost identical despite the presence of the uncertainty and the time-delay. So, using the proposed methods of SMO design and FDI, the problem of robust sensor fault reconstruction for uncertain delayed system is solved.

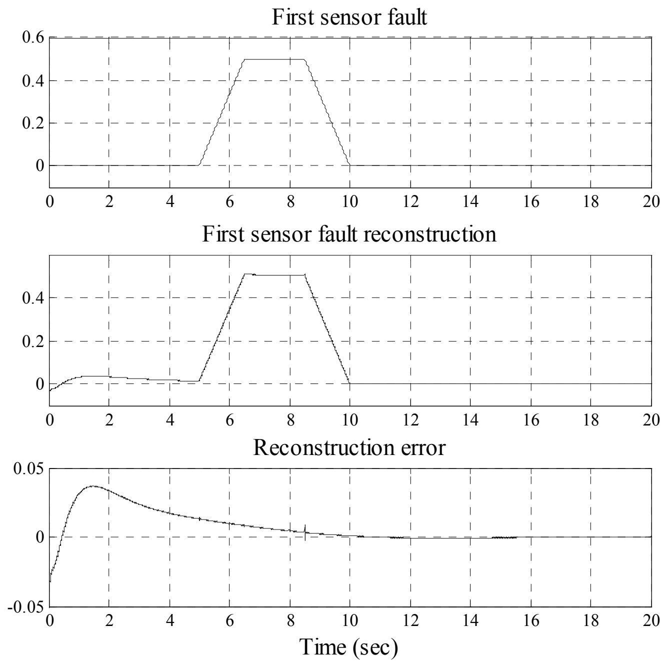

Fig. 2. Robust first sensor fault reconstruction.

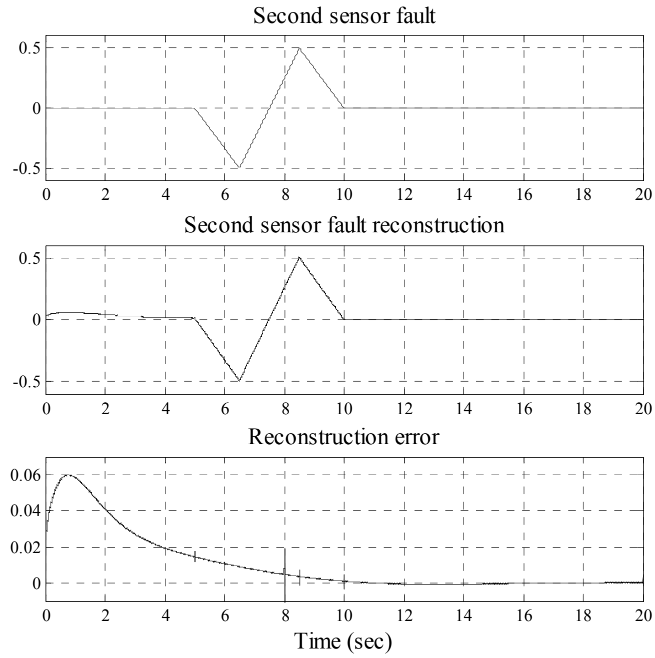

Fig. 3. Robust second sensor fault reconstruction.

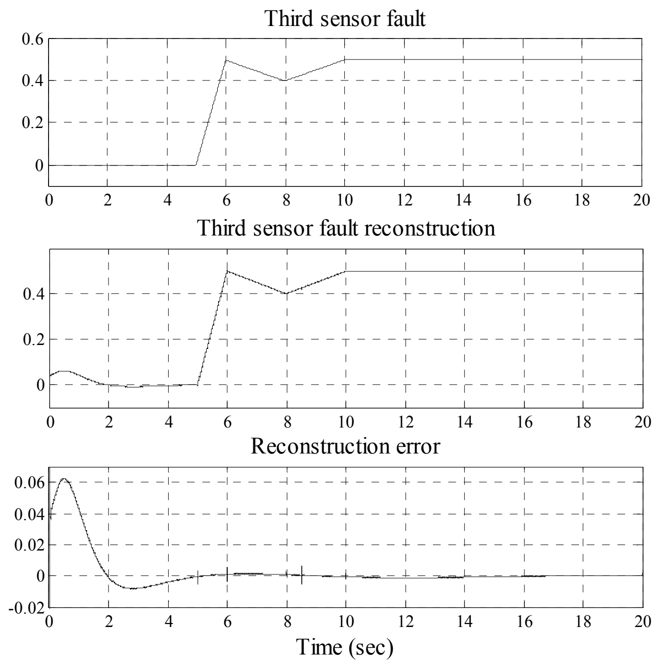

Fig. 4. Robust third sensor fault reconstruction.

7. CONCLUSION

In this paper, we proposed a new SMO design method for a class of uncertain time-delay systems. The time-delay is constant and the uncertainty is unknown and bounded. This SMO guarantees the stability of the estimation error and the reachability condition. We also developed a scheme of robust sensor fault reconstruction for this class of systems using the proposed observer and the BRL technique. A numerical example has been applied to validate the developed theoretical results.

REFERENCES

Akram, M.A., Rizvi, M.A., Bhatti, A.I., and Messai, N. (2014). Mode identification for hybrid model of si engine to detect misfire fault. *Journal of Control Engineering and Applied Informatics*, 16(3), 6574.

Andreescu, G.D. (2003). Sliding-mode observer for sensorless control of permanent magnet synchronous motor drives. *Journal of Control Engineering and Applied Informatics*, 5(1), 27–34.

Benso, A. and Carlo, S.D. (2011). The art of fault injection. *Journal of Control Engineering and Applied Informatics*, 13(4), 9–18.

Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM: Philadelphia.

Cacace, F., Germani, A., and Manes, C. (2010). An observer for a class of nonlinear systems with time varying observation delay. *Systems & Control Letters*, 59, 305–312.

Chen, M., Jiang, C.S., and Wu, Q.X. (2008). Sensor fault diagnosis for a class of time delay uncertain nonlinear systems using neural network. *International Journal of Automation and Computing*, 05(4), 401–405.

Dhahri, S., Sellami, A., and Hmida, F.B. (2012). Robust H∞ sliding mode observer design for fault estimation in a class of uncertain nonlinear systems with LMI optimization approach. *International Journal of Control, Automation, and Systems*, 10(5), 1032–1041.

Edwards, C. and Spurgeon, S.K. (1994). On the development of discontinuous observers. *International Journal Control*, 59, 1211–1229

Edwards, C. and Spurgeon, S.K. (1998). *Sliding Mode Control : Theory and Applications*. Taylor & Francis.

Fattouh, A., Sename, O., and Dion, J.M. (1999). Robust observer design for time–delay systems: a riccati equation approach. *KYBERNETIKA*, 35, 753–764.

Gahinet, P., Nemirovski, A., Laub, A., and Chilali, M. (1995). *LMI Control toolbox, User Guide*. Math Works, Inc.

Gu, K., Kharitonov, V.L., and Chen, J. (2003). *Stability of Time-delay Systems*. Birkh¨auser: Boston.

Han, X., E.Fridman, and Spurgeon, S.K. (2014). Sampled-data sliding mode observer for robust fault reconstruction: A time-delay approach. *Journal of the Franklin Institute*, 351(4), 2125-2142.

Jafarov, E.M. (2005). Design modification of sliding mode observer for uncertain mimo systems without and with time-delay. *Asian Journal of Control*, 7(4), 380–32.

Jafarov, E.M. (2009). *Variable structure control and time-delay systems*. A Series of Reference Books and Textbooks, WSEAS Press.

Koshkouei, A.J. and Burnham, K.J. (2009). Discontinuous observer for nonlinear time-delay systems. *International Journal of Systems Science*, 40, 383–392.

Li, T., Guo, L., and Xin, X. (2009). Improved delay-dependent bounded real lemma for uncertain time-delay systems. *Information Sciences*, doi:10.1016/j.ins.2009.06.015.

Liu, M., Shi, P., Zhang, L., and Zhao, X. (2011). Fault tolerant control for nonlinear markovian jump systems via proportional and derivative sliding mode observer technique. *IEEE Transactions on Circuits and Systems*, 58(11), 2755–2764.

Niculescu, S.I. (2002). *Delay Effects on Stability: A Robust Approach*. Springer-Verlag, New York, U.S.A.

Niu, Y. and Ho, D.W.C. (2006). Robust observer design for it stochastic time-delay systems via sliding mode control. *Systems and Control Letters*, 55(10), 781–793.

Pearson, A. and Fiagbedzi, Y. (1989). An observer for time-delay systems. *IEEE Transactions on Automatic Control*, 34(7), 775–787.

Raoufi, R., Marquez, H.J., and Zinober, A.S.I. (2010). H∞ sliding mode observers for uncertain nonlinear lipschitz systems with fault estimation synthesis. *International Journal of Robust and Nonlinear Control*, 20, 1785–1801.

Richard, J.P. (2003). Time-delay systems: An overview of some recent advances and open problems. *Automatica*, 39, 1667–1694.

Sharma, R. and Aldeen, M. (2011). Fault and disturbance reconstruction in non-linear systems using a network of interconnected sliding mode observers. *IET Control Theory and Applications*, 5(6), 751–763.

Tan, C.P. and Edwards, C. (2001). An LMI approach for designing sliding mode observers. *International Journal of Control*, 74, 1559–1568.

Tan, C.P. and Edwards, C. (2002). Sliding mode observers for detection and reconstruction of sensor faults. *Automatica*, 38, 1815–1821.

Tan, C.P. and Edwards, C. (2003). Sliding mode observers for robust detection and reconstruction of actuator and sensor faults. *International Journal of Robust and Nonlinear Control*, 13(5), 433–463.

Utkin, V.I. (1992). *Sliding mode in Control Optimization*. Berlin: Springer-Verlag.

Veluvolu, K.C., Kim, M., and Lee, D. (2011). Nonlinear sliding mode high-gain observers for fault estimation. *International Journal of Systems Science*, 42(7), 1065–1074.

Wu, Q. and Saif, M. (2010). Robust fault diagnosis of a satellite system using a learning strategy and second order sliding mode observer. *IEEE Systems Journal*, 4(1), 112–121.

Yan, X, G. and Edwards, C. (2008). Robust sliding mode observer-based actuator fault detection and isolation for a class of nonlinear systems. *International Journal of Systems Science*, 39(4), 349–359.

Yan, X.G. and Edwards, C. (2007). Nonlinear robust fault reconstruction and estimation using a sliding mode observer. *Automatica*, 43, 1605–1614.

Zong, Q., Zeng, F., Liu, W., Ji, Y., and Tao, Y. (2012). Sliding mode observer-based fault detection of distributed networked control systems with time delay. *Circuits Systems Signal Process*, 31, 203–222.