Indirect-based Approach for Optimal Swing up and Stabilization of a Single Inverted Pendulum with Experimental Validation

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Abstract: The optimal swing up of a single inverted pendulum (SIP) on a cart is presented, where the indirect method is used to obtain the feedforward command in the swing up mode and the LQR controller beside the high gain observer is used to stabilize the system in the stabilizing mode. After extracting the optimality conditions for a dynamic system in general formulation, the necessary conditions for optimality are derived for a SIP using the fundamental theorem of the calculus of variations (FTCV) which leads to a two-point boundary value problem (TPBVP). This problem is solved for obtaining the optimal values of states and control. To demonstrate the applicability of proposed method, after simulation study, a single pendulum setup is constructed and experimental realization is presented. In order to complete the swing up maneuver, LQR controller is designed to stabilize the system, and the high gain observer is applied to estimate the link angular velocity. Finally, a comparison between experimental and simulation results is presented and the efficiency of proposed method to reduce the used effort is illustrated.

Keywords: Inverted pendulum, optimal swing up, stabilization, indirect method, calculus of variations, LQR, high gain observer, experiment.

1. INTRODUCTION

Inverted pendulums are nonlinear, unstable and under-actuated electromechanical systems widely used in linear and nonlinear control education and research. Generally, the control of inverted pendulums can be divided into two aspects. In the first aspect, swinging up the link or links from hanging position to upright state is considered. The second aspect is stabilizing the link or links in upright position, while the cart must also be homed to a reference position. For stabilization problem, there are numerous control techniques such as feedback linearization (Chen et al., 2004), adaptive control and fuzzy learning control (Duka et al., 2007), sliding-mode approach (Tsai et al., 2010), LQR controller (Kumar et al., 2013), mixed H2/H∞ PID controller (Duy at al., 2014), fuzzy fractional order sliding mode controller (Bouarroudj et al., 2015), feedback linearization method for tracking control of constrained double inverted pendulum (Lari et al., 2015) and many other control algorithms. Besides the stabilization aspect, the swing up problem has gained increasing attention during the recent past. All the research works dealing with the swing up problem, can be divided into two categories: Non-optimal swing up and optimal swing up.

As the non-optimal swing up, different methods such as energy based method (Astorm et al., 2000; Chatterjee et al., 2002), smooth controllers (Astorm et al., 2008), fuzzy control (Wu at al., 2011), sliding mode control (Park et al., 2009; Wang, 2012), passivity-based control (Icaza, 2011), inversion-based method (Graichen et al., 2007; Gluck et al., 2013), event-Based Control (Durand et al., 2013), integral back-stepping sliding mode control (Adhikary et al., 2013) and many other methods have been performed by different researchers up to now. Unlike the non-optimal methods, in the optimal swing up a given objective function must also be minimized. The approaches used to solve the optimal control problems (OCP) are broadly classified as either indirect or direct method.

In the direct method, which is known as the “first discretize, then optimize” approach, at first, dynamic variables (states and controls) are discretized to obtain a parameter optimization problem. Then this problem is solved using different methods such as genetic algorithm, particle swarm optimization, and sequential quadratic programming approach. While, the indirect method is known as the “first optimize, then discretize” approach. In the indirect method, the optimality conditions are derived using the fundamental theorem of the calculus of variations which leads to a TPBVP. Then, discretization is used to solve the obtained TPBVP. So the indirect method results in the accurate solution of OCP, whereas by the direct method, an approximate solution is achieved (Nikoobin et al., 2013; Nikoobin et al., 2017). Most of the previous works dealing with optimal swing up are on the base of the direct method. Genetic algorithm tuned bang-bang controller (Zhao et al., 2003), linear and nonlinear programming (Cruz et al., 2013; Kahvecioglu et al., 2009), iterative impulsive control (Wang et al., 2004) and ant colony optimization method (Ast et al., 2009) have been reported for the optimal swing up of the inverted pendulum. A suboptimal nonlinear control law based on passivity analysis and dynamic programming has been presented for the Pendubot and rotary pendulum by (Oliver et al., 2012) in which switch control law is not required. Indirect method has also been applied to solve the optimal swing up problem, but this method is often used for solving the time optimal control of SIP without experimental validation (Xu et al., 2001; Chernousko et al., 2007; Mason et al., 2008; Paoletti et al., 2011; Merakeb et al., 2013). The main challenge of an indirect solution of optimal control problem is finding the proper initial guess for the solution. Different methods such as homotopy continuation method (Hermant, 2011, Nikoobin et al., 2017) have been proposed to overcome this problem. In this paper, the solution of the inversion based method is used as the initial guess of the optimal control method, and the problem is solved easily.

In this paper, the general dynamic system is considered. On the base of the indirect method, the necessary conditions for optimality are derived from the FTCV. The obtained equations establish a TPBVP solved by the bvp4c command in MATLAB. In order to verify the method, a single pendulum setup is constructed and experimental implementation for both optimal and inversion-based method (Graichen et al., 2007) is presented. In order to complete the swing up maneuver, a stabilization controller is also required to stabilize the link in upright position. To this end, the LQR state feedback controller is designed to stabilize the system besides the high gain observer to estimate the link angular velocity. The paper is outlined as follows: the next section describes the dynamic equations and optimality conditions for a general dynamic system. Optimality conditions of a SIP are derived in Section 3. Section 4 addresses the simulation results for SIP. Finally, in Section 5, the experimental validation is presented.

2. NECESSARY CONDITIONs FOR OPTIMALITY

The general dynamic equation of a mechanical system can be described as

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{h}(\mathbf{q})=\boldsymbol{\tau}, \tag{1}$$

where $\mathbf{q}\in R^n$ is the vector of joint positions, $\dot{\mathbf{q}}\in R^n$ is the vector of joint velocities, $\mathbf{D}\in R^{n\times n}$ is the inertia matrix, $\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})\in R^n$ is the centripetal, Coriolis and friction forces, $\mathbf{h}(\mathbf{q})\in R^n$ describes the gravity effects and $\tau\in R^n$ represents the force vector. By defining the state vector as:

$$\mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}, \tag{2}$$

Equation (1) can be rewritten in state space form as

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\boldsymbol{\tau}) \Rightarrow \begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix}=\begin{bmatrix}\mathbf{f}_1(\mathbf{x},\boldsymbol{\tau})\\ \mathbf{f}_2(\mathbf{x},\boldsymbol{\tau})\end{bmatrix}, \tag{3}$$

where

$$\mathbf{f}_1=\mathbf{x}_2,\ \mathbf{f}_2=\mathbf{p}(\mathbf{x}_1,\mathbf{x}_2)+\mathbf{K}(\mathbf{x}_1)\boldsymbol{\tau}, \tag{4}$$

$$\begin{aligned}&\mathbf{p}(\mathbf{x}_1,\mathbf{x}_2)=-\mathbf{D}^{-1}(\mathbf{x}_1)\left[\mathbf{b}(\mathbf{x}_1,\mathbf{x}_2)+\mathbf{h}(\mathbf{x}_1)\right],\\ &\mathbf{K}(\mathbf{x}_1)=\mathbf{D}^{-1}(\mathbf{x}_1).\end{aligned} \tag{5}$$

Optimal swing up of an inverted pendulum is an optimization problem which can be stated in the form of the optimal control problem. The goal is to swing up the pendulum from the hanging down to the standing up position in which a predefined objective function must be minimized. So, the optimal control problem for a dynamic system can be stated as follows (Hull, 1998): Find the continuous admissible control history $\boldsymbol{\tau}:\left[t_0,t_f\right]\to\Omega\subseteq R^m$ generating the corresponding state trajectory $\mathbf{x}:\left[t_0,t_f\right]\to R^n$ which minimizes the cost function

$$J=\phi(\mathbf{x}_f,t_f)+\int_{t_0}^{t_f}L(\mathbf{x},\boldsymbol{\tau},t)dt\,, \tag{6}$$

subject to the system dynamics

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\boldsymbol{\tau})\,, \tag{7}$$

the given initial condition

$$\mathbf{x}(t_0)=\mathbf{x}_0\,, \tag{8}$$

and the prescribed final conditions

$$\mathbf{x}(t_f)=\mathbf{x}_f\,. \tag{9}$$

Here, $\mathbf{x}\in R^n$ is the state vector, $\boldsymbol{\tau}\in R^m$ is the control vector, Ω is an acceptable region in R^m, t_0 and t_f are initial and final time, $\mathbf{x}_0$ and $\mathbf{x}_f$ are predefined initial and final state, respectively. L and ϕ are scalar continuously differentiable functions in which L is the integrand of the cost function and ϕ is the final state penalty term. By introducing the costate vector $\boldsymbol{\lambda}\in R^n$, the Hamiltonian function of the system can be defined as follows

$$H=L+\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{f}\,. \tag{10}$$

According to the FTCV, for the optimal trajectory $\mathbf{x}^*(t)$ and $\boldsymbol{\tau}^*(t)$, there is a non-zero costate vector $\boldsymbol{\lambda}^*(t)$ such that the following conditions along the optimal solution must be satisfied (Hull, 1998)

$$\dot{\mathbf{x}}^*=\frac{\partial H}{\partial\boldsymbol{\lambda}},\ \dot{\boldsymbol{\lambda}}^*=-\frac{\partial H}{\partial\mathbf{x}},\ \frac{\partial H}{\partial\boldsymbol{\tau}}=\mathbf{0} \tag{11}$$

where the symbol (*) denotes the extremals of $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$. By substituting (4) into (10) and by defining $\boldsymbol{\lambda}=\begin{bmatrix}\boldsymbol{\lambda}_1^{\mathrm{T}} & \boldsymbol{\lambda}_2^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, the necessary condition (11) can be rewritten as follows

$$\begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix}=\begin{bmatrix}\mathbf{x}_2\\ \mathbf{p}(\mathbf{x}_1,\mathbf{x}_2)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{K}(\mathbf{x}_1)\boldsymbol{\tau}\end{bmatrix}, \tag{12}$$

$$\dot{\boldsymbol{\lambda}}(t)=-\begin{bmatrix}\dfrac{\partial L}{\partial\mathbf{x}_1}+\dfrac{\partial}{\partial\mathbf{x}_1}\left[\mathbf{p}+\mathbf{K}\boldsymbol{\tau}\right]^{\mathrm{T}}\boldsymbol{\lambda}_2\\ \dfrac{\partial L}{\partial\mathbf{x}_2}+\boldsymbol{\lambda}_1+\dfrac{\partial}{\partial\mathbf{x}_2}\left[\mathbf{p}\right]^{\mathrm{T}}\boldsymbol{\lambda}_2\end{bmatrix}, \tag{13}$$

$$\frac{\partial L}{\partial\boldsymbol{\tau}}+\mathbf{K}^{\mathrm{T}}\boldsymbol{\lambda}_2=0. \tag{14}$$

Equations (12) and (13) represents $4n$ equations dealing with states and costates respectively, and (14) leads to n equations dealing with control $\boldsymbol{\tau}$. According to Pontryagin minimum principle, the constraint on control signal can be applied as follows (Korayem et al., 2009)

$$\boldsymbol{\tau} = \begin{cases} U^+ & \boldsymbol{\tau} > U^+ \\ \boldsymbol{\tau} & U^- \le \boldsymbol{\tau} \le U^+ \\ U^- & \boldsymbol{\tau} < U^- \end{cases} \tag{15}$$

where U^- and U^+ are the lower and upper limit of the control and $\boldsymbol{\tau}$ is the solution of the algebraic equation (14). Also for the swing up problem, there are 4n fixed boundary conditions as follows

$$\mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_f) = \mathbf{q}_f, \dot{\mathbf{q}}(t_0) = \dot{\mathbf{q}}_0, \dot{\mathbf{q}}(t_f) = \dot{\mathbf{q}}_f. \tag{16}$$

By substituting the control value $\boldsymbol{\tau}$ obtained from (14) into (12) and (13), a set of the 4n ordinary differential equation is obtained which besides the 4n boundary conditions (16), forms a TPBVP. Finally, the derived TPBVP is solved to obtain 2n states and 2n costates. In the next section, the optimality conditions for a SIP are derived in details.

3. DERIVING THE EQUATIONS FOR SIP

In this section, a schematic model of SIP similar to experimental setup presented in section 5 is considered. This model consists of a link, cart, belt and one pulley. The cart is actuated by a motor through a belt, and the link is free to rotate in a vertical plane as shown in Fig. 1.

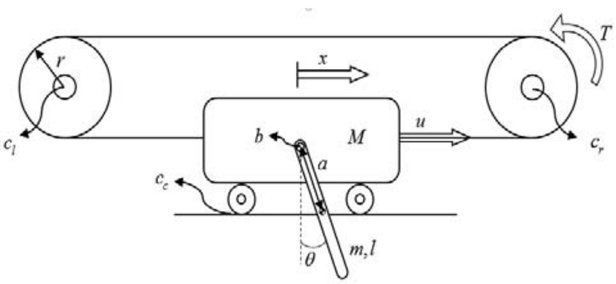

Fig. 1. Schematic model of SIP.

The dynamic equations of the system can be obtained using the Euler-Lagrange method. To this end, by selecting x and θ as the generalized coordinates, their corresponding generalized forces become

$$Q_1 = \frac{T}{r} - c_c\dot{x} - \left(\frac{c_r + c_l}{r^2}\right)\dot{x},\ Q_2 = -b\dot{\theta}, \tag{17}$$

where T is the motor torque, r is the pulley radius, c_r, c_l, c_c and b are the damping coefficient of the right pulley, left pulley, cart and joint of the link, respectively. By defining the parameter u as the force exerted on the cart through the belt, and parameter c as the total damping coefficient, the first generalized force is simplified as $Q_1 = u - c\dot{x}$.

It is assumed that the total torque produced by motor is transferred to the cart through belt, and by neglecting the pulleys mass and inertia, the kinetic and potential energies are obtained as follow

$$K = \frac{1}{2}\left[M\dot{x}^2 + m\left(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right) + I\dot{\theta}^2\right], \tag{18}$$
$$V = -mag\cos\theta,$$

where M is the cart mass, m is the link mass, I is the moment of inertia of link about its center of mass and a is the center of mass of the link. Using the Euler-Lagrange equations, the dynamic equation for the SIP can be derived as follows

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix} \tag{19}$$

where

$$D_{11} = M + m,\ D_{12} = ma\cos\theta,\ D_{22} = ma^2 + I$$
$$b_1 = -ma\dot{\theta}^2\sin\theta + c\dot{x},\ b_2 = b\dot{\theta},\ h_1 = 0,\ h_2 = mga\sin\theta.$$

By defining the state vector as

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^{\mathrm{T}}, \tag{20}$$

and using Eqs. (2)-(5), dynamic equation (19) can be rewritten in state space form as below

$$\begin{aligned} \dot{x}_1 &= f_{11} = x_3 \\ \dot{x}_2 &= f_{12} = x_4 \\ \dot{x}_3 &= f_{21} = \frac{1}{A_1}\begin{bmatrix} \left(I + ma^2\right)\left(cx_3 - max_4^2\sin x_2 - u\right) \\ -maA_2\cos x_2 \end{bmatrix} \\ \dot{x}_4 &= f_{22} = \frac{1}{A_1}\begin{bmatrix} m^2a^2x_4^2\cos x_2\sin x_2 + mau\cos x_2 - \\ macx_3\cos x_2 + \left(M + m\right)A_2 \end{bmatrix} \end{aligned} \tag{21}$$

where

$$A_1 = m^2a^2\left(\cos^2 x_2 - 1\right) - M\left(I + ma^2\right) - mI$$
$$A_2 = \left(mag\sin x_2 + bx_4\right)$$

In the next step, the proper objective function must be chosen. The commonly used objective functions for optimal control problem are minimum time, minimum effort and minimum energy. In this paper, swinging up the link from pendant mode to inverted state, with minimum effort which leads to a smooth trajectory have been considered, so the performance objective is chosen as

$$J = \int_0^{t_f} 0.5u^2 dt. \tag{22}$$

Now, by defining the costate vector as $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}^{\mathrm{T}}$ and dynamic equations in state space form as $\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{21} & f_{22} \end{bmatrix}^{\mathrm{T}}$, the Hamiltonian function can be determined using Eqs. (10), (21) and (22) as follows

$$H = 0.5u^2 + \lambda_1 f_{11} + \lambda_2 f_{12} + \lambda_3 f_{21} + \lambda_4 f_{22}, \tag{23}$$

After that, equation (11) is used to derive the equations dealing with optimality conditions. Differentiating the Hamiltonian function with respect to the costates, leads to the dynamic equations in state space form given in (21). The costate equations are obtained by differentiating the Hamiltonian with respect to the states as follows

$$
\begin{aligned}
&\dot{\lambda}_1 = 0, \\
&\dot{\lambda}_2 = -\frac{\partial}{\partial x_2}\left(\frac{\left(u - h_1 - b_1\right)\left(D_{12}\lambda_4 - D_{22}\lambda_3\right) + \left(h_2 + b_2\right)\left(-D_{12}\lambda_3 + D_{11}\lambda_4\right)}{-D_{11}D_{22} + D_{12}^2} \right), \\
&\dot{\lambda}_3 = -\lambda_1 - \frac{1}{A_1}\left[c\lambda_3\left(I + ma^2\right) - c\lambda_4 ma\cos\left(x_2\right)\right], \\
&\dot{\lambda}_4 = -\lambda_2 + \frac{1}{A_1}\left[\begin{array}{l} 2ma\lambda_3 \sin x_2 \left(Ix_4 + ma^2 x_4\right) \\ + ma\lambda_3 b\cos x_2 \\ -\lambda_4 \begin{pmatrix} 2m^2 a^2 x_4 \cos x_2 \sin x_2 \\ + b\left(M + m\right) \end{pmatrix} \end{array} \right].
\end{aligned} \tag{24}
$$

Then, the optimal control law is obtained from (14) by differentiating the Hamiltonian function with respect to control u. So, one can write

$$
u = \frac{1}{A_1}\left[\lambda_3\left(I + ma^2\right) - \lambda_4 ma\cos x_2 \right]. \tag{25}
$$

Finally, substituting (25) into (21) and (24) leads to eight nonlinear ordinary differential equations in terms of states and costates. These equations with the following eight boundary conditions

$$
\begin{aligned}
&x_1(0) = 0, x_2(0) = 0,\ x_3(0) = 0,\ x_4(0) = 0, \\
&x_1(t_f) = 0, x_2(t_f) = \pi, x_3(t_f) = 0,\ x_4(t_f) = 0,
\end{aligned} \tag{26}
$$

construct a two-point boundary value problem. This problem can be solved using the bvp4c command in MATLAB®.

4. SIMULATION RESULTS

Here, using the equations derived in the previous section, the optimal swing up problem is solved for SIP. In order to compare the obtained results with a non-optimal method, the inversion-based feedforward control proposed recently by (Graichen et al., 2007) is also presented. In order to show the efficiency of the method, the obtained results of optimal swing up are compared with those obtained by the inversion-based method.

The SIP is considered as shown in Fig. 1. All required parameters are given in **Table 1**. The swing up problem is solved according to boundary conditions (26) within the time interval $t \in [0\ \ t_f]$, $t_f = 1.3\,s$.

For the inversion-based solution of swing up problem (Graichen et al., 2007), the equations of motion are decomposed into two dynamics: input-output dynamics and internal dynamics. In this case by considering the cart acceleration as input to the system, $u = \ddot{x}$, the model of the single pendulum given in Eq. (19) can be rewritten as

$$
\begin{aligned}
&\ddot{x} = u \\
&\ddot{\theta} = -D_{22}^{-1}\left(D_{12}u + b_2 + h_2\right),
\end{aligned} \tag{27}
$$

Table 1. Mechanical parameters of the single pendulum.

Parameters	Values	Unit
Cart mass	M =1.02	Kg
Link mass	m=0.49	Kg
Center of mass of link	a=0.20	m
Moment of inertia of link	I=0.0056	kg.m^2
Cart viscous friction coefficient	c=21.0653	N.sec/m
link viscous friction coefficient	b=0.009	N.m.sec/rad
Pulley radius	r=0.01745	m

where the first equation represents the input–output dynamics, and the second one forms the internal dynamic. According to the inversion-based feedforward control proposed by Graichen, the following function is constructed using the cosine series

$$
x = -p_1 - p_2\cos\left(\frac{\pi t}{t_f}\right) + \sum_{i=1}^{2} p_i \cos\left(\frac{\left(1+i\right)\pi t}{t_f}\right), \tag{28}
$$

where p_1 and p_2 are free parameters. This function satisfies the four boundary conditions dealing with the cart trajectory given in (26). By substituting (28) into (27), the internal dynamic besides the four remained boundary conditions dealing with the link trajectory, constructs a TPBVP which can be solved to obtain the internal dynamic trajectory $\theta(t)$ and the free parameters as $p_1 = -0.1473, p_2 = -0.1537$.

For optimal swing up, the TPBVP consists of (21), (24), (25) and (26) are solved to obtain the cart and link trajectory as well as the optimal control history applied to the cart.

The angular position and velocity of link for both optimal and inversion-based swing up are shown in Fig. 2 and Fig. 3, respectively. As shown in Fig. 2 the link starts from its natural stable hanging position to its unstable upright inverted position. The linear position and velocity of cart are shown in Fig. 4 and Fig. 5. The cart displacement is a significant factor in experimental setup. For the larger cart displacement, the longer rail is required. As it can be seen from Fig. 4 the cart displacement for inversion-based method is 0.53m and for optimal method is 0.36m. So, the cart displacement for optimal swing up has reduced approximately 32% in comparison with inversion-based method

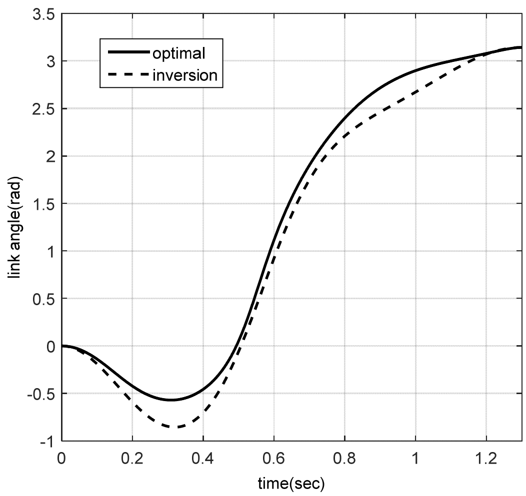

Fig. 2. Angular position of link.

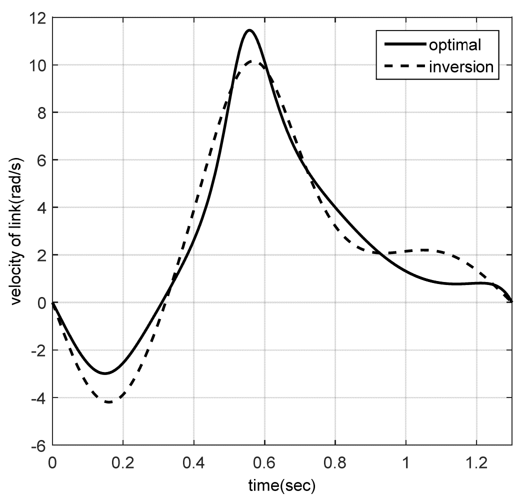

Fig. 3. Angular velocity of link.

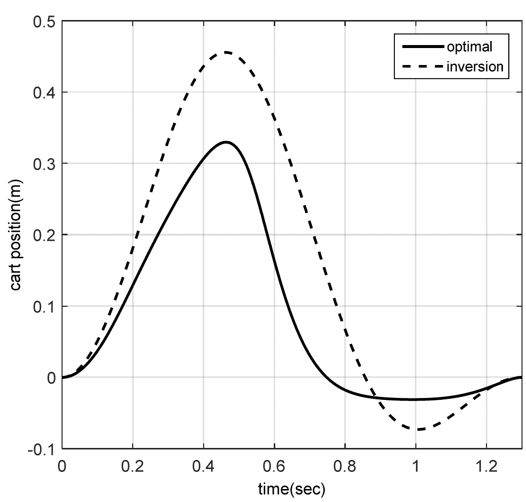

Fig. 4. Linear position of cart.

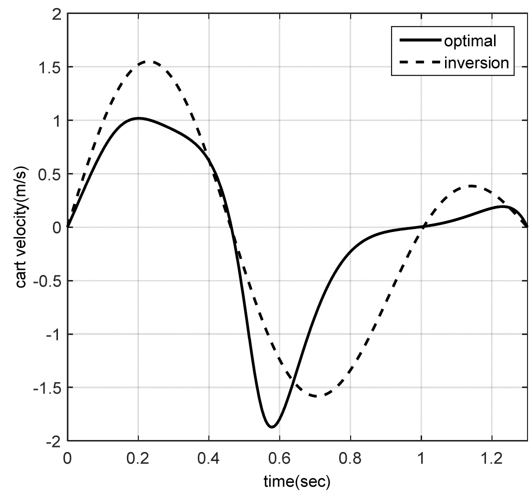

Fig. 5. Linear velocity of cart.

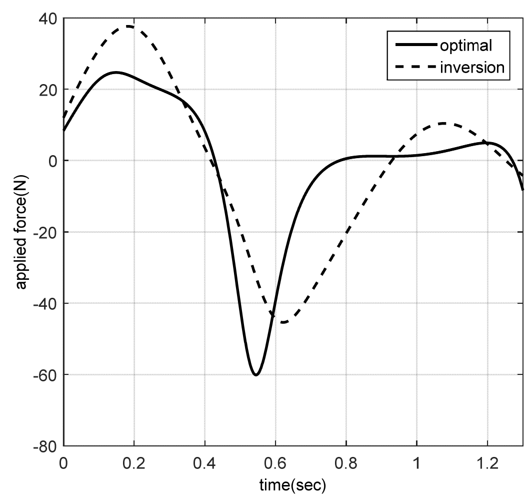

Fig. 6. Applied force to the cart.

Fig. 6 shows the applied force to the cart. Using (22), the performance index is calculated for optimal and inversion-based method as follows

$$J_{inversion} = \int_0^{1.3} 0.5u^2 dt = 370.26, \tag{29}$$

$$J_{optimal} = \int_0^{1.3} 0.5u^2 dt = 261.77.$$

So, the performance index for optimal swing up has reduced approximately 30 percent in comparison with inversion-based method. The experimental validation for this system is presented in section 5.

5. EXPERIMENTAL VALIDATION

5.1 Experimental setup description

The swing up maneuver is experimentally realized with the single pendulum in Fig. 7 corresponding to the model parameters in Table 1. The cart is actuated via a motor, driven by a lenze 940 series servo driver, connected to a PC through a network cable. The cart position is measured by a 1024 ppr Autonics incremental rotary encoder. By using a ball bearing, the link is pivoted to the cart such that it can rotate freely in a vertical plane. Its angle is measured by a 2048 ppr Autonics incremental rotary encoder. Also, the controller is implemented by the MATLAB software on a 2.69GHz PC with sampling time 10 KHz.

The interconnections of different parts of SIP system are shown in Fig. 8. The measured cart position and link angle are sent to the controller through Advantech PCI-1710HG I/O board. The motor driver is adjusted in velocity mode. This velocity signal is sent to the driver via I/O interface as the command input. Driver tries to eliminate the presence error between the cart velocity and the velocity command input via its internal PI controller.

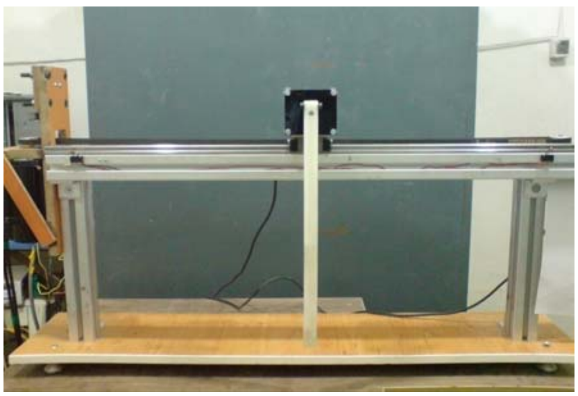

Fig. 7. Experimental setup of the single pendulum on a cart.

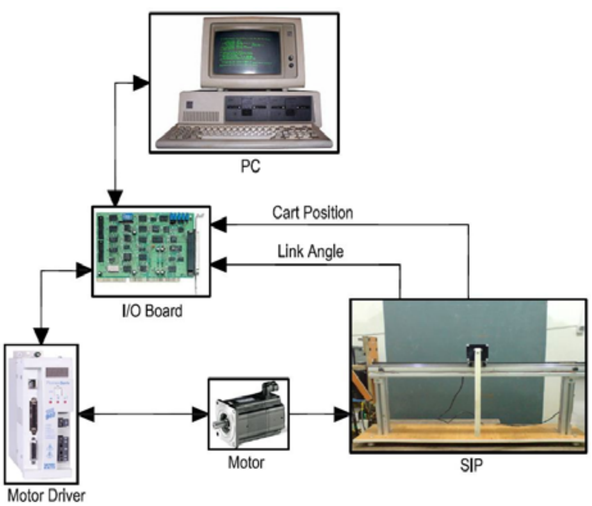

Fig. 8. Schematic architecture of SIP setup.

5.2 Control system design

The control strategy of the SIP system is composed of the swing up control, mode switching and stabilizing control. For the first step, a feedforward controller obtained of the indirect solution of optimal control problem presented in section 4 is used to swing up the pendulum with minimum effort. In order to show the efficiency of proposed method, the results of optimal swing up are compared with the result of the inversion-based method. Since only the open-loop feedforward control is used for swing up mode, it is required to have an accurate model of the pendulum. To this end, for parameters estimation, different sine functions as the applied force on the cart are exerted on both the pendulum experimental setup and the pendulum dynamic equation given in (21) to obtain the cart position and link angular position as the outputs. By minimizing the error between the experimental setup output and the dynamic equation output, the estimated parameters of the pendulum given in Table 1 are obtained.

Since the motor driver is set to velocity mode, so the cart velocity will be the feedforward command in the swing up maneuver. This signal which must be sent to the motor driver is computed offline and stored in a lookup table. The used servo motor has a large velocity bandwidth, so it tracks the pre-calculated feedforward velocity command accurately. For the inversion based method, the first derivative of cart position (27) gives the cart velocity, and for the optimal swing up method, after solving the TPBVP obtained in section 3, the cart velocity can be computed as shown in Fig 3.

After swing up phase, an extra stabilization controller is needed to grab the link and hold it upright. The stabilization controller is addressed in the following subsection.

5.3 Controller and observer design

Just as the link reaches the neighborhood of the inverted status, the controller is switched from open-loop feedforward control to the closed loop feedback control. Here the linear quadratic regulator (LQR) is used to regulate the system about the upright equilibrium point. This method has been used to stabilize the inverted pendulum systems frequently (Kumar et al., 2013). Since the motor driver works in velocity mode, so the SIP input changes from force to cart acceleration. In order to design the linear state-feedback controller, a linearized model of SIP equations is required. To this end, after rewriting the SIP equation (19), with cart acceleration as input, the dynamic equations of cart and link are decoupled as

$$\begin{aligned} &\ddot{x} = u, \\ &(ma^2 + I)\ddot{\theta} + ma\ddot{x}\cos\theta + mag\sin\theta + b\dot{\theta} = 0, \end{aligned} \tag{30}$$

Then, the linearized model about the unstable equilibrium point can be derives as follows

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{mag}{I+ma^2} & -\dfrac{b}{I+ma^2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{ma}{I+ma^2} \end{bmatrix} u, \\ \mathbf{y} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}, \end{aligned} \tag{31}$$

where $\mathbf{x}=\begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^{\mathrm{T}}$ denotes the state vector and $y=\begin{bmatrix} x & \theta \end{bmatrix}^{\mathrm{T}}$ denotes the output vector. Now, one can write the structure of linear state feedback controller as

$$u=-\mathbf{k}^{\mathrm{T}}\mathbf{x}=-\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}\begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^{\mathrm{T}}, \tag{32}$$

which minimizes the performance index

$$J=\int_0^{\infty}\left[\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x}+u^{\mathrm{T}}\mathbf{R}u\right]dt, \tag{33}$$

where **R** and **Q** are symmetric and **R**>0, $\mathbf{Q}\geq 0$. Here by choosing **R**=0.1 and **Q**=diag(9,9,3,3), the controller gain using the lqr command in MATLAB® can be calculated as $\mathbf{k}^{\mathrm{T}}$= [-9.48 -14.14 66.63 11.81].

In the experimental setup, all the state variables are not available. In fact, only two encoders are used to measure the cart position, *x*, and the pendulum angle, *θ*. In the other word, the cart velocity and the pendulum angular velocity are not immediately available for the controller law. Here, a high gain state observer is designed to estimate the required states. This observer is simple in design and provides an accurate estimation of all the states in both swing up and stabilization phases. Consider the following system

$$\begin{aligned}\dot{\mathbf{y}}&=\mathbf{A}\mathbf{y}+\mathbf{g}(\mathbf{w},\mathbf{u}),\\ \mathbf{w}&=\mathbf{c}\mathbf{y},\end{aligned} \tag{34}$$

where $(\mathbf{A},\mathbf{c})$ is observable, the observer equation is

$$\dot{\hat{\mathbf{y}}}=\mathbf{A}\hat{\mathbf{y}}+\mathbf{g}(\mathbf{w},\mathbf{u})+\mathbf{H}(\mathbf{w}-\mathbf{c}\hat{\mathbf{y}}), \tag{35}$$

where for robustness the gain $\mathbf{H}$ must be large enough (Vasiljevic et al., 2006). By using acceleration signal as input, the dynamics equations of cart and link decoupled from each other as seen in (31). So, two observers can be designed to estimate the cart velocity and pendulum angular velocity, separately. On the other hand, via the done tests, it is clearly seen that the motor tracks the velocity command input accurately, so the cart velocity can be obtained by integrating of controller output. Therefore the observer design procedure is reduced to estimate only the link angular velocity. Using (31), the dynamic equation of link can be written as below

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}&=\begin{bmatrix}0 & 1\\ \dfrac{mag}{I+ma^2} & -\dfrac{b}{I+ma^2}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{ma}{I+ma^2}\end{bmatrix}u,\\ w&=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}.\end{aligned} \tag{36}$$

This equation is in the form of the (34), therefore by using (35), the observer equation can be defined as follow

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\hat{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}&=\begin{bmatrix}0 & 1\\ \dfrac{mag}{I+ma^2} & -\dfrac{b}{I+ma^2}\end{bmatrix}\begin{bmatrix}\hat{\theta}\\ \dot{\hat{\theta}}\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{ma}{I+ma^2}\end{bmatrix}u\\ &+H\left(\theta-\hat{\theta}\right),\end{aligned} \tag{37}$$

where the observer gain H is tuned as follows

$$\mathbf{H}=\begin{bmatrix}\dfrac{\alpha_1}{\varepsilon}\\ \dfrac{\alpha_2}{\varepsilon^2}\end{bmatrix},\ \alpha_1=2,\ \alpha_2=3,\ \varepsilon=0.005. \tag{38}$$

The parameters α_1, α_2 and ε are obtained by trial and error. The pole placement approach could have been used, based on the reduced order dynamics given in (36). Figure 9 shows the observer performance. There are some mismatches in transient that can be also due to the observer tuning.

The final step to complete the control system design is selecting a proper criterion to switch from swing up controller to stabilization controller. These conditions are considered as follows:

$$\begin{aligned}&\theta<0.5\ rad,\\ &\dot{\theta}<2.5\ rad/\sec.\end{aligned} \tag{39}$$

In the following, experimental tests are performed in two steps. At first step, the pendulum stabilization is done, and then swing up maneuver is presented.

5.4 Pendulum stabilization

This test is performed to verify the performance of the state feedback controller and observer. In this experiment, the link is started from the inverted position and the controller stabilizes it. To investigate the observer performance, the link angular velocity obtained by differentiating the link angular position after pre and post filtering besides the estimated link angular velocity by the observer are shown in Fig. 9. As it can be seen, the observer is able to estimate the angular velocity of the link as well.

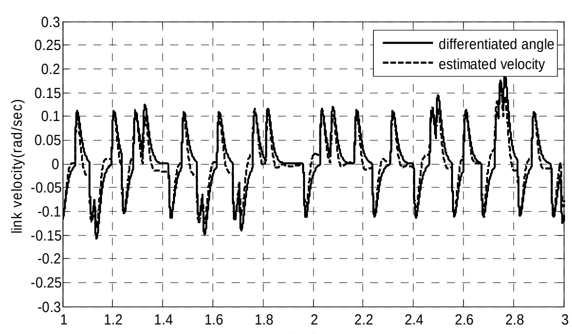

Fig. 9. Differentiated angular position and estimated angular velocity.

The position and velocity of cart and link are shown in Fig. 10 - Fig. 13. It is observed from Fig. 10, the deviation of $\pm 0.01\ \mathrm{m}$ is maintained after 3.5 seconds. As it can be seen from Fig. 12, the link is held in upright position with $\pi \pm 0.02\ \mathrm{rad}$ deviation after 3 seconds.

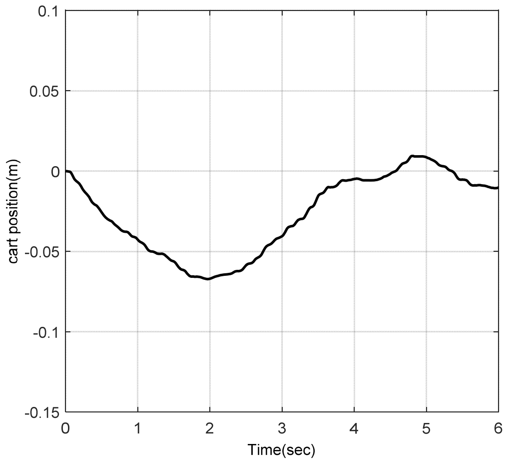

Fig. 10. Cart position in upright position.

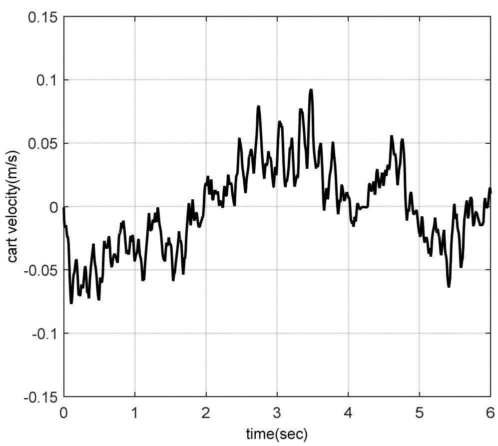

Fig. 11. Cart velocity in upright position.

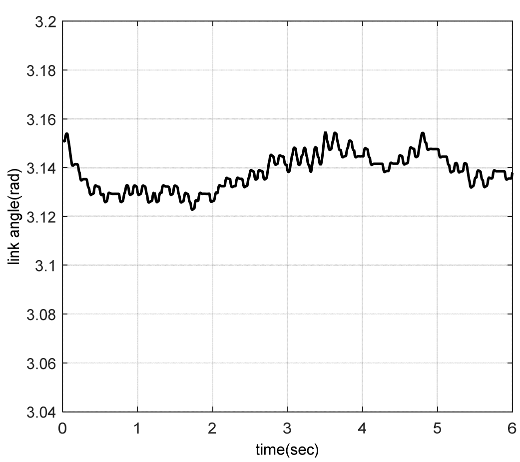

Fig. 12. Link angle in upright position.

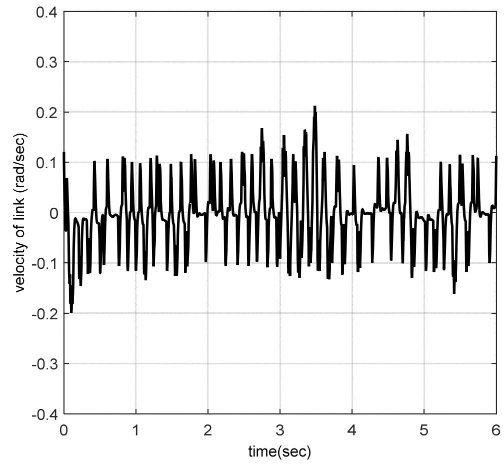

Fig. 13. Link velocity in upright position.

5.5 Pendulum swing up and stabilization

In this part, the cart velocities obtained in section 4 with optimal control method and inversion based method are used to swing up the pendulum. As the link reaches the neighborhood of the inverted status, the controller is switched from open-loop feedforward control to the closed loop feedback control. After swinging up mode, the observer and controller tuned in the previous test are used to stabilize the pendulum. The switching time between two controllers for the inversion-based method is obtained to be 0.96s and for the optimal method is obtained to be 0.9s.

The snapshots of the optimal swing up maneuver in twenty sequences with time step 0.065 sec are shown in Fig. 14. At first frame, the link and cart are at the reference point, and in the last frame, the link reaches upright position and the cart returns to its initial position. The simulation and experimental results of inversion-based swing up and optimal swing up are shown in Fig. 15 and Fig. 16 respectively. The switching time in which feedforward controller switches to stabilizing controller is also shown in these figures. As it can be seen from these figures, the experimental results are very close to the simulation results.

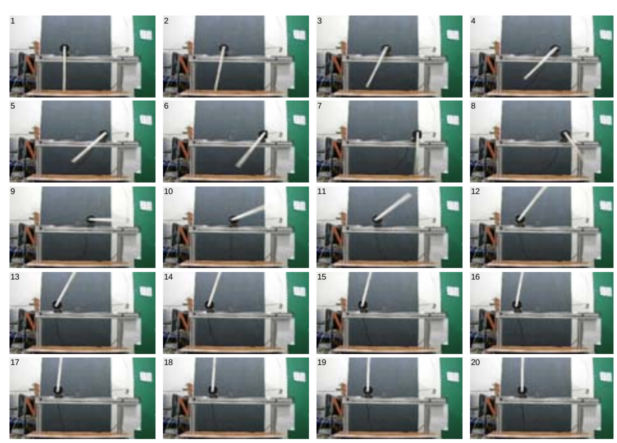

Fig. 14. Snapshots of the optimal swing up maneuver.

By measuring the motor current and multiplying it by the motor torque constant, the torque produced by the motor can be calculated. Dividing the obtained torque by radius of the pulley gives the applied force to the cart. The force exerted on the cart obtained of simulation and experimental results for inversion based swing up and optimal swing up beside the switching time are shown in Fig. 17 and Fig. 18, respectively.

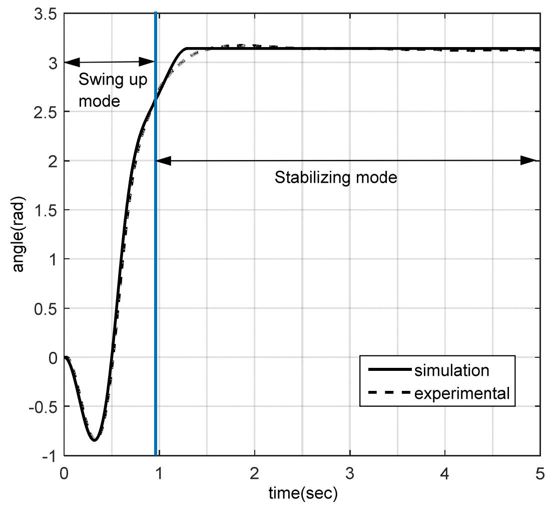

Fig. 15. Link angle for inversion-based swing up.

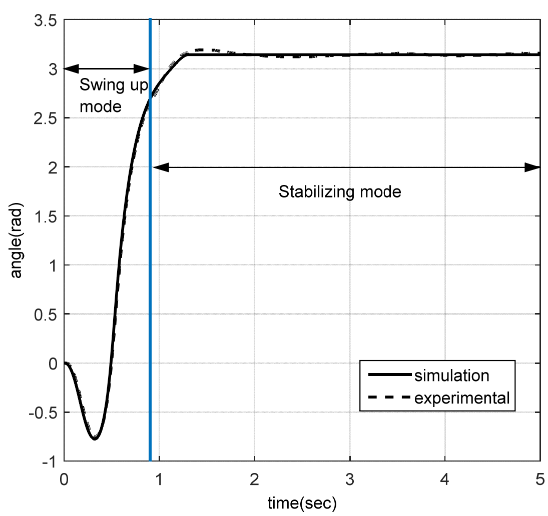

Fig. 16. Link angle for optimal swing up.

In order to compare the objective function arising from the two different swing up methods, the following performance index can be defined.

$$J = 0.5\left(\frac{k_t}{r}\right)^2 \int_0^{1.3} i_m{}^2 dt, \tag{40}$$

where k_t is motor torque constant which is equal to $0.8\, N.m/A$, r is the radius of the pulley and i_m is measured current. Since the measured current is often a noisy signal, the swing up test is done for eight times. The performance indices obtained of optimal swing up and inversion based swing up for eight tests are listed in the first and second column of **Table 2**. **Table 2** shows that the values of the performance index for all eight tests are reduced by using the optimal swing up strategy. The mean value of the first column is 304.8 and the mean value of the second column is 361.5. The results are summarized in Table 3. From the experimental results, the performance index for optimal swing up has reduced approximately 15.6 percent in comparison with the inversion-based method. For the inversion-based method, the error percent between the experimental and simulation is 2.4% and for the optimal method, this error is 14% which shows the greater sensitivity of the proposed method in comparison with inversion-based one. As a future work the robust optimal trajectory planning approach (Boscariol et al., 2016) can improve the robustness of the proposed method.

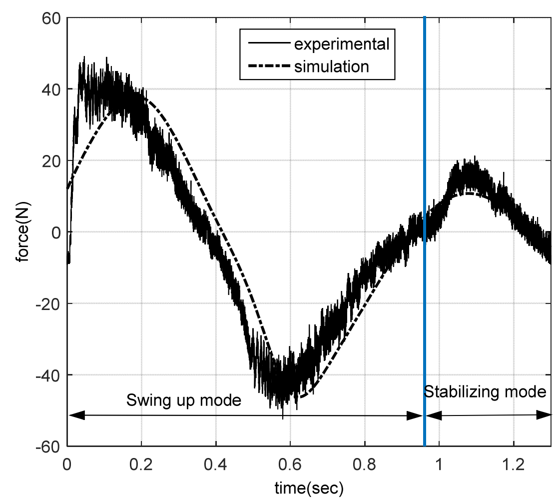

Fig. 17. Experimental and simulated force for inversion-based swing up.

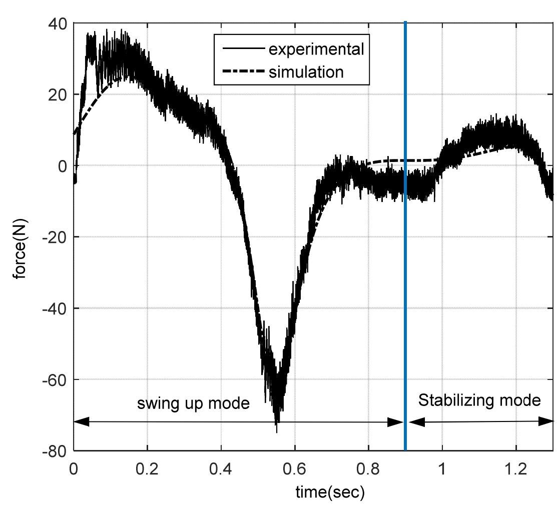

Fig. 18. Experimental and simulated force for optimal swing up.

Table 3 indicates a greater sensibility of the proposed control approach than of the inversion-based one; the experimental validation shows that the performances are degraded of 14% compared to 2.4% performance degradation for the inversion-based method. The work can be improved by a robustness analysis of the control solution.

Table 2. Performance indices obtained from conducted. Experiments.

Optimal method	Inversion based method
310.45	370.09
311.04	369.05
305.80	360.69
302.01	360.63
304.13	361.44
304.03	359.45
300.85	356.15
300.57	354.47

Table 3. The reduction and error percent of conducted experiments.

Performance index	Inversion based method	Optimal method	Reduction percent
experimental	361.5	304.8	15.6%
simulation	370.3	261.8	29.3%
Error percent	2.4%	14%	

6. CONCLUSION

In this paper, swing up of SIP with minimum effort is considered. To this end, the optimal swing up is formulated as the optimal control problem. To solve it the indirect approach based on FTCV is used that leads to a TPBVP solved numerically with the MATLAB function bvp4c. From the simulation results for the SIP, approximately 30% reduction in the used effort is observed in comparison with the inversion-based method. In order to verify the method experimentally, an experimental SIP is constructed. However, the experimental realization of swing up needs the stabilization of the SIP in the upright position. To this end, the LQR state feedback controller is designed to stabilize the system besides the high gain observer to estimate the link angular velocity. In the complete swing up maneuver, the feedforward control is used to steer the pendulum from pendant position to the neighborhood of upright position, and the designed LQR regulator is used to hold the pendulum in the inverted state. Finally, the swing up maneuver is performed several times, and the performance index is obtained by measuring the motor current in each test. It is shown that the performance index of optimal swing up in comparison with inversion based swing up is reduced 29.3% in simulation and 15.6% in experiments.

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