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An H_∞ Filtering Problem for Systems With State-Dependent Noise

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Abstract: The paper presents an H_∞-type filtering problem for stochastic systems subject to state-dependent (multiplicative) noise. Necessary and sufficient conditions for the existence of a fixed (reduced) order filter are derived both for continuous-time and for the discrete-time cases. These conditions depend on the solutions of some specific systems of linear matrix inequalities subject to a rank minimization condition. For the considered problem it is shown that the rank constraint can be solved as a semi-definite programming problem using a particular affine transformation.

Keywords: Filtering techniques, Multiplicative noise, H-infinity, Reduced-order models

1. INTRODUCTION

The H_∞ filtering received an increasing attention over the last two decades in the control literature. This interest is motivated by the insensitivity of H_∞ filters to the noise statistic properties. Indeed the H_∞ filter does not make any assumption about the noise, except the one that it has bounded energy, and it minimizes the L_2-gain from the exogenous noise signals to the estimation error. Among the early references treating the H_∞ filtering problem one mentions [4], [7], [10], [12]. More recently, a number of papers addressed the H_∞ filtering problem by linear matrix inequalities (LMI) method (see e.g. [2] and[6]) using Bounded Real Lemma approach in the deterministic case. An important aspect in any filtering problem is the robustness of the filtering performance with respect with the modelling uncertainty, knowing that accurate system models are not as readily available in most applications. Since the filtering performance deteriorates in the presence of modelling uncertainties, many papers have been devoted to robust filtering methods (see e.g. [4], [8] and their references). A possible approach to improve the robustness filtering performance is based on models with multiplicative (state-dependent) noise in which the system parameters are subject to random perturbations. The use of models with multiplicative noise is not related only with robustness issues. There are many applications including aerial and space navigation and control, where such models arise in a natural way (see e.g. [5]). In fact the systems with state-dependent noise are used from long time ago (see e.g. [15]) and actually many results of advanced control of such systems are available ([11]).

The purpose of the present paper is to solve an H_∞-type filtering problem for systems with state-dependent noise. The structure of the H_∞ filter considered is a very general one, only its order being assumed fixed. This order constraint aims at the possibility to get reduced-order filters, when the specific of the application allows this. Reduced-order design is desirable to reduce the computational complexity in real-time filtering process. A similar filtering problem but in the continuous-time case was treated in [13] considering the measured output not corrupted with state-dependent noise and in [14] one can find a Kalman-type filtering problem for discrete-time systems with multiplicative noise. Besides the specific solvability conditions derived for the H_∞ filtering in the continuous-time and in the discrete-time cases, new developments to determine a computationally tractable reduced-order solution are presented in this paper.

The paper is organized as follows: the filtering problem in the continuous-time case is treated in Section 2 which is devised in two subsections. The first of them contains the filtering problem statement together with some useful known results. The solution in the continuous-time framework is given in the second subsection where necessary and sufficient conditions for the existence of a filter with specified order are given. A similar structure has been adopted for Section 3 of the paper which presents the discrete-time version of the filtering problem. In Section 4 one considers a case study illustrating some of the theoretical results. The paper ends with some concluding remarks and future developments.

Throughout the paper the superscript $'T'$ stands for matrix transposition, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, I is the identity matrix of appropriate dimensions and for a real matrix $P, P>0$ means that P is symmetric and positive definite.

2. THE CONTINUOUS-TIME CASE

2.1 Notations, problem formulation and useful results

Consider the following stochastic system with state-dependent noise both in its dynamics and in the measured output:

$$dx(t) = \left(A_0 x(t) + Bv(t)\right)dt + \sum_{j=1}^{r} A_j x(t)\, d\xi_j(t)$$
$$dy(t) = \left(C_0 x(t) + Dv(t)\right)dt + \sum_{j=1}^{r} C_j x(t)\, d\xi_j(t), \tag{1}$$

with $x \in \mathbb{R}^n$ is the state system, $v \in \mathbb{R}^m$ is an exogenous disturbance, $y \in \mathbb{R}^p$ is the measured output and $\xi_i(t)$, $i = 1,...,r$, $t \geq 0$ denote independent standard Wiener processes. It is assumed that $v \in L^{2,m}$ where $L^{2,m}$ stands for the space of measurable functions with bounded energy, that is

$$\|u\|^2 = \int_0^\infty E|u(t)|^2\, dt < \infty,$$

where E denotes the expectation and $|u|^2 = \sum_{i=1}^{m} u_i^2$, u_i, $i = 1,...,m$ being the components of u.

Definition 1. The system (1) is called *exponentially stable in mean square* (ESMS) if there exist $\alpha > 0$ and $\beta \geq 1$ such that $E|\Phi(t,t_0)|^2 \leq \beta e^{-\alpha(t-t_0)}$ for all $t \geq t_0$, where $\Phi(t,t_0)$ is the fundamental (random) matrix solution associated with the system:

$$dx(t) = A_0 x(t) dt + \sum_{j=1}^{r} A_j x(t) d\xi_j(t).$$

Necessary and sufficient conditions for ESMS in the considered continuous-time case are given by the following known result (see e.g. [1], [11]).

Proposition 1. *The system with state-dependent noise* (1) *is ESMS if and only if there exists a symmetric matrix* $X > 0$ *such that*

$$A_0^T X + XA_0 + \sum_{j=1}^{r} A_j^T X A_j < 0.$$

The H_∞ type filtering problem in the considered in this section has the following statement: given the ESMS system (1) and a level of attenuation $\gamma > 0$, determine if possible, a stable filter n_f-order deterministic filter

$$dx_f(t) = A_f x_f(t) + B_f dy(t)$$
$$y_f(t) = C_f x_f(t) \tag{2}$$

such that

$$E\int_0^\infty \left[|z(t)|^2 - \gamma^2 |v(t)|^2\right] dt < 0, \tag{3}$$

for all $v \in L^{2,m}$, where $z(t) := y_f(t) - Hx(t)$, $H \in \mathbb{R}^{s\times n}$ is a given matrix and $x(t)$ represents the solution of (1) with null initial condition.

A key result allowing to solve the above filtering problem is the following version of the Bounded Real Lemma for systems with state dependent noise ([1], [11]).

Theorem 1. *Assume that the system*

$$dx(t) = \left(A_0 x(t) + Bv(t)\right)dt + \sum_{j=1}^{r} A_j x(t)\, d\xi_j(t)$$
$$y(t) = Cx(t)$$

is ESMS. Then the γ*-attenuation condition* $E\int_0^\infty \left[|y(t)|^2 - \gamma^2 |v(t)|^2\right] dt < 0$ holds if and only *if there exists* $X > 0$ *such that*

$$A_0^T X + XA_0 + \sum_{j=1}^{r} A_j^T X A_j + \gamma^{-2} XBB^T X$$
$$+ C^T C < 0. \tag{4}$$

Remark 1. The filter (2) has a deterministic structure. A more complex structure including state-dependent noise terms in (2) may provide better filtering performance but in such situation implementation problems occur since these noises cannot be directly measured.

2.2. The H_∞*-type filter in the continuous-time case*

From (1) and (2) one obtains the resulting system

$$d\tilde{x}(t) = \left(\mathsf{A}_0 \tilde{x}(t) + \mathsf{B} v(t)\right)dt + \sum_{j=1}^{r} \mathsf{A}_j \tilde{x}(t)\, d\xi_j(t)$$
$$z(t) = \mathsf{C}\tilde{x}(t), \tag{5}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_f \end{bmatrix}, \mathsf{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_f C_0 & A_f \end{bmatrix}, \mathsf{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix},$$
$$\mathsf{A}_j = \begin{bmatrix} A_j & 0 \\ B_f C_j & 0 \end{bmatrix}, \mathsf{C} = \begin{bmatrix} -H & C_f \end{bmatrix}.$$

Applying Theorem 1 for the system (5) it follows that the γ attenuation condition (3) is fulfilled if and only if there exists $\mathsf{X} \in \mathbb{R}^{(n+n_f)\times(n+n_f)}$, $\mathsf{X} > \mathbf{0}$ such that

$$\mathsf{A}_0^T \mathsf{X} + \mathsf{X}\mathsf{A}_0 + \sum_{j=1}^{r} \mathsf{A}_j^T \mathsf{X} \mathsf{A}_j + \gamma^{-2} \mathsf{X}\mathsf{B}\mathsf{B}^T \mathsf{X}$$
$$+ \mathsf{C}^T \mathsf{C} < 0$$

which based on Schur complement formula ([1]), can be written in the equivalent form

$$\begin{bmatrix} \mathsf{A}_0^T \mathsf{X} + \mathsf{X}\mathsf{A} + \sum_{j=1}^{r} \mathsf{A}_j^T \mathsf{X} \mathsf{A}_j + \mathsf{C}^T \mathsf{C} & \mathsf{X}\mathsf{B} \\ \mathsf{B}^T \mathsf{X} & -\gamma^2 I \end{bmatrix} < 0. \tag{6}$$

With the partition

$$\mathsf{X}=\begin{bmatrix} R & M \\ M^T & S \end{bmatrix},\ R\in\mathbb{R}^{n\times n}, \tag{7}$$

the above condition becomes:

$$\begin{bmatrix} \mathsf{E}_{11} & \mathsf{E}_{12} & \mathsf{E}_{13} \\ \mathsf{E}_{12}{}^T & \mathsf{E}_{22} & \mathsf{E}_{23} \\ \mathsf{E}_{13}{}^T & \mathsf{E}_{23}{}^T & -\gamma^2 I_m \end{bmatrix}<0, \tag{8}$$

where

$$\begin{aligned}
\mathsf{E}_{11} &:= A_0{}^T R + RA_0 + MB_f C_0 + C_0{}^T B_f{}^T M^T + H^T H \\
&\quad + \sum_{j=1}^{r}\Big(A_j{}^T RA_j + A_j{}^T MB_f C_j + C_j{}^T B_f{}^T M^T A_j \\
&\quad + C_j{}^T B_f{}^T SB_f C_j \Big) \\
\mathsf{E}_{12} &:= MA_f + A_0{}^T M + C_0{}^T B_f{}^T S - H^T C_f \\
\mathsf{E}_{13} &:= RB + MB_f D \\
\mathsf{E}_{22} &:= SA_f + A_f{}^T S + C_f{}^T C_f \\
\mathsf{E}_{23} &:= M^T B + SB_f D.
\end{aligned}$$

In order to remove the nonlinear term $C_f{}^T C_f$ from E_{22} one can replace the condition (8) with the equivalent one:

$$\begin{bmatrix} \mathsf{E}_{11} & \mathsf{E}_{12} & \mathsf{E}_{13} & 0 \\ \mathsf{E}_{12}{}^T & A_f{}^T S + SA_f & \mathsf{E}_{23} & C_f{}^T \\ \mathsf{E}_{13}{}^T & \mathsf{E}_{23}{}^T & -\gamma^2 I_m & 0 \\ 0 & C_f & 0 & -I_s \end{bmatrix}<0. \tag{9}$$

This equivalence immediately follows computing the Schur complement of $-I_s$ in the matrix of the left hand side in the above inequality. Further assume that B_f is full rank. This is not a restrictive assumption since if this assumption is not satisfied then for a small enough perturbation of B_f such that it becomes full rank, condition (9) remains valid. Thus B_f is no more an unknown variable and (9) can be rewritten as

$$Z+\mathsf{P}^T\Omega\mathsf{Q}+\mathsf{Q}^T\Omega^T\mathsf{P}<0, \tag{10}$$

where:

$$Z=\begin{bmatrix} \mathsf{E}_{11} & (2,1)^T & (3,1)^T & 0 \\ M^T A_0 + SB_f C_0 & 0 & (3,2)^T & 0 \\ B^T R + D^T B_f{}^T S & B^T M + D^T B_f{}^T S & -\gamma^2 I_m & 0 \\ 0 & 0 & 0 & -I_s \end{bmatrix} \tag{11}$$

$$\mathsf{P}=\begin{bmatrix} M^T & S & 0 & 0 \\ -H & 0 & 0 & I_s \end{bmatrix},\ \Omega=\begin{bmatrix} A_f \\ C_f \end{bmatrix},\ \mathsf{Q}=\begin{bmatrix} 0 & I_{n_f} & 0 & 0 \end{bmatrix}.$$

According with the Projection Lemma (e.g. [1]), the inequality (10) is feasible with respect to Ω if and only if

$$W_{\mathsf{P}}{}^T Z W_{\mathsf{P}}<0 \tag{12}$$

and

$$W_{\mathsf{Q}}{}^T Z W_{\mathsf{Q}}<0, \tag{13}$$

where $W_{\mathsf{P}}, W_{\mathsf{Q}}$ are bases of the null subspaces of P and Q, respectively. From the above expression of P it results that a basis of its null subspace is:

$$W_{\mathsf{P}}=\begin{bmatrix} \hat{R} & 0 \\ N^T & 0 \\ 0 & I \\ H\hat{R} & 0 \end{bmatrix},$$

where $\hat{R}, N$ are given by the partition of

$$\mathsf{X}^{-1}=\begin{bmatrix} \hat{R} & N \\ N^T & \hat{S} \end{bmatrix},\ \hat{R}\in\mathbb{R}^{n\times n}. \tag{14}$$

Then, with the above expression of W_{P} and with the conditions $NM^T = I-\hat{R}R$ and $NS=-\hat{R}M$ obtained from the equality $\mathsf{X}^{-1}\mathsf{X}=I$, direct computations give:

$$\begin{aligned}
&\begin{bmatrix} \hat{R}^{-1} & 0 \\ 0 & I \end{bmatrix} W_{\mathsf{P}}{}^T Z W_{\mathsf{P}} \begin{bmatrix} \hat{R}^{-1} & 0 \\ 0 & I \end{bmatrix} \\
&=\begin{bmatrix} A_0{}^T\hat{R}^{-1}+\hat{R}^{-1}A_0+\mathsf{L}(R,M,S) & \hat{R}^{-1}B \\ B^T\hat{R}^{-1} & -\gamma^2 I_m \end{bmatrix}<0
\end{aligned} \tag{15}$$

where the following notation has was introduced:

$$\begin{aligned}
\mathsf{L}(R,M,S) &:= \sum_{j=1}^{r}\Big(A_j{}^T RA_j + A_j{}^T MB_f C_j \\
&\quad + C_j{}^T B_f{}^T M^T A_j + C_j{}^T B_f{}^T SB_f C_j \Big).
\end{aligned} \tag{16}$$

Notice that (15) and (12) are equivalent. Further, a basis of the null subspace of Q is:

$$W_{\mathsf{Q}}=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

for which one obtains that (13) is equivalent with:

$$\begin{bmatrix} A_0{}^T R + RA_0 + MB_f C_0 + C_0{}^T B_f{}^T M^T \\ +H^T H + \mathsf{L}(R,M,S) & (2,1)^T \\ B^T R + D^T B_f{}^T M^T & -\gamma^2 I_m \end{bmatrix}<0. \tag{17}$$

The above developments are summarized by the following result:

Theorem 2. *The filtering problem formulated above for the system* (1) *has an* n_f *-order solution of form* (2) *if and only if there exist the matrices* $P,R \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n_f \times n_f}$, $P>0, R>0,$ $S>0$ *and* $M \in \mathbb{R}^{n\times n_f}$ *satisfying the condition (17) and:*

$$\begin{bmatrix} A_0^T P + PA_0 + \mathsf{L}(R,M,S) & PB \\ B^T P & -\gamma^2 I_m \end{bmatrix} < 0 \tag{18}$$

$$\begin{bmatrix} R & M \\ M^T & S \end{bmatrix} > 0 \tag{19}$$

$$\operatorname{rank}\begin{bmatrix} P-R & M \\ M^T & -S \end{bmatrix} \le n_f, \tag{20}$$

where $\mathsf{L}(R,M,S)$ *is defined in (16).*

Proof. The inequality directly follows from (18) where $P := \hat{R}^{-1}$. The rank condition (20) follows from the relationship between X and X^{-1} which gives $P = \hat{R}^{-1} = R - MS^{-1}M^T$ and (19) shows that $\mathsf{X} > 0$.

Remark 2. *If the above feasibility conditions are fulfilled then the matrices* A_f, C_f *of the filter are obtained solving* (10) *with respect to* Ω.

3. THE DISCRETE-TIME CASE

3.1 Notations, problem formulation and useful results

Consider the discrete-time system with state-dependent noise

$$\begin{aligned} x(k+1) &= A_0 x(k) + \sum_{j=1}^{r} A_j x(k)\xi(k) + Bv(k), \\ y(k) &= C_0 x(k) + \sum_{j=1}^{r} C_j x(k)\xi_j(k) + Dv(k), \\ &k = 0,1,... \end{aligned} \tag{21}$$

where $x(k) \in \mathbb{R}^n$ is the state vector at moment k, $y(k) \in \mathbb{R}^p$ represents the measured output and $\xi_j(k) \in \mathbb{R},\ j=1,...,r,\ k=0,1,..$ are independent random variables with zero mean and unit covariance. It is assumed that the exogenous signals are energy bounded, namely $v(k) \in \ell^{2,m}$ where $\ell^{2,m}$ denotes the Lebesgue space of bounded norm square summable $\mathbb{R}^m$-valued functions.

Definition 2. The discrete-time system (21) is called *exponentially stable in mean square* (ESMS) if there exists $\beta > 0$ and $\alpha \in (0,1)$ such that $E\left[|x(k)|^2\right] \le \beta\alpha^k |x_0|^2$ for all $k>0$ and for any initial condition $x_0 \in \mathbb{R}^n$ at $k=0$, where x_k is the solution of (21) with $v(k) \equiv 0$.

The stability in mean square of a system with state-dependent noise can analyzed using the following result (e.g. [1]).

Proposition 2. The system (21) is ESMS if and only if there exists a symmetric matrix $X>0$ verifying the following Lyapunov inequality

$$X > A_0^T X A_0 + A_1^T X A_1. \quad \square$$

The problem considered in this section has the following statement: given the ESMS system (21) determine if possible, a stable system of form

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f y(k),\ k=0,1,... \\ y_f(k) &= C_f x(k) \end{aligned} \tag{22}$$

with the specified order $n_f \ge 1$, such that

$$J_\gamma := \sum_{k=0}^{\infty} E\left[|z(k)|^2 - \gamma^2 |v(k)|^2\right] < 0 \tag{23}$$

for all $v \in \ell^{2,m}$, where

$$z(k) := y_f(k) - Hx(k) \tag{24}$$

denotes a quality output, $H \in \mathbb{R}^{s\times n}$, $\gamma > 0$ is a given level of attenuation and $x(k)$ is the solution of (21) with zero initial condition.

A useful auxiliary result for the following developments is the next Bounded Real Lemma version for systems with state-dependent noise ([3]).

Theorem 3. Assume that the system with multiplicative noise

$$\begin{aligned} x(k+1) &= A_0 x(k) + \sum_{j=1}^{r} A_j x_j \xi_j(k) + Bv(k), \\ y(k) &= Cx(k), \quad k=0,1,... \end{aligned}$$

is ESMS. Then $\sum_{k=0}^{\infty} E\left[|y(k)|^2 - \gamma^2 |v(k)|^2\right] < 0$ *for a given* $\gamma > 0$, *where* $x(k)$ *is the solution of the above system with zero initial condition, if and only if there exists a symmetric matrix* $X>0$ *such that*

$$\begin{aligned} &-X + A_0^T X A_0 + \sum_{j=1}^{r} A_j^T X A_j \\ &+ A_0^T XB\left(\gamma^2 I - B^T XB\right)^{-1} B^T X A_0 + C^T C < 0. \end{aligned} \tag{25}$$

3.2. The H_∞-type discrete-time filter

From (21), (22), and (24) one obtains the resulting system

$$\begin{aligned}&\tilde{x}(k+1)=\mathsf{A}_0\tilde{x}(k)+\sum_{j=1}^{r}\mathsf{A}_j\tilde{x}(k)\xi_j(k)+\mathsf{B}v(k),\\&z(k)=\mathsf{C}\tilde{x}(k),\quad k=0,1,...\end{aligned} \tag{26}$$

where $\tilde{x}:=\begin{bmatrix}x^T & {x_f}^T\end{bmatrix}^T$ and

$$\begin{aligned}&\mathsf{A}_0=\begin{bmatrix}A_0 & 0\\ B_fC_0 & A_f\end{bmatrix},\ \mathsf{A}_j=\begin{bmatrix}A_j & 0\\ B_fC_j & 0\end{bmatrix},\\&\mathsf{B}=\begin{bmatrix}B\\ B_fD\end{bmatrix},\ \mathsf{C}=\begin{bmatrix}-H & C_f\end{bmatrix}.\end{aligned} \tag{27}$$

Applying Theorem 3 for the system (26) it follows that the H_∞ filtering problem is feasible if and only if there exists a symmetric matrix $\mathsf{X}\in\mathbb{R}^{(n+n_f)\times(n+n_f)}$, $\mathsf{X}>0$ such that

$$\begin{aligned}&-\mathsf{X}+\mathsf{A}_0{}^T\mathsf{X}\mathsf{A}_0+\sum_{j=1}^{r}\mathsf{A}_j{}^T\mathsf{X}\mathsf{A}_j\\&\quad+\mathsf{A}_0{}^T\mathsf{X}\mathsf{B}\left(\gamma^2I-\mathsf{B}^T\mathsf{X}\mathsf{B}\right)^{-1}\mathsf{B}^T\mathsf{X}\mathsf{A}_0+\mathsf{C}^T\mathsf{C}<0.\end{aligned} \tag{28}$$

Based on the Schur complement formula, the above condition is equivalent with

$$\begin{bmatrix}-\mathsf{X}+\mathsf{C}^T\mathsf{C} & \mathsf{A}_0{}^T & \cdots & \mathsf{A}_r{}^T & \mathsf{A}_0{}^T\mathsf{X}\mathsf{B}\\ \mathsf{A}_0 & -\mathsf{X}^{-1} & \cdots & 0 & 0\\ \mathsf{A}_1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathsf{A}_r & 0 & \cdots & -\mathsf{X}^{-1} & 0\\ \mathsf{B}^T\mathsf{X}\mathsf{A}_0 & 0 & \cdots & 0 & -\left(\gamma^2I-\mathsf{B}^T\mathsf{X}\mathsf{B}\right)\end{bmatrix}<0 \tag{29}$$

In the above inequality the unknown variables are X, A_f, B_f and C_f. As in the previous section without reducing the generality of the problem, one can chose B_f as an arbitrary full rank matrix. Denoting

$$\Omega:=\begin{bmatrix}A_f\\ C_f\end{bmatrix},\quad \mathsf{X}:=\begin{bmatrix}R & M\\ M^T & S\end{bmatrix}, \tag{30}$$

direct algebraic computations together with the Schur complement formula show that condition (29) can be written as

$$Z+P^T\Omega^TQ+Q^T\Omega P<0 \tag{31}$$

where the following notations have been used

$$Z:=$$
$$\begin{bmatrix}Z_{11} & -M & Z_{13} & Z_{14} & \cdots & Z_{1(4+r)} & Z_{1(5+r)} & Z_{1(6+r)} & 0\\ * & -S & Z_{23} & Z_{24} & \cdots & 0 & 0 & Z_{2(6+r)} & {C_f}^T\\ * & * & -R & -M & \cdots & 0 & 0 & 0 & 0\\ * & 0 & * & -S & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ * & 0 & 0 & 0 & \cdots & -R & -M & 0 & 0\\ * & 0 & 0 & 0 & \cdots & * & -S & 0 & 0\\ * & * & 0 & 0 & \cdots & 0 & 0 & Z_{(6+r)(6+r)} & 0\\ 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & -I\end{bmatrix}$$

where ‘*’ denotes the corresponding elements such that Z is symmetric and

$$\begin{aligned}&Z_{11}:=-R+H^TH\\&Z_{13}:={A_0}^TR+{C_0}^T{B_f}^TM^T\\&Z_{14}:={A_0}^TM+{C_0}^T{B_f}^TS\\&Z_{1(4+j)}:={A_j}^TR+{C_j}^T{B_f}^TM^T\\&Z_{1(5+j)}:={A_j}^TM+{C_j}^T{B_f}^TS,\ j=1,...,r\\&Z_{1(6+r)}:=\left({A_0}^TR+{C_0}^T{B_f}^TM^T\right)B\\&\qquad\qquad+\left({A_0}^TM+{C_0}^T{B_f}^TS\right)B_fD\\&Z_{2(6+r)}:={A_f}^T\left(M^TB+SB_fD\right)\\&Z_{(6+r)(6+r)}:=-\left(\gamma^2I-B^TRB-D^T{B_f}^TM^TB\right.\\&\qquad\qquad\left.-B^TMB_fD-D^T{B_f}^TSB_fD\right)\end{aligned}$$

and where

$$\mathsf{P}=$$
$$\begin{bmatrix}0 & 0 & M^T & S & 0 & 0 & \cdots & M^TB+SB_fD & 0\\ -H & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I\end{bmatrix} \tag{32}$$

and

$$\mathsf{Q}=\begin{bmatrix}0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0\end{bmatrix}. \tag{33}$$

According with the Projection Lemma, there exists Ω satisfying (31) if and only if the following two conditions are accomplished

$${W_\mathsf{P}}^TZW_\mathsf{P}<0 \tag{34}$$

and

$${W_\mathsf{Q}}^TZW_\mathsf{Q}<0, \tag{35}$$

where W_P and W_Q denote bases of the null-spaces of P and Q, respectively. Further, conditions (34) and (35) will be explicit. Firstly, taking into account (32) it results that a basis of the null-space of P is

$$W_{\mathsf{P}}=\begin{bmatrix} 0 & 0 & 0 & 0\cdots & I & 0 \\ 0 & I & 0 & 0\cdots & 0 & 0 \\ -B & 0 & 0 & 0\cdots & 0 & I \\ -B_fD & 0 & 0 & 0\cdots & 0 & -S^{-1}M^T \\ 0 & 0 & I & 0\cdots & 0 & 0 \\ 0 & 0 & 0 & I\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & 0 & 0\cdots & H & 0 \end{bmatrix}.$$

Then direct algebraic computations together with Schur complement arguments give that the inequality (34) is equivalent with

$$\begin{bmatrix} -\hat{R}+\gamma^{-2}BB^T & 0 & 0 & \cdots & 0 & 0 & A_0\hat{R} \\ 0 & -\hat{R} & -N & \cdots & 0 & 0 & A_1\hat{R} \\ 0 & -N^T & -\hat{S} & \cdots & 0 & 0 & B_fC_1\hat{R} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\hat{R} & -N & A_r\hat{R} \\ 0 & 0 & 0 & \cdots & -N^T & -\hat{S} & B_fC_r\hat{R} \\ * & * & * & \cdots & * & * & -\hat{R} \end{bmatrix}<0 \qquad (36)$$

where $\hat{R}\in\mathbb{R}^{n\times n}$, $N\in\mathbb{R}^{n\times n_f}$ and $\hat{S}\in\mathbb{R}^{n_f\times n_f}$ are the block elements of X^{-1}, namely

$$\mathsf{X}^{-1}:=\begin{bmatrix} \hat{R} & N \\ N^T & \hat{S} \end{bmatrix}. \qquad (37)$$

Further, since

$$W_{\mathsf{Q}}=\begin{bmatrix} I & 0 & 0 & 0\cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 \\ 0 & I & 0 & 0\cdots & 0 & 0 & 0 \\ 0 & 0 & I & 0\cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & I\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots & I & 0 & 0 \\ 0 & 0 & 0 & 0\cdots & 0 & I & 0 \\ 0 & 0 & 0 & 0\cdots & 0 & 0 & I \end{bmatrix},$$

one obtains that condition (35) is equivalent with

$$\begin{bmatrix} -R+H^TH & 0 & A_0{}^T & C_0{}^TB_f{}^T & \cdots A_r{}^T & C_r{}^TB_f{}^T \\ 0 & -\gamma^2I & B^T & D^TB_f{}^T & \cdots 0 & 0 \\ A_0 & B & -\hat{R} & -N & \cdots 0 & 0 \\ B_fC_0 & B_fD & -N^T & -\hat{S} & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_r & 0 & 0 & 0 & -\hat{R} & -N \\ B_fC_r & 0 & 0 & 0 & -N^T & -\hat{S} \end{bmatrix}<0. \qquad (38)$$

The above developments are concluded in the following result.

Theorem 4. *The* H_∞ *-type filtering problem in the discrete-time case has an* n_f *-order solution if and only if there exists a symmetric matrix* $\mathsf{X}>0,\mathsf{X}\in\mathbb{R}^{(n+n_f)\times(n+n_f)}$ *such that its block element* R *from partition* (30) *and the block elements* $\hat{R},N$ *and* $\hat{S}$ *of* X^{-1} *defined by* (37), *verify the inequalities* (36) *and* (38).

Remark 3. As in the standard design methodology based on linear matrix inequalities, one firstly must solve the system (36), (38) with respect with $R,\hat{R},N$ and $\hat{S}$, and then determine Ω from the LMI (31) The system (36), (38) cannot be solved using the usual semi-definite programming based methods since R and $\hat{R},N,\hat{S}$ are related from the condition that X^{-1} is the inverse of X .

A simple method to avoid the computational problem mentioned in Remark 3 is to take into account that $R\geq\hat{R}^{-1}$. Indeed this follows using the Schur complement formula in the obvious inequality

$$\begin{bmatrix} \mathsf{X} & I \\ I & \mathsf{X}^{-1} \end{bmatrix}\geq 0.$$

Then one can state the following corollary.

Corollary 1. *If the system obtained from* (36), (38) *replacing* R *from the block* (1,1) *of* (38) *by* $\hat{R}^{-1}$ *is feasible, then the filtering problem has an* n_f *-order solution.*

The inequality (38) in which R is replaced by $\hat{R}^{-1}$ can be pre- and post-multiplied by $diag\left(\hat{R},I,I,I,\ldots,I,I\right)$ and using again a Schur complement argument, one obtains

$$\begin{bmatrix} -\hat{R} & 0 & \hat{R}A_0{}^T & \hat{R}C_0{}^TB_f{}^T\cdots & \hat{R}A_r{}^T & \hat{R}C_r{}^TB_f{}^T & \hat{R}H^T \\ 0 & -\gamma^2I & B^T & D^TB_f{}^T\cdots & 0 & 0 & 0 \\ A_0\hat{R} & B & -\hat{R} & -N\cdots & 0 & 0 & 0 \\ B_fC_0\hat{R} & B_fD & -N^T & -\hat{S}\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_r\hat{R} & 0 & 0 & 0 & -\hat{R} & -N & 0 \\ B_fC_r\hat{R} & 0 & 0 & 0 & -N^T & -\hat{S} & 0 \\ H\hat{R} & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}<0. \quad (39)$$

The system (36), (39) can be then solved with respect with $\hat{R},N$ and $\hat{S}$ such that $\mathsf{X}^{-1}>0$ using semi-definite programming based algorithms.

Remark 4. Corollary 1 gives sufficient feasibility conditions for the considered H_∞ filtering problem and therefore the results obtained solving the system (36),(39) instead of (36), (38) can be conservative.

A possibility to avoid the conservativeness mentioned above is to use the following result proved in [9].

Theorem 5. *If* $Q \le 0$ *and* C *is invertible, then the rank minimization problem* $\min \operatorname{rank} X$ *with respect with* $X \ge 0$ *satisfying the constraint*

$$Q + CXC^T - \sum_i M_i X M_i^T \ge 0 \qquad (40)$$

can be solved as a semi-definite programming problem replacing the rank minimization problem with the trace minimization of X *subject to the same constraints.* □

This result can be used defining the matrix

$$X = \begin{bmatrix} \hat{R} & N & I \\ N^T & \hat{S} & 0 \\ I & 0 & R \end{bmatrix}$$

and imposing the condition $rank\, X \le n + n_f$ which guarantees that $R^{-1} = \hat{R} - N\hat{S}^{-1}N^T$, namely $R, \hat{R}, N$ and $\hat{S}$ are related by the fact that (37), is the inverse of X defined in (30). Then the conditions (36), (38) are expressed in the form (40) which allows according to Theorem 5, to solve a trace minimization problem using semi-definite programming based numerical algorithms.

4. A CASE STUDY

In order to illustrate some of the above developments consider a navigation problem consisting in determining an estimation of an airplane altitude using measurements from a barometric altimeter and from a RADAR altimeter. Due to some inherent sources of error, the barometric altimeter indication is altered by a bias error and by a small additive white noise ([5]). Its continuous-time dynamics may be approximated by the following state space equations

$$\dot{h} = -\frac{1}{\tau}(h - v)$$
$$h_{baro} = h + b + \eta\,, \qquad (41)$$

where v represents the commanded altitude, h is the true altitude, b denotes the bias and η is a standard zero-mean white noise with known intensity R_1. On the other hand, the RADAR altimeter determines the altitude without bias but with a measurement noise which intensity increases with the altitude as follows

$$h_{radar} = h(1 + \xi) + \nu \qquad (42)$$

with $E[\nu(t)\nu(\tau)] = R_2\delta(t - \tau)$ and $E[\xi(t)\xi(\tau)] = R_3\delta(t - \tau)$. If the bias b in the second equation (41) is approximated by $\dot{b} = \bar{w}$ where $\bar{w}$ is a standard white noise independent from ξ, ν and η, the following model is obtained:

$$\begin{bmatrix} \dot{h} \\ \dot{b} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h \\ b \end{bmatrix} + \begin{bmatrix} -\frac{1}{\tau} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \bar{w} \\ \eta \\ \nu \end{bmatrix}$$

$$y = \begin{bmatrix} h_{baro} \\ h_{radar} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h \\ b \end{bmatrix} + \begin{bmatrix} 0 & 0 & \sqrt{R_1} & 0 \\ 0 & 0 & 0 & \sqrt{R_2} \end{bmatrix} \begin{bmatrix} v \\ \bar{w} \\ \eta \\ \nu \end{bmatrix} \qquad (43)$$

$$+ \begin{bmatrix} 0 & 0 \\ \sqrt{R_3} & 0 \end{bmatrix} \begin{bmatrix} h \\ b \end{bmatrix} \xi.$$

The estimated state is the true altitude h. Sampling the above continuous-time system above one obtains a discrete-time system of form (21). Since the above system is not stable, a small negative perturbation of the zero (2,2) element in the state matrix has been introduced.

For a sampling period $T = 0.25\sec$ and for $\tau = 30\sec$, $R_1 = R_2 = 0.4$ and $R_3 = 0.0016$ one obtains using Corollary 1 with the attenuation level $\gamma = 1.07$, an H_∞ filter which response is depicted in Fig. 1.

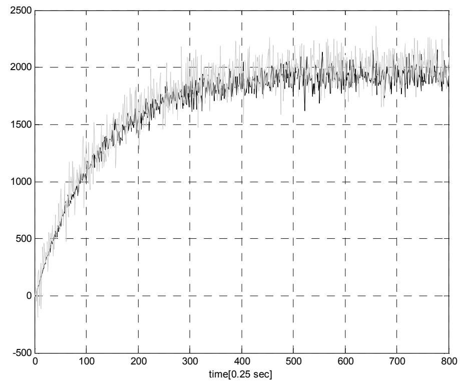

Fig. 1. Filtered signal (black) and unfiltered (grey).

5. CONCLUSIONS

An H_∞ filtering problem for systems with multiplicative (state-dependent) noise has been considered. It was shown that necessary and sufficient solvability conditions with a filter of specified (reduced) order can be expressed as a system of linear matrix inequalities together with a rank constraint both in the continuous-time case and in the discrete-time version. The numerical difficulty of determined by this rank constraint can be overcame either using a result given in Section 3 which provide sufficient feasibility conditions, or to solve a trace minimization problem via semi-definite programming, based on the results proved in ([9]). This later approach has the advantage that it is not conservative and detailed computations will be given in the forthcoming papers.

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