Stability Analysis of Fractional Order Multiple Linear Uncertain Systems: Exposed Edge Sampling Approach

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Abstract: Multiple linear uncertainty models mainly appear at linear time invariant system modelling of cascaded process and this uncertainty structure complicates parametric robust stability analyses of real control systems. To address this problem, this study presents a robust stability analysis scheme for fractional order control system models including multiple linear uncertainty structures. A numerical method, which is based Edge Theorem, has been utilized for the robust stability analysis of fractional order multiple linear uncertain control systems. For this purpose, exposed edge scan sampling approach is used to obtain a image of root region boundaries of fractional order multiple linear uncertain control systems in complex s-domain, and an extension of exposed edge scan sampling is proposed for root locus analyses of fractional order closed loop control systems involving multiple linear uncertainty structures. In the paper, types of linear uncertainties are discussed briefly and multiple versions of these uncertainties are studied for fractional order closed loop control in detail. An application of proposed method for closed loop control systems is numerically demonstrated via illustrative examples. It is observed in this study that exposed edge sampling approach can be an effective method for robust stability analysis of complicated uncertainty structures and it presents potential of decreasing computational complexity in robust stabilization problems of uncertain fractional order control systems.

Keywords: Fractional-order systems, multiple linear uncertainties, robust stability, roots region, edge sampling.

1. INTRODUCTION

Nowadays, fractional order calculus (FOC) drew great interest since its superior consistency to represent real world processes comparing to classical mathematics. (Podlubny, 1999a; Podlubny, 1999b; Petras, 2009). Increasing number of studies related to FOC have been come up since the first idea about the topic in late 17th century. Since then, particularly in the last decades, FOC has found applications in numerous fields such as physics (Parada et al., 2007; Hilfer, 2000), mathematics (Miller and Ross, 1993; Gutierraz et al., 2010), chemistry (Oldham and Spanier, 1974), electrics (Arena et al., 2000), bioengineering (Magin, 2012), signal processing (Vinagre et al., 2003) etc. Last decades also brought so many applications of FOC on control theory (Caponetto et al., 2010; Petras et al., 2002; Chen et al., 2009; Caponetto and Dongola, 2013).

One of the challenging issues in control practice is the uncertainties in control systems. Uncertainties in parameters or models mostly yield to unpredictable results in that, responses of simulations and real processes may differ from each other casually (Senol et al., 2014). In order to prevent the system from this undesired behavior, one has to assure the system's response robustly within possible ranges of system parameters. Robust control of classical systems has been widely studied in the literature (Bhattacharyya et al., 1995; Matusu and Prokop, 2011; Bhattacharyya et al, 2010) as well as systems of fractional orders (Tan et al., 2009; Yeroglu et al., 2010; Senol and Yeroglu, 2013; Senol and Yeroglu, 2012a; Senol and Yeroglu, 2012b). Various types of uncertainties have been taken into account in many works. A study related to the stability of fractional order polynomials (FOPs) with different types of uncertainties using the value set concept was presented in (Senol and Yeroglu, 2012b). A simplification method related to frequency response computation with uncertain parameters was presented in (Senol and Yeroglu, 2012a; Senol and Yeroglu, 2012b; Shaw and Jayasuriya, 1996; Tan and Atherton, 2000) and another frequency response study of uncertain systems with interval plants was given in (Yeroglu et al., 2010). Robust stability analysis of linear fractional order systems (FOS) with interval uncertainties can be found in (Senol et al., 2014). Magnitude and phase envelopes of control systems with affine linear uncertainty were shown in (Tan and Atherton, 1998). There are also some studies on robust stability and stabilization of uncertain FOS (Lu and Chen, 2010a; Lu and Chen, 2010b; Lu and Chen, 2009). Convex polytopic uncertainties were dealt with in (Lu and Chen, 2010a). FOS of the order $0 < \alpha < 1$ with interval uncertainties was studied in (Lu and Chen, 2010b) and a Linear Matrix Inequalities (LMI) approach on robust stability and stabilization of fractional order interval systems was presented in (Lu and Chen, 2009). A frequency response study on the stability of FOS with nonlinear uncertainties can also be found (Senol and Yeroglu, 2013). Stability analysis of FOS with interval coefficients and interval orders was given in (Senol et al. 2014).

This paper deals with the roots region based stability analysis of FOS with multiple linear uncertainty structures which is composed of the multiplication of two or more FOS with linear uncertainties. There can be found some papers in the literature which studied different types of uncertainties (Tan, 2002; Hamelin and Boukhobza, 2005; Tan and Atherton, 2002; Yeroglu and Senol, 2013). Previously, robust stability of multilinear affine polynomials was addressed in (Tan and Atherton, 2002) and computation of the frequency response of multilinear affine systems was investigated in (Tan, 2002; Yeroglu and Senol, 2013). A fault detection approach for multilinear affine systems was presented in (Hamelin and Boukhobza, 2005). A simplified robust stability analysis for FOS with multilinear affine uncertainties can be found in (Yeroglu and Senol, 2013).

Stability analysis in this paper is based on the root locus analysis of the characteristic equation of the multiple linear uncertain FOS. Fractional orders of the characteristic equation are extended to minimal integer orders with a Least Common Multiplier (LCM). Then, roots of this new polynomial are computed and their absolute phase values are found. Stability is decided considering the locations of these roots. There can be found some studies based on the related method in the literature (Caponetto et al., 2010; Senol et al., 2014; Radwan et al., 2009). Roots region based stability analysis of FOS with interval coefficients and interval orders is presented in (Senol et al., 2014).

Computational complexity of a multiple linear system exponentially grows as the number of uncertain parameters increase; therefore, this is one of the most challenging problems of control systems with uncertainties. As a FOS with multiple linear uncertainties is built up of two or more uncertain FOS, it is possible to find great number of roots form the roots region of such systems. This study utilizes the Edge theorem to decrease this complexity. Using the Edge Theorem, only the roots which are on the exposed edges of the roots region are computed, thus, the computational complexity is reduced substantially. There can be found wide application areas of the Edge Theorem in the literature (Senol et al., 2014; Bhattacharyya et al., 1995; Matusu and Prokop, 2011; Bhattacharyya et al., 2010; Senol and Yeroglu, 2012a; Senol and Yeroglu, 2012b). Usage of the Edge Theorem in roots region computation (Senol et al., 2014), value set computation (Bhattacharyya et al., 1995), Kharitonov's Theorem (Matusu and Prokop, 2011), etc. are some of the instances.

Multiple linear uncertainty structures are one of the complicated parametric uncertainty structures, which mainly appear at model of cascaded process or system blocks. This type of uncertainty structures frequently emerge in modelling of uncertain Linear Time Invariant (LTI) system models because coefficients of such models can be formed by linear combinations of interval uncertain model parameters of real systems. Cascading of such models complicates uncertainty structures of models and other methods such as Kharitonov's Theorem may not be effective to have a definite result about the robust stability of such complex structures. In the current study, we demonstrated that application of Edge Theorem via exposed edge polinomial sampling method allows more accurate results for parametric robust stability analyses of such complex system models. Illustrative examples were given to validate the proposed method for analysis of fractional order closed loop control systems.

Paper organized as follow: Section 2 summarizes fractional order uncertain systems, linear uncertainty types, multiple linear uncertainty structure and roots region based stability analysis for fractional order interval polynomials. Stability investigation of fractional order multiple linear uncertain polynomials is studied in Section 3 and roots region of the characteristic equation of fractional order multiple linear uncertain systems is given in Section 4. Section 5 has illustrative examples and Section 6 has the concluding remarks.

2. PROBLEM STATEMENT AND PRELIMINARIES

2.1. Fractional-order uncertain systems with multiple linear uncertainty structures

This section gives brief information about fractional-order uncertain systems and multiple linear uncertainty structure. Consider the fractional-order polytopic polynomial family of the following form (Yeroglu and Senol, 2013).

$$P(s,\mathbf{q}) = I_0(\mathbf{q})s^{\alpha_0} + I_1(\mathbf{q})s^{\alpha_1} + \cdots + I_n(\mathbf{q})s^{\alpha_n} \tag{1}$$

where, the coefficients $I_i(\mathbf{q})$ linearly depend on $\mathbf{q} = [q_0, q_1, \cdots, q_q]^T$ and the uncertainty box is

$$Q = \{\mathbf{q} : q_i \in [\underline{q_i} \;\; \overline{q_i}],\quad i = 1,2,\cdots,q\}. \tag{2}$$

$\underline{q_i}$ and $\overline{q_i}$ specify the lower and upper bounds of the i-th perturbation q_i respectively. $\alpha_0 < \alpha_1 < ... < \alpha_n$ are non-integer orders of the polynomial. Let $I_k(\mathbf{q})$ has linear uncertainty of general form. Then, uncertain parameters of Eq. 1 can be written as,

$$I_k(\mathbf{q}) = a_{k,0} + a_{k,1}q_1 + a_{k,2}q_2 + ... + a_{k,q}q_q \tag{3}$$

where, $a_{k,i}$ are constants and q_i are uncertain parameters ($k = 0,1,2,...,n$ and $i = 1,2,\cdots,q$). Referring to Eq. 3, different types of $I_k(\mathbf{q})$ form different types of linear uncertainties. Three types of linear uncertainties have been represented in this paper. For instance, following form of $P(s,\mathbf{q})$ builds the single parameter uncertainty.

$$P(s,\mathbf{q}) = a_0 + (a_1 + b_1 q)s^{\alpha_1} + (a_2 + b_2 q)s^{\alpha_2} + ... + (a_n + b_n q)s^{\alpha_n} \tag{4}$$

where, a_i, b_i, $i = 1,2,...,n$ are constants and q is the uncertain parameter. Similarly, interval uncertainty can be defined as follows.

$$P(s,\mathbf{q}) = q_1 s^{\alpha_1} + q_2 s^{\alpha_2} + ... + q_n s^{\alpha_n} \tag{5}$$

where, q_i, $i = 1,2,...,n$ are uncertain parameters that lay in the uncertainty interval $\left[\underline{q_i}, \overline{q_i}\right]$. $\underline{q_i}$ and $\overline{q_i}$ are the lower and the upper limits of the uncertain parameters respectively.

Similarly, following form of $P(s,\mathbf{q})$ can be named as affine linear uncertainty structure.

$$P(s,\mathbf{q}) = (a_{1,0} + a_{1,1}q_1 + a_{1,2}q_2 + \ldots + a_{1,k}q_k)s^{\alpha_1} + (a_{2,0} + a_{2,1}q_1 + a_{2,2}q_2 + \ldots + a_{2,k}q_k)s^{\alpha_2} + \ldots + (a_{n,0} + a_{n,1}q_1 + a_{n,2}q_2 + \ldots + a_{n,k}q_k)s^{\alpha_n} \tag{6}$$

where, $a_{i,j}$ are constants, α_i are arbitrary real orders and q_j are uncertain parameters, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,k$. As a fractional order uncertain plant can have fractional order uncertain polynomials in its numerator and denominator, one can say that a fractional order plant with any linear uncertainty structure can be written as follows.

$$G(s,\mathbf{a}_i,\mathbf{b}_k) = \frac{N_k(s,\mathbf{b}_k)}{D_i(s,\mathbf{a}_i)} = \frac{N_0(\mathbf{b})s^{\alpha_{N0}} + N_1(\mathbf{b})s^{\alpha_{N1}} + \cdots + N_m(\mathbf{b})s^{\alpha_{Nm}}}{D_0(\mathbf{a})s^{\alpha_{D0}} + D_1(\mathbf{a})s^{\alpha_{D1}} + \cdots + D_n(\mathbf{a})s^{\alpha_{Dn}}} \tag{7}$$

where, $N_k(s,\mathbf{b}_k)$ and $D_i(s,\mathbf{a}_i)$ are fractional order uncertain polynomials of the numerator and the denominator respectively ($i = 0,1,2,\cdots,n$ and $k = 0,1,2,\cdots,m$).

Now, consider a fractional order polytopic family of polynomials with the multiple linear uncertainty structure, which is multiplication of two or more linear uncertain family of polynomials as follows.

$$F(s,\mathbf{h}) = A(s)P_1(s,\mathbf{h}_1)P_2(s,\mathbf{h}_2)\cdots P_n(s,\mathbf{h}_n) \tag{8}$$

where, $\mathbf{h}_i = [q_{i1}, q_{i2}, \cdots, q_{ik}]$, $A(s)$ is a fixed polynomial and $P_i(s,\mathbf{h}_i)$, $i = 1,2,\cdots,n$ are polynomials with any linear uncertainty structure. Thus, considering Eq. 7 and Eq. 8, a fractional order system with multiple linear uncertainty structure can be given in the following form.

$$L(s,\mathbf{c}) = \frac{L_N(s,\mathbf{b})}{L_D(s,\mathbf{a})} = \frac{C_N(s)}{C_D(s)} \cdot \frac{P_{N1}(s,\mathbf{b_1})P_{N2}(s,\mathbf{b}_2)\ldots P_{Nm}(s,\mathbf{b_m})}{P_{D1}(s,\mathbf{a_1})P_{D2}(s,\mathbf{a_2})\ldots P_{Dn}(s,\mathbf{a_n})} \tag{9}$$

where, $C_N(s)$ and $C_D(s)$ are numerator and denominator polynomials of the controller, $P_{Ni}(s,\mathbf{b}_i)$ and $P_{Di}(s,\mathbf{a}_i)$, $i = 1,2,\ldots,t$ are numerator and denominator polynomials of the plant with any linear uncertainty and $\mathbf{c} = [\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_m, \mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n]$.

Closed loop characteristic equation of the system in Eq. 9 can be given in the following form.

$$\Delta(s,\mathbf{c}) = C_N(s)P_{N1}(s,\mathbf{b_1})P_{N2}(s,\mathbf{b}_2)\ldots P_{Nm}(s,\mathbf{b}_m) + C_D(s)P_{D1}(s,\mathbf{a_1})P_{D2}(s,\mathbf{a_2})\ldots P_{Dn}(s,\mathbf{a}_n) \tag{10}$$

Fig. 1 shows the typical block diagram of a fractional order system with multiple linear uncertainty. This study addresses problem of robust stability analyses of this kind of complex fractional order control system models. Real systems established by the cascaded process or blocks mainly yield this kind of system models and robust stability analysis of such system is beneficial for control engineering problems.

2.2 Exposed Edge Based Stability Analysis for Fractional Order Interval Polynomials

This section introduces the exposed edge stability analysis to fractional order interval systems (Senol et al., 2014). Consider the fractional order interval polynomial in Eq. 5. In order to obtain the roots region, one can apply $s = v^m$ mapping to the fractional order polynomial. This converts the fractional orders to expanded integer orders, thus, the fractional order stability problem turns into an integer order stability problem within the Riemann sheets. Assuming $s = v^m$, m sheets of Riemann surfaces are defined in the complex root space (Chen et al., 2006; Ahn et al., 2007; Monje et al., 2010). Then, the stability analysis can be performed by root locus analysis of the expanded integer order polynomials in their root spaces. For $s = v^m$ transformation, the expanded integer order form of Eq. 5 can be written as follows.

$$P^m(v,\mathbf{q}) = q_1 v^{\beta_1} + q_2 v^{\beta_2} + \ldots + q_n v^{\beta_n} \tag{11}$$

where, $P^m(v,\mathbf{q})$ is the expanded order polynomial and $\beta_i = m\alpha_i$ are expanded integer orders of the polynomial, $i = 1,2,\cdots,n$. $\mathbf{q} = [q_1, q_2, \cdots, q_n]$ is the set of uncertain parameters. Then, lower and upper limits of the uncertain parameter vector $\mathbf{q}$ define a hypercube Q in the coefficient space as follows.

$$Q = \{q : \underline{q}_i \le q_i \le \overline{q}_i,\ i = 1,2,\cdots,n\} \tag{12}$$

Let $\mathbf{q}$ be the set of uncertain parameters of the polynomial in Eq. 1 and consider the hypercube given in Eq. 12. Hypercube Q represents a polynomial family defined by perturbations of the uncertain parameter q in lower and upper bounds of elements. Let us denote this polynomial family set by $\Omega \in R^n$ .Then, the following theorem holds.

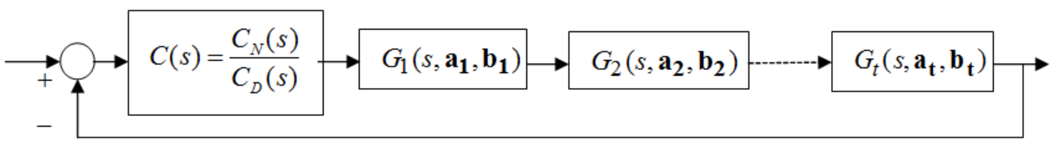

Fig. 1. Closed loop fractional order system with multiple linear uncertainty.

Theorem 1: The complex roots of the polynomial family Ω forms a set of root points which compose a roots region as,

$$R(\Omega)=\{v_r : P_u^m(q,v_r)=0,\ \forall q\in Q,\, r\in 1,2,\cdots,\xi\} \tag{13}$$

where, $R(\Omega)$ is defined as the root region of the fractional order polynomial in Eq. 1 (Senol et al., 2014). This root region determines the stability of fractional order uncertain polynomial.

Proof: This proof is summarized from (Senol et al., 2014). Expansion coefficient is the least common multiplier (LCM) which expands the fractional orders to minimal integer orders. Degree of the new polynomial family in this case can be written as $\xi=\max\{\beta_i\}$ thus, every member of the polynomial family has roots on the Riemann surface (Monje et al., 2010). Let us denote as this roots as $v_1,v_2,\cdots,v_\xi$. Roots located in the first Riemann sheet, which coincides with the phase range $\varphi\in(-\pi/m,\pi/m)$ are physically meaningful and this region is used for the stability analysis of fractional order uncertain polynomial (Senol et al., 2014; Radwan et al., 2009; Monje et al., 2010). Thus, the roots region of the expanded integer order polynomial is obtained.

Based on the stability analysis given in Appendix 1, followings are integrated in proof. Corner points of the hypercube Q are formed by considering all lower and upper limits of the uncertain coefficients, $q_i\in\left[\underline{q}_i,\overline{q}_i\right]$ thus, the hypercube has 2^n corners and $n2^{n-1}$ exposed edges. n is the number of uncertain coefficients. Exposed edges are the outer line segments connecting the corner points. Fig. 2 shows an instance hypercube of a fractional order interval polynomial with 3 uncertain coefficients.

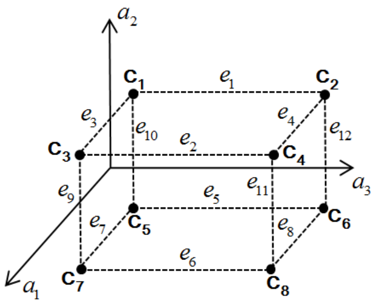

Fig. 2. Corner points and exposed edges of hypercube Q for a fractional order interval polynomial with 3 uncertain coefficients.

Set of corner points and exposed edges are denoted by $c=\left[c_1,c_2,\cdots,c_{2^n}\right]$ and $e=\left[e_1,e_2,\cdots,e_{n2^{n-1}}\right]$ respectively. The edge between two corners can be obtained using the following equation.

$$e_{i-j}=\left(\lambda c_i+(\lambda-1)c_j\right),\lambda\in[0,1] \tag{14}$$

where, c_i and c_j are the corner polynomials and e_{i-j} is the edge between these corners. Thus, the roots region which is constituted by exposed edges, compose the exact roots region of fractional order interval polynomials. Then, the stability of the interval polynomial can be investigated with the aid of Appendix *1. This completes the proof.*

3. STABILITY ANALYSIS FOR FRACTIONAL ORDER MULTIPLE LINEAR UNCERTAIN POLYNOMIALS

This section applies exposed edge stability analysis for fractional order polynomials of multiple linear uncertainties. Let us consider the fractional order multiple linear uncertain polynomial given in Eq. 8. Extended integer order form of the multiple linear uncertain polynomial family can be written as follows.

$$F^m(v,\mathbf{h})=A^m(s)P_1^m(v,\mathbf{h}_1)P_2^m(v,\mathbf{h}_2)\cdots P_n^m(v,\mathbf{h}_n) \tag{15}$$

where, $P_i^m(v,\mathbf{h}_1)$ are extended integer order polynomials of any linear uncertainty. Considering the lower and upper limits of the uncertain parameters vector $\mathbf{h}$, the hypercube $Q_\mathbf{h}$ is defined in the following form.

$$Q_\mathbf{h}=\{\mathbf{h}:\underline{q}_{ij}\le q_{ij}\le\overline{q}_{ij},\, i=1,2,\cdots,n\quad j=1,2,\cdots,k\} \tag{16}$$

Hypercube $Q_\mathbf{h}$ stands for a polynomial family defined by perturbations of lower and upper limits of uncertain parameters q_{ij}. Let us denote this polynomial family by $\Omega_\mathbf{h}\in R^n$. Then, the following corollary is valid.

Corollary 1: The complex roots of the multiple linear polynomial family $\Omega_\mathbf{h}$ forms a set of root points defined as,

$$R(\Omega_\mathbf{h})=\{v_r : F_u^m(\mathbf{h},v_r)=0,\ \forall\mathbf{h}\in Q_\mathbf{h},\, r\in 1,2,\cdots,\xi\} \tag{17}$$

where, $R(\Omega_\mathbf{h})$ is defined as the roots region of the fractional order polynomial in Eq. 8, which is used to investigate the stability of fractional order multiple linear polynomial.

Proof: $\Omega_\mathbf{h}$ is a subset of interval polynomial family Ω. In this case, the root region of $\Omega_\mathbf{h}$, denoted by $R(\Omega_h)$, is also subset of $R(\Omega)$. So, if all elements of $R(\Omega)$ are stable roots, all elements of $R(\Omega_h)$ are also stable roots. ¤

Then, stability investigation of fractional order polynomials with multiple linear uncertainties can be done as given in Appendix 1. Similar to fractional order polynomials with interval uncertainty, corner points of the hypercube $Q_\mathbf{h}$ are formed by considering all lower and upper limits of the uncertain coefficients, $q_{ij}\in\left[\underline{q}_{ij},\overline{q}_{ij}\right]$ thus, the hypercube has 2^n corners and $n2^{n-1}$ exposed edges. n is the number of uncertain coefficients.

4. STABILITY ANALYSIS FOR FRACTIONAL ORDER MULTIPLE LINEAR UNCERTAIN SYSTEMS

This section presents exposed edge stability analysis for fractional order systems of multiple linear uncertainties. Let us consider the fractional order multiple linear uncertain plant given in Eq. 9. Extended integer order form of the multiple linear uncertain plant family can be written as follows.

$$L^m(v,\mathbf{c}) = \frac{L_N^m(v,\mathbf{b})}{L_D^m(v,\mathbf{a})} =$$
$$\frac{C_N^m(v)}{C_D^m(v)} \cdot \frac{P_{N1}^m(v,\mathbf{b_1})P_{N2}^m(v,\mathbf{b}_2)...P_{Nt}^m(v,\mathbf{b_t})}{P_{D1}^m(v,\mathbf{a_1})P_{D2}^m(v,\mathbf{a_2})...P_{Dt}^m(v,\mathbf{a_t})} \tag{18}$$

where, $P_{Ni}^m(v,\mathbf{b}_i)$ and $P_{Di}^m(v,\mathbf{a}_i)$ are extended integer order polynomials of any linear uncertainty. $L_N^m(v,\mathbf{b})$ and $L_D^m(v,\mathbf{a})$ are extended integer order multiple linear polynomial families. Closed loop characteristic equation of this system can be written as,

$$\Delta_L^m(v,\mathbf{c}) = C_N^m(v)P_{N1}^m(v,\mathbf{b_1})P_{N2}^m(v,\mathbf{b}_2)...P_{Nt}^m(v,\mathbf{b_t})$$
$$+C_D^m(v)P_{D1}^m(v,\mathbf{a_1})P_{D2}^m(v,\mathbf{a_2})...P_{Dt}^m(v,\mathbf{a_t}) \tag{19}$$

Considering the lower and upper limits of the uncertain parameters vector $\mathbf{c}$, the hypercube $Q_\mathbf{c}$ is defined in the following form.

$$Q_\mathbf{c} = \{\mathbf{c} : \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij} \ \ and \ \ \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij},$$
$$i = 1,2,\cdots,n \qquad j = 1,2,\cdots,k\} \tag{20}$$

Hypercube $Q_\mathbf{c}$ stands for a polynomial family defined by perturbations of lower and upper limits of uncertain parameters b_{ij} and a_{ij}. Let us denote this polynomial family by $\Omega_\mathbf{c} \in R^n$. Then, the following corollary holds.

Corollary 2: The complex roots of the multiple linear characteristic equation $\Omega_\mathbf{c}$ forms a set of root points defined as,

$$R(\Omega_\mathbf{c}) = \{v_r : \Delta_L^m(\mathbf{c},v_r) = 0, \ \forall \mathbf{c} \in Q_\mathbf{c}, \ r \in 1,2,\cdots,\xi\} \tag{21}$$

where, $R(\Omega_\mathbf{c})$ is defined as the roots region of the fractional order polynomial in Eq. 8.

Proof: $\Omega_\mathbf{c}$ is a subset of interval polynomial family Ω. So, $R(\Omega_h)$ becomes a subset of $R(\Omega)$. Therefore, if all elements of $R(\Omega)$ are stable roots, all elements of $R(\Omega_\mathbf{c})$ are also stable roots.

Then, stability investigation of fractional order systems with multiple linear uncertainties can be done as given in Appendix 1. Stability investigation processes can be better understood with the illustrative examples given in the next section.

5. ILLUSTRATIVE EXAMPLES

This section includes illustrative examples to verify the results given in the previous sections. The first example has two fractional order systems with interval uncertainty. Second example has two fractional order systems with affine linear uncertainty.

Example 1: Consider the fractional order systems with interval uncertainty and PI controller.

$$C(s) = K_p + \frac{K_i}{s}$$
$$G_{11}(s,\mathbf{a}) = \frac{1}{a_{11}s^{1.1} + a_{12}}$$
$$G_{12}(s,\mathbf{b}) = \frac{1}{b_{11}s^{0.6} + b_{12}} \tag{22}$$

where, $K_p = 1$, $K_i = 2$ and the uncertain parameters are considered as $a_{11} \in [1,1.2]$, $a_{12} \in [0.7,0.9]$, $b_{11} \in [1.8,2]$ and $b_{12} \in [0.9,1]$. Connecting these systems in series will give the following transfer function.

$$G_1(s,\mathbf{a},\mathbf{b}) = C(s)G_{11}(s,\mathbf{a})G_{12}(s,\mathbf{b}) =$$
$$\frac{s+2}{a_{11}b_{11}s^{1.7} + a_{11}b_{12}s^{1.1} + a_{12}b_{11}s^{0.6} + a_{12}b_{12}} \tag{23}$$

Thus, the characteristic equation of the system found in Eq. 23 can be written as,

$$\Delta(s) = a_{11}b_{11}s^{1.7} + a_{11}b_{12}s^{1.1} +$$
$$a_{12}b_{11}s^{0.6} + s + a_{12}b_{12} + 2 = 0 \tag{24}$$

Applying $s = v^m$ mapping for $m = 10$, one can rewrite the characteristic equation of the system as,

$$\Delta_u^{10}(v) = a_{11}b_{11}v^{17} + a_{11}b_{12}v^{11} + v^{10} +$$
$$a_{12}b_{11}v^6 + a_{12}b_{12} + 2 = 0 \tag{25}$$

For this example, members of the characteristic equation which compose the corners of the hypercube can be found with the following limits of the uncertain parameters.

$$\Delta_{g_1} = \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \underline{b}_{11}\ \underline{b}_{12}],v),$$
$$\Delta_{g_2} = \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \underline{b}_{11}\ \overline{b}_{12}],v),$$
$$\Delta_{g_3} = \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \overline{b}_{11}\ \underline{b}_{12}],v),$$
$$\Delta_{g_4} = \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \overline{b}_{11}\ \overline{b}_{12}],v),$$
$$\vdots$$
$$\Delta_{g_{13}} = \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \underline{b}_{11}\ \underline{b}_{12}],v),$$
$$\Delta_{g_{14}} = \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \underline{b}_{11}\ \overline{b}_{12}],v),$$
$$\Delta_{g_{15}} = \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \overline{b}_{11}\ \underline{b}_{12}],v),$$
$$\Delta_{g_{16}} = \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \overline{b}_{11}\ \overline{b}_{12}],v) \tag{26}$$

Similarly, edge polynomials can be obtained for $h \in [0,1]$ as follows,

$$\begin{aligned}
\Delta_{e_1} &= \Delta^m([\underline{a}_{11}\,\underline{a}_{12}\,\underline{b}_{11}\,S(b_{12},h)],v),\\
\Delta_{e_2} &= \Delta^m([\underline{a}_{11}\,\underline{a}_{12}\,S(b_{11},h)\,\underline{b}_{12}],v),\\
\Delta_{e_3} &= \Delta^m([\underline{a}_{11}\,S(a_{12},h)\,\underline{b}_{11}\,\underline{b}_{12}],v),\\
\Delta_{e_4} &= \Delta^m([S(a_{11},h)\,\underline{a}_{12}\,\underline{b}_{11}\,\underline{b}_{12}],v),\\
&\vdots\\
\Delta_{e_{29}} &= \Delta^m([\overline{a}_{11}\,\overline{a}_{12}\,\underline{b}_{11}\,S(b_{12},h)],v),\\
\Delta_{e_{30}} &= \Delta^m([\overline{a}_{11}\,\overline{a}_{12}\,S(b_{11},h)\,\underline{b}_{12}],v),\\
\Delta_{e_{31}} &= \Delta^m([\overline{a}_{11}\,\overline{a}_{12}\,S(b_{11},h)\,\overline{b}_{12}],v),\\
\Delta_{e_{32}} &= \Delta^m([\overline{a}_{11}\,\overline{a}_{12}\,\overline{b}_{11}\,S(b_{12},h)],v).
\end{aligned} \tag{27}$$

Fig. 3(a) shows the roots region $R(\Omega)$ of the system in Eq. 23. Roots in the region $R(\Omega)$ ensures the condition $\pi/20 < |\arg\{R(\Omega)\}| < \pi/10$ and therefore the system is robustly stable for the given uncertain parameters. Fig. 3(b) shows the roots of corner and edge polynomials in the root spaces and indicates the robust stability of the polynomial family given by Eq. 23. Fig. 3(c) demonstrates the step-responses of corner polynomials and confirms robust stability of the system. Root region in Fig. 3(a) is built by using 65.536 roots and exposed edge based root region in Fig. 3(b) is built by using 1600 roots. Thus, the proposed method dramatically decreases the number of roots which are used to represent the exact root region. So, the computational cost for stability investigation is considerably reduced as a result of considering only exposed edge polinomials of interval systems.

Example 2: Consider the fractional order systems with affine linear uncertainty.

$$\begin{aligned}
G_{21}(s,\mathbf{a}) &= \frac{1}{(2a_{21}+3a_{22})s^{2.3}+(a_{21}+a_{22})}\\
G_{22}(s,\mathbf{b}) &= \frac{1}{(b_{21}+2b_{22})s^{0.6}+(b_{21}+b_{22})}
\end{aligned} \tag{28}$$

where, the uncertain parameters are $a_{21}\in[1.5,2]$, $a_{22}\in[0.3,1.3]$, $b_{22}\in[0.3,1.4]$ and $b_{21}\in[1.4,2.5]$. Multiplication of these systems gives the following fractional order system with multiple affine linear uncertainty.

$$\begin{aligned}
G_2(s,\mathbf{a},\mathbf{b}) &= G_{21}(s,\mathbf{a})G_{22}(s,\mathbf{b}) = \frac{N_2(s)}{D_2(s)}\\
N_2(s) &= 1\\
D_2(s) &= (2a_{21}b_{21}+4a_{21}b_{22}+3a_{22}b_{21}+6a_{22}b_{22})s^{2.1}+\\
&(2a_{21}b_{21}+2a_{21}b_{22}+3a_{22}b_{21}+3a_{22}b_{22})s^{1.3}+\\
&(a_{21}b_{21}+2a_{21}b_{22}+a_{22}b_{21}+2a_{22}b_{22})s^{0.8}+\\
&(a_{21}b_{21}+a_{21}b_{22}+a_{22}b_{21}+a_{22}b_{22})
\end{aligned} \tag{29}$$

Applying $s=v^m$ mapping for $m=10$, one can rewrite the characteristic equation of the system as,

$$\begin{aligned}
\Delta(v) &= (2a_{21}b_{21}+4a_{21}b_{22}+3a_{22}b_{21}+6a_{22}b_{22})v^{21}+\\
&(2a_{21}b_{21}+2a_{21}b_{22}+3a_{22}b_{21}+3a_{22}b_{22})v^{13}+\\
&(a_{21}b_{21}+2a_{21}b_{22}+a_{22}b_{21}+2a_{22}b_{22})v^{8}+\\
&(a_{21}b_{21}+a_{21}b_{22}+a_{22}b_{21}+a_{22}b_{22})+1=0
\end{aligned} \tag{30}$$

Fig. 4(a) shows the roots region of the system in Eq. 29. Roots in the region $R(\Omega)$ do not ensure the condition $\pi/20 < |\arg\{R(\Omega)\}| < \pi/10$ and therefore the system is unstable for the given uncertain parameters. Fig. 4(b) shows the roots of corner and edge polynomials in the root spaces and Fig. 4(c) demonstrates the step-responses of corner polynomials. It is clear in Fig. 4 that the system in Eq. 29 is unstable.

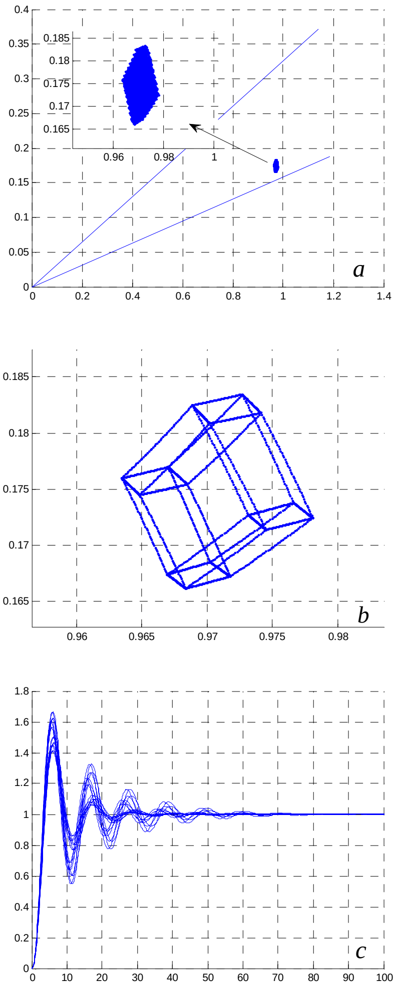

Fig. 3. Stability test results for the uncertain fractional order control system given by Eq. 23.

(a) Root space obtained for region- sampled robust stability analysis, (b) Root space obtained for edge-sampled robust stability analysis, (c) Step responses of corner polynomials.

Root region given in Fig. 4(a) is built by 1.048.576 roots and exposed edge based root region given in Fig. 4(b) is built by 3.200 roots. Like the previous example, the propsed method reduces computation cost for root region investigation. Reduction of computaional complexity is an important criteria for development of computer-aided system stabilization tools for complex real systems.

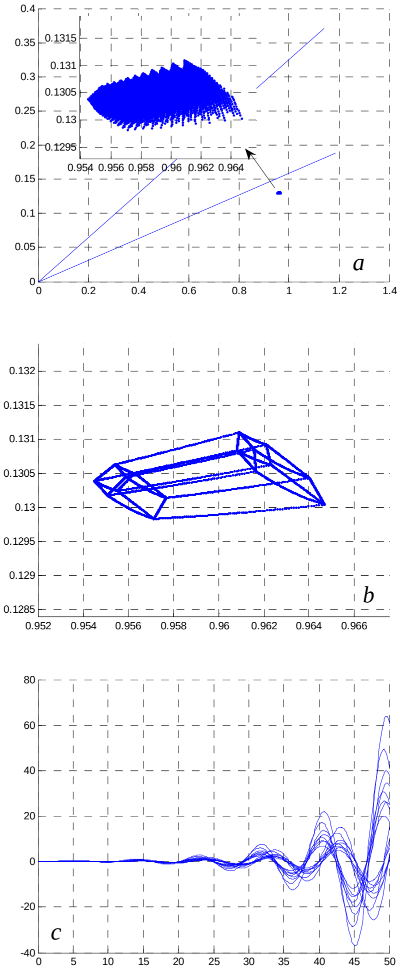

Fig. 4. Stability test results for the uncertain fractional order control system given by Eq. 29.

(a) Root space obtained for region- sampled robust stability analysis, (b) Root space obtained for edge-sampled robust stability analysis, (c) Step responses of corner polynomials.

6. CONCLUSION

This study extents exposed edge based robust stability analysis method for fractional order closed loop control systems with multiple linear uncertainties, which may contain two or more fractional order systems with any linear uncertainty structures. We observed that proposed Edge Theorem based scheme provides straightforward solution for numerical stability analysis of uncertain systems with multiple linear uncertainty structures, which, in fact, is not an so easy problem to solve by other methods. Cascaded real systems commonly lead to multiple linear uncertainty structures in practice, and stability analyses of these uncertainity models are very useful for robust control of complex real systems that are composed of cascaded, multiple subsystem components or processes.

As summary, paper demonstrates an extension of exposed edge based numerical stability analysis method for fractional order polynomials involving multiple linear uncertainty structures. Main advantage of the proposed method lay in reduction of stability analysis of very complicated roots region of fractional order multiple linear uncertain characteristic polynomial to the analysis of exposed edge boundaries. This simplification makes robust stability analysis of fractional order system structures almost independent of complexity of their uncertainity structures, therefore the paper can contribute to the robust stabilization problem of complex fractional order control systems in practical term.

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APPENDIX 1

This appendix is summarized from (Senol et al., 2014; Radwan et al., 2009) to present roots location based stability analysis for fractional order interval polynomials. Studies related to the stability analysis of expanded degree integer order polynomial in the root space can be found in the literature (Caponetto et al., 2010; Senol et al., 2014; Radwan et al., 2009). Radwan et al. put this analysis method in a systematic form (Radwan et al., 2009) and the Radwan procedure for the stability analysis was summarized as follows.

- Expand the orders of the fractional order polynomial with m expansion coefficient and calculate the root phases of the polynomial in the first Riemann sheet, $\varphi \in (-\pi/m, \pi/m)$.
- If the root phases are in the range of $\pi/2m < |\varphi_r| < \pi/m$, the system is stable.
- If the root phases are equal to $\pi/2m$, the system oscillates.
- Otherwise, the system is unstable.

For $m=1$, the Riemann surface refers to the open left half of the complex plane and this region is known as the stability region for integer order polynomials. Thus, the stability condition for integer order systems can be written as $\pi/2 < |\varphi_v| < \pi$. Stability regions for integer and fractional order systems are given in Fig. 5.

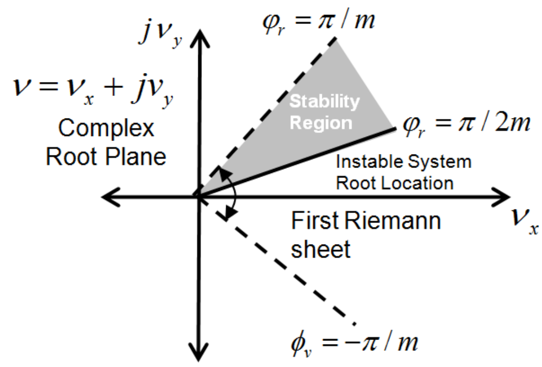

Fig. 5. Stability region for an expanded integer order polynomial in the first Riemann surface after $s = v^m$ mapping for $m > 1$.