Controller with time-delay to stabilize first-order processes with dead-time

Francisco Medina-Dorantes*, Raúl Villafuerte-Segura**
Baltazar Aguirre-Hernández*

** Universidad Autónoma Metropolitana-Iztapalapa (UAM-Izt), México*
(e-mail: bahe@xanum.uam.mx).
*** Universidad Autónoma del Estado de Hidalgo (UAEH), México*
(e-mail: villafuerte@uaeh.edu.mx)

Abstract: This paper focuses on the stability analysis of analytic functions with two transcendental terms in order to obtain parameters that guarantee an exponential decay rate σ in the response of the linear time-invariant system associated with the analytic function. As a consequence of this stability analysis, analytic expressions to tune all of the gains of a controller with time-delay action called Proportional Integral Retarded (PIR) control law that σ-stabilizes a first-order process with dead-time are obtained. To illustrate the effectiveness of the theoretical results proposed, an application on a Quanser thermal platform is given. Furthermore, a comparison with a classical PID control law is made.

Keywords: D-composition method, first-order system, time delay systems, stability analysis, stabilizing feedback.

1. INTRODUCTION

It is common knowledge that most of the real-life systems (like industrial processes) have inherent delays in its dynamics, this is why stability analysis and controller design for time delay systems have become topics of great importance in practical applications. Typically, this stability analysis is carried out within the framework of two approaches: time domain and frequency domain. In the first approach, the analysis is based on the well-known Lyapunov's criteria using linear matrix inequalities (LMI's) via convex optimization (Krasovskii, 1956; Razumikhin, 1956; Gu et al., 2003; Kharitonov, 2013). However, this approach only provides sufficient conditions which are usually conservative or even non-existent. In the second approach, the analysis is based on the root locus of its corresponding characteristic equation (an analytic function with transcendental terms or a quasi-polynomial) in the complex plane. In the last decade special attention has been paid to the temporal approach while the frequency approach has been abandoned. However, unlike the temporal approach, the frequency approach can provide necessary and sufficient conditions which are not conservative. Even more, the feasibility of the LMI's is usually non-existent, when they have parameters tuned by a frequency method.

It is generally accepted that about 90% of industrial processes can be approximated by dynamical models of first or second order with dead-time (Silva et al., 2005). Due to its simple structure and ease of implementation, the most used control law for tracking and stabilizing this kind of systems is the conventional PID controller in conjunction with its different actions (P, I, PI, PD), see (Astrom and Hagglund, 1995). In the past decade, several techniques for tuning the parameters of the classical control laws PI or PID to stabilize first or second order processes with dead-time have been proposed (O'Dwyer, 2009). The omission of the derivative action is recommended because of its poor performance against high-frequency noise (Farkh et al., 2009a; Farkh et al., 2009b; Hwang and Hwang, 2004; Oliveira et al., 2005; Roy and Iqbal, 2005; Wang et al., 2014; Marquez-Rubio et al., 2014). The PID has proven to be very efficient when the dead-time is small while the response of the closed-loop system is usually poor when the dead-time is large (Silva et al., 2001). In last case, it is convenient to introduce a dead-time compensating structure. The best-known dead-time compensating structures are the Smith predictor (Smith, 1957), and its extensions (Palmor, 1996; Watanabe and Ito, 1981; O'Dwyer, 2005).

The derivative action is of great importance in a classical control law. Therefore, the use of encoders, observers, estimators or high-pass filters is usually an alternative to the use of this action in the control laws. However, the implementation of this action has at least three natural disadvantages. Firstly, the estimation algorithms typically increase the controller design. Secondly, the application of filters or compensators often increase the order of the closed-loop system. Thirdly, the use of measurement tools (encoders) are usually very noisy. Another alternative to the derivative action is the use of delays in the control law (retarded action). The controllers with time-delays have a simple structure and are easy to implement, like the PID. The controllers with retarded action have proven to be superior to those employing a derivative action. Among the advantages of these controllers are the following: noise attenuation, do not require estimators or filters to approximate the time derivative, provide soft control signals so do not damage actuators, and their numerical implementation is computationally more efficient than other low-order controllers, like the PID type (Abdallah et al., 1993; Bleich, 1996; de Souza Vieira, 1996; Just et al., 1997; I. Hong Suh, 1980; Swisher and Tenqchen, 1988; Zhong, 2002; Gu et al., 2005; Olgac and Cavdaroglu, 2011).

The deliberate application of time-delays to stabilize a system is not a new topic, one of the most important contributions in this regard is the work of (Pyragas, 1992), which promoted a line of research known as TDFC (Time-delayed feedback control). Recently in (Villafuerte et al., 2013) a method for tuning the gains of a proportional retarded (PR) control law to σ-stabilize a second order system is proposed. While in (Ramirez et al., 2016) the above concept is extended to a PIR control law for second-order systems without dead-time. In (Villafuerte and Ortega, 2015) the tuning of a convex sum of PR controllers is proposed and in (Ramírez et al., 2015) a control law for providing exponential estimates as well as a guaranteed cost for a class of nonlinear time delay systems is presented. A new backstepping result for time-varying systems with one delay in the input is given in (Mazenc and Malisoff, 2016). In (Hernández-Díez et al., 2017) the design of P-δ controllers for single-input-single-output linear time-invariant systems without time-delays is proposed. However, until now this methodology has only been applied to delay-free linear time-invariant systems. Thus, the tuning of a controller with time-delay to stabilize linear time-invariant systems with dead-time is a topic with areas of opportunity.

In this paper, the stability analysis of a class of analytic functions with two transcendental terms in order to obtain exact analytic expressions for tuning all of the parameters of the PIR control law that renders σ-stabiliy of first-order processes with dead-time is presented. The tuning is a direct consequence of a stability analysis of the characteristic equation of the closed-loop system of a first-order process with dead-time and a PIR control law. Thus, the characteristic equation analyzed is an analytic function or quasi-polynomial with two transcendental terms of the form

$$Q(s,k_p,k_i,k_r,h,\tau) = P_1(s,k_p,k_i)+P_2(s,k_p,k_i)\mathbf{e}^{-\theta s}+P_3(s,k_r)\mathbf{e}^{-(\theta+h)s},$$

where $P_i(\cdot)$, $i=1,2,3$, are polynomials with real coefficients, $s\in\mathrm{C}$, $k_p,k_i,k_r,h\in\Re$, C denotes the set of complex numbers, and $\Re$ denotes the set of real numbers. The stability analysis is done in the frequency domain using the D-composition method (Neimark, 1949) and the root continuity property (Michiels and Niculescu, 2014; Xu-Guang Li, 2015). The parameters (k_p,k_i,k_r,h) of control law are determined so that the solution of the closed-loop system has a rate of decay σ. To achieve this, exact analytic expressions are given to ensure that the corresponding quasi-polynomial has a root of multiplicity at least three at $s=-\sigma$. The results proposed here are compared with a classical PID control law using two tuning approaches. The purpose of the comparison is to show that the PIR controller can be an efficient alternative to the use of classic controllers such as the PID. In addition, due to its natural structure, the PIR controller can generate a better performance in the absence of derivative actions and in the presence of noise from the processes. Further, these control laws are implemented on a Quanser thermal platform.

Part of the theoretical results and experimental studies presented by our paper have been developed during the preparation of the Master Thesis in (Medina-Dorantes F. I., 2016).

The paper is organized as follows. The system's description and preliminary results are presented in Section 2. The tuning of the PIR controller to σ-stabilize a first-order processes with dead-time is stated in Section 3. In Section 4, an illustration of the theoretical results obtained in the previous section is presented. Finally, some concluding remarks in Section 5 are given.

2. PRELIMINARY RESULTS

In this section, the statement of the problem, some definitions and results concerning the stability and stabilization of time delay systems are presented.

2.1 Problem Statement

Consider the following first-order process with a dead-time

$$G(s)=\frac{K}{T\,s+1}\mathbf{e}^{-\theta s}, \tag{1}$$

where $K\in\Re^+$ is the steady-state gain of the plant, $\theta\in\Re^+$ is the dead-time or time-delay, and $0\neq T\in\Re$ is the time constant of the plant, so the free time delay open-loop system is not assumed to be either stable or unstable.

Now, consider the following PIR control law

$$C(s)=k_p+\frac{k_i}{s}+k_r\,\mathbf{e}^{-hs}, \tag{2}$$

where k_p, k_i, and $k_r\in\Re$ are the proportional, integral and retarded gains, respectively, and $h\in\Re^+$ is a time-delay. Thus, the transfer function of the closed-loop system (1)-(2) is

$$H(s)=\frac{C(s)G(s)}{1-C(s)G(s)} = \frac{\left(K\,k_p s+K\,k_i\right)\mathbf{e}^{-\theta s}+K\,k_r s\,\mathbf{e}^{-(\theta+h)s}}{Ts^2+s+\left(K\,k_p s+K\,k_i\right)\mathbf{e}^{-\theta s}+K\,k_r s\,\mathbf{e}^{-(\theta+h)s}}, \tag{3}$$

and the characteristic quasi-polynomial of (3) is

$$q(s)=Ts^2+s+K\left(k_p s+k_i\right)\mathbf{e}^{-\theta s}+Kk_r s\,\mathbf{e}^{-(\theta+h)s}. \tag{4}$$

It is well known that the inclusion of a time-delay in a control law (retarded action) can contribute to the stabilization of a system, whereby a PIR control law (2) is an appealing alternative to the classical PID control laws for stabilizing a first-order process of the form (1). The main problem of this kind of controllers is to tune its parameters (k_p,k_i,k_r,h).

Besides, it is not enough to determine the parameters under which the system can be stabilized but also to determine the parameters that better stabilize the system. However, it is clear that a criterion for choosing the parameters of a controller that better stabilize a system is a relative one. In this paper, our criterion is related to the σ-stabilization of the closed-loop system and the assignment of dominant roots with negative real part. Results below are developed to grasp this concept.

2.2 On stability of time delay systems

In this subsection basic definitions and concepts of time delay systems are presented.

Definition 1. (Bellman and Cooke, 1963) The closed-loop system (1)-(2) is said to be stable if its corresponding characteristic quasi-polynomial (4) satisfies

$$\mathrm{Re}(s) < 0$$

for all $s \in \mathrm{C}$ such that $q(s) = 0$, where $\mathrm{Re}(s)$ denotes the real part of s.

Lemma 1. (Bellman and Cooke, 1963) Consider a quasi-polynomial of the form (4) and let

$$\alpha_0 = \max_{i=1,\ldots,\infty} \{\mathrm{Re}(s_i) : \; q(s_i) = 0; \; s_i \in \mathrm{C}\}. \tag{5}$$

Then for any $\alpha \geq \alpha_0$, there is a constant $L > 0$ such that the solution $x(t,\varphi)$ of the closed-loop system (1)-(2) satisfies the inequality

$$\| x(t,\varphi) \| \leq L\, \mathbf{e}^{\alpha t} \| \varphi \|_{(\theta+h)}, \tag{6}$$

where φ is the initial condition in the Banach space $C([-(\theta+h),0],\Re)$ with norm

$$\| \varphi \|_{(\theta+h)} = \max_{\tau \in [-(\theta+h),0]} \| \varphi(\tau) \|_2 \,.$$

Using Definition 1 and Lemma 1, we will be say that the parameters (k_p, k_i, k_r, h) of the PIR control law (2) σ-stabilize the first-order process with dead-time (1) if the quasi-polynomial (4) satisfies the condition

$$\alpha_0 = -\sigma, \; \sigma \in \Re^+.$$

Thus σ-stability implies that the system response (1)-(2) has an exponential decay σ.

In the frequency domain approach there are several techniques for tuning the gains of a control law to stabilize a system. The most popular are Bode, Nyquist, Nichols, root locus, Hermite-Biehler theorem and D-composition method. In what follows, the D-composition method is used to obtain the stability and instabilities regions of the quasi-polynomial.

Lemma 2. Consider the closed-loop system (1)-(2) where $0 \neq k_i,\; k_p \in \Re$ are given. The crossing boundaries of the quasi-polynomial in the $h - k_r$ parameter plane are defined by the parametric equations

$$h(\omega) = \frac{1}{\omega} \arctan\left(\frac{T\omega^2 - K\big(k_p \omega \sin(\theta\omega) + k_i \cos(\theta\omega)\big)}{-\omega - K\big(k_p \omega \cos(\theta\omega) - k_i \sin(\theta\omega)\big)} \right) + \frac{n\pi}{\omega} - \theta, \tag{7}$$

$$k_r(h(\omega),\omega) = \frac{\omega + K\big(k_p \omega \cos(\theta\omega) - k_i \sin(\theta\omega)\big)}{-K\,\omega \cos\big((\theta+h)\omega\big)}, \tag{8}$$

where $n = 0, \pm 1, \pm 2, , \ldots$ and $\omega \in \Re^+$.

Proof. Observe that $s = 0$ is a root of the quasi-polynomial (4) if $k_i = 0$, which is not considered at this time. Now let $s = j\omega$ with $\mathrm{j}^2 = -1$. Then

$$\begin{aligned} q(j\omega) \;&=\; -T\omega^2 + j\omega + K\big(k_p \omega j + k_i\big)\mathbf{e}^{-\theta\omega j} \\ &\quad + K k_r\, \omega j\, \mathbf{e}^{-(\theta+h)\omega j} = 0, \end{aligned}$$

if

$$\begin{aligned} \mathrm{Re}\big(q(j\omega)\big) &= -T\omega^2 + K\big(k_p\, \omega \sin(\theta\omega) + k_i \cos(\theta\omega)\big) \\ &\quad + K k_r\, \omega \sin\big((\theta+h)\omega\big) = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} \mathrm{Im}\big(q(j\omega)\big) &= \omega + K\big(k_p\, \omega \cos(\theta\omega) - k_i \sin(\theta\omega)\big) \\ &\quad + K k_r\, \omega \cos\big((\theta+h)\omega\big) = 0. \end{aligned} \tag{10}$$

It follows rom the above equations that

$$\tan\big((h+\theta)\omega\big) = \frac{-T\omega^2 + K\big(k_p \omega \sin(\theta\omega) + k_i \cos(\theta\omega)\big)}{\omega + K\big(k_p \omega \cos(\theta\omega) - k_i \sin(\theta\omega)\big)}. \tag{11}$$

Accordingly, equations (7) and (8) follow from equations (11) and (10), respectively.

An application of Lemma 2 with parameters $T = 36$, $K = 0.9$, $\theta = 1$, $k_p = 0.8634$, and $k_i = 0.62$ leads to Fig. 1 that shows the stability and instability regions in the (h, k_r) parameter plane of the quasi-polynomial. (x) denotes roots with positive real part, (x) denotes roots with negative real part and the gray area (■) contains the parameters (h, k_r) where the quasi-polynomial (4) is stable.

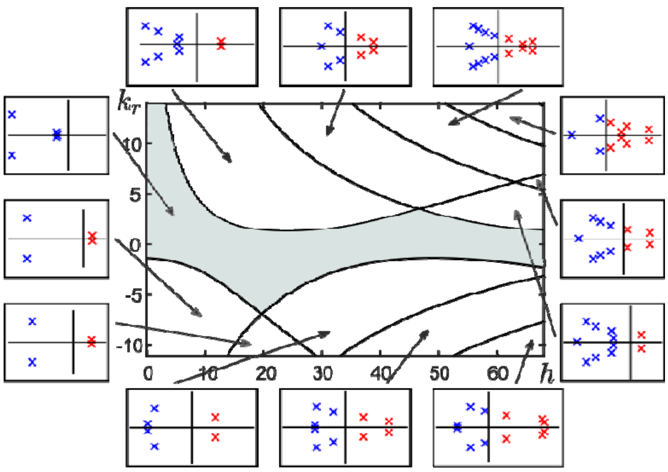

Fig. 1. Crossing boundaries of quasi-polynomial (4).

Now it is natural to ask, what are the parameters of the gray area in Fig. 1 that stabilize a first-order process with dead-time with the maximum exponential decay in its response?

To answer this question, some results for tuning the parameters of a PIR controller and σ-stabilization of are presented below.

3. TUNING OF A PIR CONTROL LAW

In this section, the σ-stability regions and a geometric characterization of the maximum exponential decay achieved by a quasi-polynomial of the form (4) are formulated. Because of the analysis of the previous regions, analytic equations to tune the parameters (k_p, k_i, k_r, h) a PIR control law are proposed.

3.1 σ-Stability Regions

By making the substitution $s \to s - \sigma$, the quasi-polynomial (4) becomes

$$q_\sigma(s) = T(s-\sigma)^2 + s - \sigma + K\left(k_p(s-\sigma) + k_i\right)\mathbf{e}^{-\theta(s-\sigma)} + K\,k_r(s-\sigma)\,\mathbf{e}^{-(\theta+h)(s-\sigma)}. \tag{12}$$

Thus σ-stability analysis of the quasi-polynomial (4) is reduced to an analysis of marginal stability of the quasi-polynomial (12) for any given $\sigma > 0$ and the following result can be enunciated.

Lemma 3. Consider the closed-loop system (1)-(2), where $0 \neq k_i$, $k_p \in \Re$, $\sigma \in \Re^+$ are given. Then the σ-stability regions of the quasi-polynomial (4) in the (h, k_r) parameter plane are delimited by the following crossing boundaries: when $s = 0$,

$$k_r(h) = \frac{T\sigma^2 - \sigma + \phi_2}{K\,\sigma \mathbf{e}^{(\theta+h)\sigma}}, \quad h \in \Re^+. \tag{13}$$

When $s = i\omega$,

$$\bar{h} := h(\omega) = \frac{\left(\arctan\left(\frac{\sigma M + \omega N}{\sigma N - \omega M}\right) + n\pi\right)}{\omega} - \theta, \tag{14}$$

$$k_r(\bar{h}, \omega) = \frac{-M\,\mathbf{e}^{-(\theta+\bar{h})\sigma}}{K\left[\omega \sin(\omega(\theta+\bar{h})) - \sigma\cos(\omega(\theta+\bar{h}))\right]}, \tag{15}$$

where $n = 0, \pm1, \pm2, \ldots$, $\omega \in \Re^+$,

$$\begin{aligned} M &= \alpha + \phi_1 \sin(\theta\omega) + \phi_2\cos(\theta\omega), \\ N &= \beta + \phi_1\cos(\theta\omega) - \phi_2 \sin(\theta\omega) \end{aligned} \tag{16}$$

with $\alpha = T(\sigma^2 - \omega^2) - \sigma$, $\beta = \omega(1 - 2T\sigma)$, $\phi_1 = \mathbf{e}^{\theta\sigma} K\,k_p\,\omega$, $\phi_2 = \mathbf{e}^{\theta\sigma} K\left(k_i - k_p\sigma\right)$, and $\varphi = \arctan(\sigma/\omega)$.

Proof. Note that the crossing boundaries of the quasi-polynomial (12) delimit the σ-stability region of the quasi-polynomial (4). Observe that $s = 0$ is a root of the quasi-polynomial (12) if

$$q_\sigma(0) = T\sigma^2 - \sigma + \phi_2 - \sigma K\,k_r \mathbf{e}^{(\theta+h)\sigma} = 0. \tag{17}$$

Thus (13) is obtained by solving the above equation for k_r. Now suppose that $s = j\omega$. Then $q_\sigma(j\omega) = 0$ if

$$0 = \mathrm{Re}\left(q_\sigma(j\omega)\right) = \alpha + \phi_1\sin(\theta\omega) + \phi_2\cos(\theta\omega) + K\,k_r\,\mathbf{e}^{(\theta+h)\sigma}\left(\omega\sin\left((\theta+h)\omega\right) - \sigma\cos\left((\theta+h)\omega\right)\right), \tag{18}$$

and

$$0 = \mathrm{Im}\left(q_\sigma(j\omega)\right) = \beta + \phi_1\cos(\theta\omega) - \phi_2\sin(\theta\omega) + K\,k_r\,\mathbf{e}^{(\theta+h)\sigma}\left(\omega\cos\left((\theta+h)\omega\right) + \sigma\sin\left((\theta+h)\omega\right)\right), \tag{19}$$

where α, β, ϕ_1 and ϕ_2 are given by (16). Since σ and ω are positive, then the following identities holds

$$\omega\sin\left((\theta+h)\omega\right) - \sigma\cos\left((\theta+h)\omega\right) = \sqrt{\omega^2+\sigma^2}\,\sin\left((\theta+h)\,\omega - \varphi\right), \tag{20}$$

and

$$\omega\cos\left((\theta+h)\omega\right) + \sigma\sin\left((\theta+h)\omega\right) = \sqrt{\omega^2+\sigma^2}\,\cos\left((\theta+h)\,\omega - \varphi\right) \tag{21}$$

Substitution of this identities into (18) and (19) gives

$$\sin\left((\theta+h)\omega - \varphi\right) = \frac{-\alpha - \phi_1\sin(\theta\omega) - \phi_2\cos(\theta\omega)}{K\,k_r\,\mathbf{e}^{(\theta+h)\sigma}\sqrt{\omega^2+\sigma^2}}, \tag{22}$$

$$\cos\left((\theta+h)\omega - \varphi\right) = \frac{-\beta - \phi_1\cos(\theta\omega) + \phi_2\sin(\theta\omega)}{K\,k_r\,\mathbf{e}^{(\theta+h)\sigma}\sqrt{\omega^2+\sigma^2}}. \tag{23}$$

From equations (22) and (23),

$$\tan\left((\theta+h)\omega - \varphi\right) = \frac{M}{N},$$

where M and N are defined in (16). Thus

$$h = \left(\arctan\left(\frac{M}{N}\right) + \arctan\left(\frac{\sigma}{\omega}\right) + n\pi\right)/\omega - \theta, \quad n = 0, \pm1, \pm2, \ldots.$$

The result follows using trigonometric identities, and algebraic manipulations.

An application of Lemma 3 with parameters $T = 36$, $K = 0.9$, $\theta = 1$, $k_p = 0.8635$, and $k_i = 0.62$ produces the Figs 2 and 3 that show the σ-stability regions of quasi-polynomial (4) when $\sigma \in [0, \sigma^*]$. In this case $\sigma^* = 0.25$.

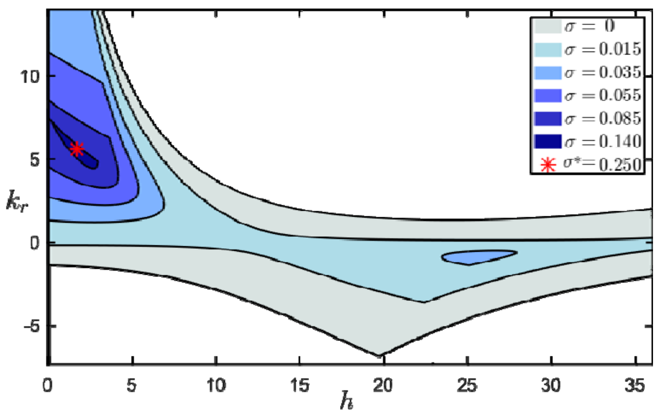

Fig. 2. σ-stability regions of quasi-polynomial (4).

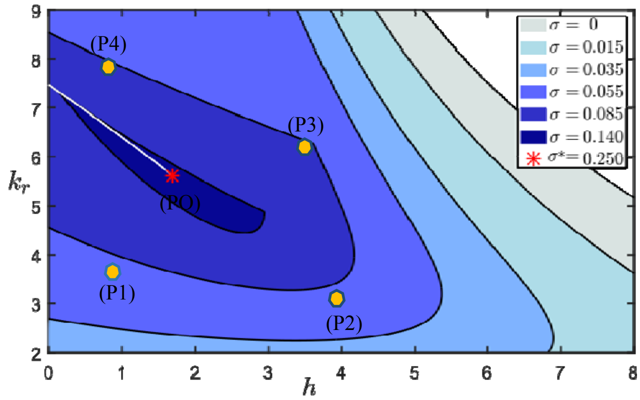

Fig. 3. Zoom in to σ-stability regions of (4).

It is clear that the σ-stability regions are concentric and that they collapse to a point as σ tends to $\sigma^* = 0.25$. Therefore, it can be assumed that σ^* is the maximum exponential decay achieved by closed-loop system (1)-(2) for k_p and k_i given. The coordinates of the collapse point $(h,k_r) = (1.6757,5.6166)$ are the parameter values of the retarded action for the PIR control tuning. By a root locus of the parameters of Fig. 3, it can be observed that the quasi-polynomial (4) has three dominant roots at $-\sigma^*$, when $(k_p,k_i,k_r,h) = (0.8635,0.62,5.6166,1.6757)$. Therefore, the σ-stabilization of the closed-loop first-order process (1) with the PID controller (2) is guaranteed, using the previous parameters (k_p,k_i,k_r,h). In Figs. 2 and 3, a complete map of all the parameters h vs k_r of the PIR controller that stabilizes a first-order process with dead-time is shown, for k_p and k_i given. While in Fig. 3a the performance of the system response with respect to the selection of the points (h,k_r) used in the PIR controller is presented. Here, the points employed for the exemplification are the points (P1), (P2), (P3), (P4) and (PO) given in Fig. 3. We choose the parameters (h,k_r) such that the response of the closed-loop system has the maximum exponential decay σ*, this point is (PO). However, the user of the PIR controller has the opportunity to choose the parameters of the control law that best suit to obtain the desired response of the closed-loop system, see Fig. 3a.

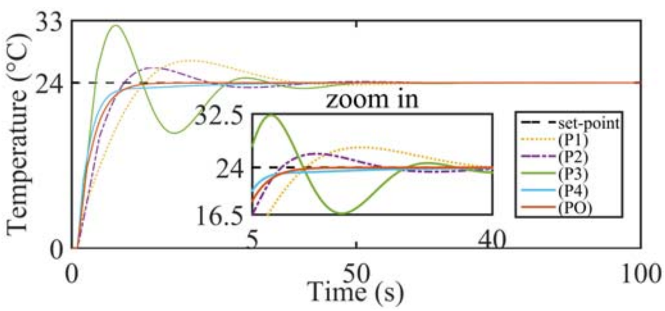

Fig. 3a. System response (1)-(2) when σ, k_p and k_i are fixed and k_r, h are varied.

Next the equations of h and k_r are determined analytically, when k_p and k_i are given.

3.2 Analytical equations for tuning a PID control law

Lemma 4. Consider the closed-loop system (1)-(2), where $0 \neq k_i,\ k_p \in \Re$ and $\sigma > 0$ are given. Then, quasi-polynomial (4) has a root of multiplicity at least two at $s = -\sigma$ if

$$h = \frac{Kk_i\mathbf{e}^{\sigma\theta} + T\theta\sigma^3 - (T+\theta)\sigma^2}{\left((K\sigma k_p - Kk_i)\mathbf{e}^{\sigma\theta} - T\sigma^2 + \sigma\right)\sigma}, \tag{24}$$

$$k_r = \frac{K(k_i - k_p\sigma)e^{\theta\sigma} + T\sigma^2 - \sigma}{K\sigma e^{(\theta+h)\sigma}}. \tag{25}$$

Proof. The quasi-polynomial (4) has two dominant roots at $s = -\sigma$ if the quasi-polynomial (12) has two dominant roots at $s = 0$. The conditions $q_\sigma(0) = 0$, and $\frac{d}{ds}q_\sigma(s)|_{s=0} = 0$ become

$$0 = T\sigma^2 - \sigma + \mathbf{e}^{\sigma\theta}K(k_i - \sigma k_p) - Kk_r\sigma\mathbf{e}^{(\theta+h)\sigma}, \tag{26}$$

and

$$\begin{aligned} 0 = &-2T\sigma + 1 + K\mathbf{e}^{\theta\sigma}\left((k_p\sigma - k_i)\theta + k_p\right) \\ &+ Kk_r\mathbf{e}^{(\theta+h)\sigma}\left((\theta+h)\sigma + 1\right), \end{aligned} \tag{27}$$

Solving equation (27) for k_r, we obtain

$$k_r = \frac{2T\sigma - 1 - K\mathbf{e}^{\theta\sigma}\left((k_p\sigma - k_i)\theta + k_p\right)}{K\mathbf{e}^{(\theta+h)\sigma}\left((\theta+h)\sigma + 1\right)},$$

and substituting this into (26) yields

$$T\sigma^2 - \sigma + \mathbf{e}^{\sigma\theta}K(k_i - \sigma k_p) - \frac{2T\sigma - 1 - K\mathbf{e}^{\theta\sigma}\left((k_p\sigma - k_i)\theta + k_p\right)}{(\theta+h)\sigma + 1} = 0.$$

Then (24) follows by solving the equation above for h and (25) follows by solving equation (26) for k_r.

In Lemma 4 were obtained analytic expressions of the parameters h and k_r for which the quasi-polynomial (4) has at least two dominant roots. A white line indicates the curve (h,k_r) in Fig. 3. Note that (h,k_r) approaches (h^*,k_r^*) which in turn is the point at which the σ-regions collapse and it is there where the quasi-polynomial (4) has three dominant roots at $-\sigma^*$. The following result is formulated in this sense.

Proposition 5. Consider the closed-loop system (1)-(2) with given $0 \neq k_i$ and $k_p \in \Re$. Then, quasi-polynomial (4) has a root of multiplicity at least three at $s = -\sigma$ if k_r and h satisfy the equations (24) and (25), and σ is a positive solution of the quasi-polynomial

$$\begin{aligned} f(\sigma,k_i,k_p) = &\left(a_5\sigma^5 + a_4\sigma^4 + a_3\sigma^3 + a_2\sigma^2 + a_1\sigma\right)\mathbf{e}^{\theta\sigma} \\ &+ (b_2\sigma + b_1)\mathbf{e}^{2\theta\sigma} + c\sigma^4 = 0, \end{aligned} \tag{28}$$

$$k_i \neq \frac{\sigma\left(K\mathbf{e}^{\theta\sigma}k_p - T\sigma + 1\right)}{K\mathbf{e}^{\theta\sigma}}. \tag{29}$$

Here

$$\begin{array}{ll} a_1 = 2\mathbf{e}^{\theta\sigma} K k_i, & a_5 = \theta^2 K k_p T, \\ a_2 = -4 K k_i T - 2\theta K k_i, & b_1 = 2K^2 k_i k_p, \\ a_3 = K\left(4T\theta k_i + \theta^2 k_i\right), & b_2 = -K k_i^{\ 2}, \\ a_4 = -K\left(\left(k_p + k_i T\right)\theta^2 + 2T\theta k_p\right), & c = T^2. \end{array}$$

Proof. By Lemma 2, $s = -\sigma$ is a root of multiplicity at least two of the quasi-polynomial (4). The equation $\frac{\partial^2}{\partial s^2} q_\sigma(s)|_{s=0} = 0$ implies

$$\begin{aligned} &- Kk_r\,\mathbf{e}^{(\theta+h)\sigma}\left(\left(h^2 + 2h\theta + \theta^2\right)\sigma + 2(h+\theta)\right) \\ &+ K\mathbf{e}^{\sigma\theta}\left(-\sigma\theta^2 k_p + \theta^2 k_i - 2\theta k_p\right) + 2T = 0. \end{aligned} \tag{30}$$

From (26),

$$Kk_r\sigma\,\mathbf{e}^{(\theta+h)\sigma} = -K\sigma k_p \mathbf{e}^{\sigma\theta} - K\mathbf{e}^{\sigma\theta} k_i - T\sigma^2 + \sigma.$$

Substituting this and (24) into (30) yields

$$\frac{1}{Q} f(\sigma, k_i, k_p) = 0,$$

where $f(\sigma, k_i, k_p)$ is given by (28) and

$$Q = \sigma^2\left(\left(K\sigma k_p - Kk_i\right)\mathbf{e}^{\sigma\theta} - T\sigma^2 + \sigma\right) \neq 0$$

as long as k_i satisfies (29).

Up to now, analytic expressions for tuning of the parameters h, k_r and σ for which the quasi-polynomial (4) has at least three roots at $-\sigma$ are characterized. However, it is necessary to give preliminary values to k_p and k_i. The following result provides analytic equations for tuning of the parameters $(\hat{k}_i, \hat{k}_p, \hat{h}, \hat{k}_r)$ only giving the value of a desired σ.

Proposition 6. Consider the closed-loop system (1)-(2) with characteristic quasi-polynomial (4), and let

$0 < \sigma \neq \frac{4T+\theta\pm\sqrt{8T^2+\theta^2}}{2T\theta}$ be a given constant. Then a quasi-polynomial of the form (4) has a root of multiplicity at least three at $s = -\sigma$ if the parameters $(\hat{k}_i, \hat{k}_p, \hat{h}, \hat{k}_r)$ satisfy the conditions:

$$\hat{k}_i \in (k_{i,\inf},\ k_{i,\sup}), \tag{31}$$

$$\hat{k}_p = \frac{K^2\hat{k}_i^{\,2}\mathbf{e}^{2\sigma\theta} + K\hat{k}_i\varsigma\mathbf{e}^{\sigma\theta} - T^2\sigma^4}{K\left(\left(T\sigma^5\theta^2 - \left(2T\theta + \theta^2\right)\sigma^4\right)\mathbf{e}^{\sigma\theta} + 2\hat{k}_i\mathbf{e}^{2\sigma\theta}\right)}, \tag{32}$$

$$\hat{h} = \frac{K\hat{k}_i\mathbf{e}^{\sigma\theta} + T\theta\sigma^3 - (T+\theta)\sigma^2}{\left(\left(K\sigma\hat{k}_p - K\hat{k}_i\right)\mathbf{e}^{\sigma\theta} - T\sigma^2 + \sigma\right)\sigma} \tag{33}$$

$$\hat{k}_r = \frac{K\left(\hat{k}_i - \hat{k}_p\sigma\right)e^{\theta\sigma} + T\sigma^2 - \sigma}{K\sigma e^{(\theta+\hat{h})\sigma}}, \tag{34}$$

where $k_{i,\inf} = \min\{R_1, R_2\}$, $k_{i,\sup} = \max\{R_1, R_2\}$,

$R_1 = \frac{(-T\sigma\theta+2T+\theta)\theta\sigma^3}{2K\mathbf{e}^{\sigma\theta}}$, $R_2 = \frac{(-T\sigma\theta+T+\theta)\sigma^2}{K\mathbf{e}^{\sigma\theta}}$ *and*

$$\varsigma = T\sigma^4\theta^2 - \left(4T\theta + \theta^2\right)\sigma^3 + \left(4T + 2\theta\right)\sigma^2 - 2\sigma.$$

Proof. Suppose that $\hat{k}_i$ and $\hat{k}_p$ satisfy equations (31) and (32), respectively. Note that if $\hat{k}_i \neq R_1$ then $\hat{k}_p$ given by (32) is well defined, and if $\hat{k}_i \neq R_2$, the restriction (29) is satisfied. Substituting (32) into (28) yields $f(\sigma, \hat{k}_i, \hat{k}_p) = 0$, where h and k_r are given by (33) and (34), respectively. Therefore, Proposition 5 holds, namely the quasi-polynomial (4) has one root of multiplicity at least three at $s = -\sigma$.

Next we prove that h is well defined. It follows from (33) that

$$\hat{h} = \frac{\theta^2 T\sigma^4 - \left(\theta^2 + 2T\theta\right)\sigma^3 + 2e^{\theta\sigma}K\hat{k}_i}{-T\theta\sigma^4 + (T+\theta)\sigma^3 - e^{\theta\sigma}K\hat{k}_i\sigma}, \tag{35}$$

Observe that the denominator of the expression above is different from zero if $\hat{k}_i \neq R_2$. Also, it is positive if

$$\begin{aligned} 0 &< \left(\theta^2 T\sigma^4 - \left(\theta^2 + 2T\theta\right)\sigma^3 + 2e^{\theta\sigma}K\hat{k}_i\right) \\ &\quad\times\left(-T\theta\sigma^4 + (T+\theta)\sigma^3 - e^{\theta\sigma}K\hat{k}_i\sigma\right) \\ &= A\hat{k}_i^2 + B\hat{k}_i + C, \end{aligned} \tag{36}$$

where

$$\begin{aligned} A &= -2K^2\mathbf{e}^{2\theta\sigma}\sigma, \\ B &= -K\mathbf{e}^{\theta\sigma}\sigma^3\left(\sigma^2\theta^2 T - \sigma\theta^2 - 2(T+\theta)\right) \\ C &= -\sigma^6\theta(T\theta\sigma - \theta - 2T),(T\theta\sigma - T - \theta). \end{aligned}$$

On the other hand, if $\sigma \neq \dfrac{4T+\theta\pm\sqrt{8T^2+\theta^2}}{2T\theta}$ then $\eta = T\sigma^2\theta^2 - \left(4\theta T + \theta^2\right)\sigma + 2(T+\theta) \neq 0$, and since $A < 0$, the discriminant

$$B^2 - 4AC = K^2\mathbf{e}^{2\theta\sigma}\sigma^6\eta^2 > 0,$$

is positive. So there is an open interval where $\hat{h}$ is positive. The solutions of (36) are the endpoints of this interval, and the endpoints are curiously $k_{i,\inf}$ and $k_{i,\sup}$. Therefore, $\hat{h}$ is well defined if $\hat{k}_i$ satisfies (31).

3.3 Summary to tune a PIR control law

The following steps can be used for tuning a PIR controller for first-order plants with time delay:

Step 1. Give a positive value of σ. This value corresponds to the desired σ-stability of the closed loop system (1)-(2).

Step 2. Choose a non-zero value of $\hat{k}_i$ in the interval given by (31). Take into account the root locus in Fig. 4 and the values of $\hat{k}_p$ and $\hat{h}$ given by equations (35) and (32). That is, the choice of $\hat{k}_i$ may imply that the values of $\hat{k}_p$ and $\hat{h}$ are poorly suited to the process.

Step 3. Calculate $\hat{k}_p$, $\hat{h}$ and $\hat{k}_r$ from the equations (32), (33) and (34), respectively.

Step 4. If necessary, choose another value of σ and repeat the previous steps.

3.3 Proposal for the election of k_i

As explained above, the first step to tuning is to choose a $\hat{k}_i \in (k_{i,\inf}, k_{i,\sup})$. However, it is natural to ask, which $\hat{k}_i$ is appropriate? The following is a proposal for picking $\hat{k}_i$.

According to Proposition 6, for any $\sigma > 0$ given, one value of $\hat{k}_i$ in the interval (31) must be chosen. At the same time, each value of $\hat{k}_i$ determines different values of $\hat{k}_p$, $\hat{k}_r$ and $\hat{h}$ using (32), (34) and (33), respectively.

In Fig. 4 can be seen as the roots trail are close or far from the dominant roots when $\hat{k}_i$ is incremented or decremented, respectively. Here, $T = 36$, $K = 0.9$, $\theta = 1$, $\sigma = 0.25$ and the values of $\hat{k}_i \in (k_{i,\inf}, k_{i,\sup}) = (0.43267, 1.5143)$, $\hat{k}_p$, $\hat{k}_r$ and $\hat{h}$ are given in Table 1.

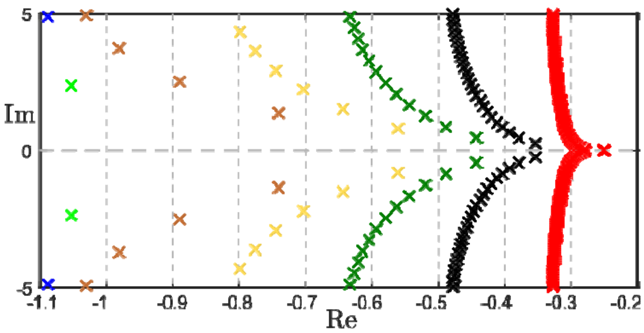

Fig. 4. Roots locus of (4) when σ is fixed and k_i is varied.

$\hat{k}_i = \{$(x) 0.5, (x) 0.66, (x) 0.82, (x) 0.97, (x) 1.1, (**x**) 1.3, (x) 1.5$\}$

Table 1. Parameters of control law (2).

$\hat{k}_i$	$\hat{k}_p$	$\hat{k}_r$	$\hat{h}$
0.5	-21.638	26.761	0.531
0.658	3.060	3.834	2.108
2.108	7.654	0.842	4.403
0.975	11.041	0.010	14.710
1.291	11.974	0.513e-4	30.867
1.450	12.714	1.497e-16	126.504

Observe that, for any σ given, when $\hat{k}_i$ is in the interval (31) the σ-stability of the closed-loop system (1)-(2) is preserved, because the dominant root remain fixed although the gain $\hat{k}_i$ is varied. However, if $\hat{k}_i$ is varied then the roots to left of the dominant root (roots trail) are affected.

4. IMPLEMENTATION OF THEORETICAL RESULTS

In this section, the implementation and validation via simulation and experimentation of the theoretical results obtained in the previous sections are presented. The simulation was done using the Matlab-Simulink R2015a while the experimentation was made using the QNET-HVACT thermal platform by Quanser shown in Fig. 5. Furthermore, to assess the effectiveness of the new control a comparison between the classical PID control law and the PIR control law is made.

Fig. 5. Quanser QNET-HVAC© Platform.

The QNET-HVACT thermal platform can be considered as a fist-order process with dead-time of the form (1), where $T = 36$, $K = 0.9$, and $\theta = 1$. Namely,

$$G(s) = \frac{0.9}{36s+1}\mathbf{e}^{-s}. \tag{37}$$

For comparison purposes, the closed-loop system of the QNET-HVACT thermal platform with the control laws PIR and PID is given. It should be mentioned that the controllers were tuned in the following way: for PID-Silva the parameters of the PID controller that had a performance similar to the PIR controller were searched, the above with the purpose to exemplify the same performance of the system response between the PID-Silva and PIR, but different saturation in the control action. While the parameters of the PID-σ controller were tuned in such a way that the closed-loop system had three dominant roots in -σ. In order to have a similar criterion to the PIR with respect to the placement of dominant roots. The PIR controller is tuned using Proposition 6. For $\sigma = 0.25$, the interval (31) is $(0.43267, 1.5143)$. Choosing $k_i = 0.62$, the gains are $k_p = 0.8635$ and $k_r = 5.6166$, and $h = 1.6757$ is the time-delay parameter of the controller (2). Meanwhile, the PID controller is tuned by using two methods:

1. PID-Silva. Tuning using the rules given in (Silva et al., 2001):

$$k_p = \frac{2T+\theta}{2K(\theta+\lambda)},\ k_i = \frac{1}{K(\theta+\lambda)},\ k_d = \frac{T\theta}{2K(\theta+\lambda)},$$

where $\lambda = 0.2T + \theta$. Here $k_p = 4.4082$, $k_i = 0.1208$, $k_d = 2.1739$.

2. PID-σ. Tuning assigning three dominant roots at $s = -\sigma$:

$$k_p = -\frac{T\sigma^3\theta^2 - 3T\sigma^2\theta - \sigma^2\theta^2 + \theta\sigma + 1}{K\mathbf{e}^{\theta\sigma}},$$

$$k_i = \frac{\sigma\left(-T\sigma^2\theta + K\mathbf{e}^{\theta\sigma}k_p + \theta\sigma + 1\right)}{2K\mathbf{e}^{\theta\sigma}},$$

$$k_d = \frac{K\mathbf{e}^{\theta\sigma}\sigma k_p - K\mathbf{e}^{\theta\sigma}k_i - T\sigma^2 + \sigma}{K\sigma^2\mathbf{e}^{\theta\sigma}}.$$

Note that the PID-σ controller parameters that guarantee the σ-stability of the closed-loop system are unique.

4.1 Simulation results

Plots of the closed-loop system of the QNET-HVACT thermal platform (37) with the controllers PIR, PID-Silva and PID-σ are shown in Fig. 6. While the signals generated by each control law are shown in Fig. 7. Observe that the convergence of the controllers PIR and PID-σ is similar, while that of the PID-Silva controller is slightly slower.

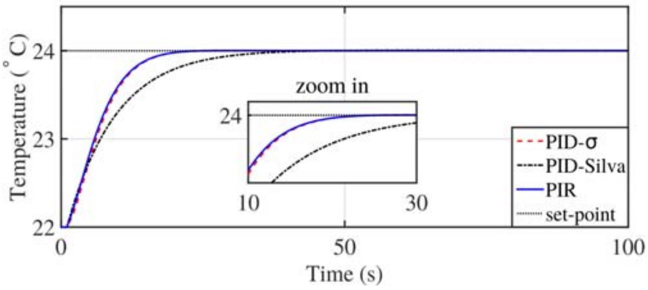

Fig. 6. Simulation system response of equation (37).

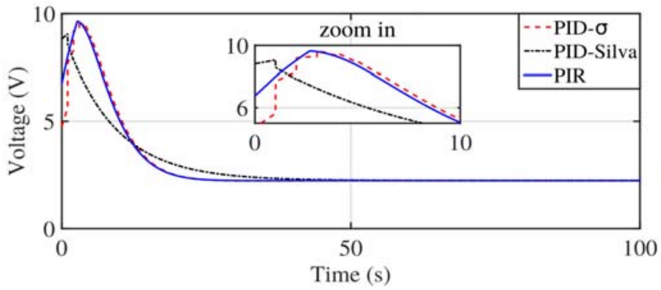

Fig. 7. Simulation control signal applied to equation (37).

4.2 Experimental results

Plots of the real responses of the closed-loop system of the QNET-HVACT thermal platform with the control laws PIR, PID-Silva and PID-σ are show in Fig. 8. Plots of the control signals applied to the platform are shown in Fig. 9. It is observed that the three system responses have similar performance. However the control signal is excessively noisy and saturated when the PID-σ control is applied. The PID-Silva saturates only in the initial stages and it has a characteristic noise. Compared with the other two control laws, the PIR control law is not saturated at any time and is not noisy. In addition, to do a better comparison of the performance of the control laws, the Table 2 shows the mean square error (MSE) of the response of the closed-loop system of the thermal platform with the controls PID-σ, PID-Silva and PIR. It is clear that the PIR control has the best performance in simulation and experimentation.

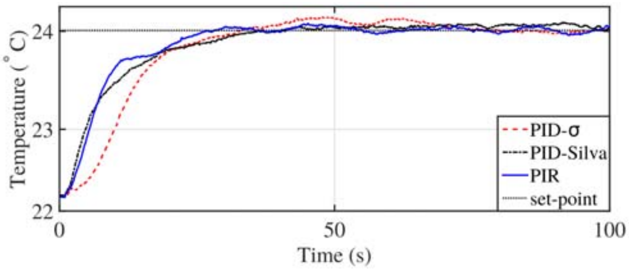

Fig. 8. Real system response of QNET-HVACT.

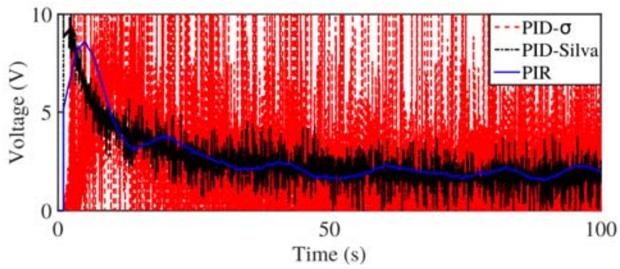

Fig. 9. Real control signal applied to QNET-HVACT.

Table 2. MSE of the system response.

	PID-σ	PID-Silva
Simulation (MSE)	0.1266	0.1402
Experimental (MSE)	3.7419	3.7528

5. CONCLUSIONS

In this paper the stability analysis of analytic functions with two transcendental terms is presented. Getting exact analytic expressions to tune the parameters (k_p, k_i, k_r, h) of a PIR control law to σ-stabilize a first-order process with dead-time is a consequence of this stability analysis. The stability analysis of the closed-loop system is carried out on its corresponding characteristic function or quasi-polynomial using the D-composition method and the root continuity property. Furthermore, the PIR controller is compared with a classical PID controller using two tuning approaches: PID-σ and PID-Silva. The parameters of the PID-Silva control were tuned in such a way that the response of the first-order process with dead-time had a similar performance to the process when use the PIR control. While the PID-s was tuned to place three dominant roots in -σ.

In general, the above results suggest that the performance of the thermal system response is relatively better using the PIR control law. This can be corroborated by measuring the MSE shown in Table 2. In addition, the PIR control law generates a smooth, noise-free and non-saturated control signal, thereby damage to actuators is less and its useful life is longer, see Fig 9. Besides, the excess of saturation in the actuators results in an increase in the energy consumption of the processes. It is important to note that the theoretical results have not been published previously and the results of the implementation have proven to be efficient and easy to apply. Accordingly, the evidence from this study suggests that the PIR controller is an alternative for tracking and stabilizing motion of first-order processes with dead-time.

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