On the Stability of Nonlinear Minimum Variance Control for a Second-Order Volterra Series Model

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Abstract: In this paper, the stability of a closed-loop system with nonlinear minimum variance controller for a second order Volterra series model is studied. It is shown that the closed-loop system with second-order Volterra series model and minimum variance control signal is a state-dependent switching system with an arbitrary switching signal. The necessary condition for asymptotic stability of this system is the stability of its all subsystems and is investigated using a linearization approach. Also, a sufficient closed-loop stability condition with nonlinear minimum variance control is introduced. It is shown that if the sufficient stability condition is violated, it can be satisfied by using a generalized output and nonlinear generalized minimum variance control.

Keywords: nonlinear minimum variance control, nonlinear generalized minimum variance control, stability, Volterra series model, state-dependent switching systems

1. INTRODUCTION

Minimum variance (MV) controllers are optimal stochastic controllers with many applications such as control performance assessment (CPA). In Minimum variance controllers, output variance of the system is minimized. It was initially introduced for control of minimum phase linear stochastic systems (K. Astrom, 1967). In order to deal with the effect of nonlinearities, MV controller has been extended to non-linear systems (Sales and Billings, 1990; Majecki and Grimble, 2004; Harris and Yu, 2007; Maboodi, Camacho and Khaki-Sedigh, 2015). Among the proposed methods in extending MV control to nonlinear systems, nonlinear minimum variance (NMV) control using the time series models approach is noticeable due to its practical implementations (Harris and Yu, 2007; Maboodi et al., 2015). However, the minimum variance control closed loop stability for Volterra series models has not been investigated.

Minimum variance controller was initially introduced for the control of minimum phase linear stochastic systems (K. Astrom, 1967). For non-minimum phase plants, the closed-loop system was unstable due to pole-zero cancelations. Later, minimum variance controller was extended to non-minimum phase systems by a minor modifications (K. J. Astrom, 1971). This basic form of MV controllers has high gains and wide bandwidth that cause large control signal variations (Jelali, 2006). The generalized minimum variance (GMV) controller was introduced to overcome these application problems (Clarke and Hastings-James, 1971). Moreover, GMV controllers overcome the stability issue in implementing MV controllers in unstable or non-minimum phase plants.

Industrial control loops inherently include nonlinearities, and linearizing non-linear models around the nominal operating point is a common solution for a small deviation from the operating point. However, linear approximation fails in a case of large deviations from the operating point. Initially, MV control was extended to nonlinear models by using the nonlinear autoregressive moving average with exogenous input (NARMAX) models (Sales and Billings, 1990). Extending MV and GMV controllers to nonlinear systems described by superposing a nonlinear system and a linear additive disturbance has been addressed by some authors (Bittanti and Piroddi, 1993; Michael J Grimble, 2005; Harris and Yu, 2007; Maboodi et al., 2015; Kazemi and Arefi, 2017). Usually, disturbance models are linear time-invariant in practice, so this representation is not a restrictive condition (Michael J Grimble, 2005). (Bittanti and Piroddi, 1993) designed a MV control for a nonlinear plant by using multilayer perceptron neural networks. Designing a nonlinear minimum variance control using nonlinear model inverses is reported in (Michael J Grimble, 2005; Alipouri and Poshtan, 2014b; Alipouri and Alipour, 2017). A vector auto regressive with exogenous input (VARX) model is used to identify a MIMO system and then a linear MV control is designed in (Alipouri and Poshtan, 2014a). It is demonstrated that some kinds of nonlinear systems can be modeled by VARX with a desired accuracy. In (Kazemi and Arefi, 2017; Pupeikis, 2014) a minimum variance control is designed for a nonlinear system with the Wiener model and furthermore in (Kazemi and Arefi, 2017) a performance assessment scheme is proposed based on this controller. Also, the predictive nonlinear minimum variance control is investigated in (Mike J Grimble and Majecki, 2010b; Mike J Grimble and Majecki, 2010a; Michael John Grimble and Majecki, 2015).

Designing MV control using the state space model was first studied in (Silveira and Coelho, 2011). (Silveira and Coelho, 2011) introduced a state space design scheme for MV control to prevent the solution of the Diophantine equation in transfer function design scheme that had a high computational cost in systems with long time delays. (Thereafter, Mike J Grimble and Majecki, 2010b) proposed a state space approach for nonlinear predictive minimum variance control introduced in (Mike J Grimble and Majecki, 2010a). In (Michael John Grimble and Majecki, 2015) a nonlinear predictive generalized minimum variance is designed for a state-dependent nonlinear system that is based on a nonlinear operator and a linear state-dependent model. In (Hur and Grimble, 2015) an observer is used instead of a Kalman filter to decrease uncertainty. This observer is designed to be used in fault monitoring applications. In (Alipouri and Alipour, 2017) a minimum variance control is designed for an independent drive electric vehicle that has a state space model. System inverse model is obtained and closed-loop stability is verified.

Nonlinear process dynamics that exhibit harmonics, asymmetric behaviour and input multiplicities can be described by a Volterra series model (FJ III, Pearson, and Ogunnaike, 2001). Using this fact, (Harris and Yu, 2007) proposed a method to design NMV control and they estimated minimum variance performance bounds using a Volterra series approximation. (Maboodi et al., 2015) extended MV and GMV to a class of nonlinear systems with an additive linear disturbance in the state space framework. In this paper, the nonlinear system is modelled by a second-order Volterra series. Then, a d-step ahead model predictor is designed based on the nonlinear Volterra series model. Control signal is derived by solving a second order equation that has two answers. It has been mentioned that the selection between these two signal controls depend on the type of plant and control strategy. In addition, two methods have been proposed; 1) selecting the control signal with the smallest absolute value 2) selecting the control signal with the smallest slew rate in the actuator. Control signal has been switched to the real part if the solution of second order equation was a complex number (Maboodi et al., 2015).

In this paper, it is shown that the strategies mentioned by (Maboodi et al., 2015) in the selection of control signals may lead to instability. Moreover, NMV control may not necessarily be a stabilizing controller. It has been proved that the selection between two (\three) control signals must depend on the stability of the system's equilibrium points. Furthermore, stabilizing NMV control by defining generalized output and replacing NGMV control has been discussed.

This paper is organized as follows. In section 2, state space representation of a system with a second order Volterra series model and a linear additive disturbance is introduced. Then, a d-step ahead predictor is obtained, and NMV (/NGMV) controller equations are derived. Section 3 discussed the stability of NMV controller by linearizing closed-loop system around operating points. In section 5, simulations are demonstrating the results of section 4. Finally, conclusions are given in section 6.

2. PROCESS DESCRIPTION

A general closed-loop system is shown in Fig. 1, where $r(k)$, $u(k)$ and $e(k)$ are the set point, control input and error signals, respectively. Unmeasured disturbance $\varepsilon(k)$ can be modelled by a zero mean white noise with variance σ_ε^2. The plant output is denoted by $m(k)$ that is described by a non-autoregressive second order Volterra series model

$$m(k) = \sum_{i=0}^{N} h_1(i)u(k-i-d) + \sum_{i=0}^{N}\sum_{j=i}^{N} h_2(i,j)u(k-i-d)u(k-j-d) \tag{1}$$

Moreover, $y(t)$ is the closed-loop system output. h_1 and h_2 are respectively linear and nonlinear term parameters, N is the common truncation order for linear and nonlinear terms (Maboodi et al., 2015).

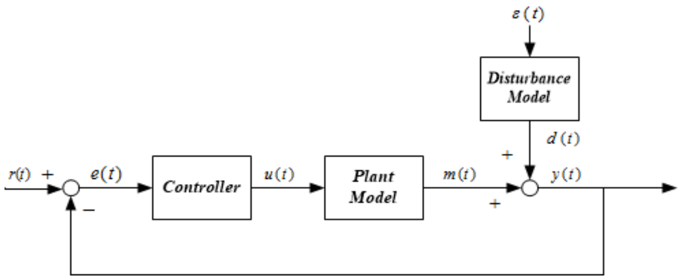

Fig. 1. General closed loop system.

State space model of the system will be obtained by choosing past input values of the Volterra series model in Eq. (1) as system states (Gruber, 2010)

$$\begin{aligned} x_1(k) &= u(k-d) \\ x_2(k) &= u(k-d-1) \\ &\vdots \\ x_N(k) &= u(k-d-N+1) \\ x_{N+1}(k) &= u(k-d-N) \end{aligned} \tag{2}$$

So nonlinear state-space model can be defined as followed

$$\begin{aligned} \boldsymbol{x_m}(k) &= \boldsymbol{A_m}\boldsymbol{x_m}(k-1) + \boldsymbol{B_m}u(k-d) \\ m(k) &= \boldsymbol{H_1}\boldsymbol{x_m}(k) + \boldsymbol{x_m^T}(k)\boldsymbol{H_2}\boldsymbol{x_m}(k) \end{aligned} \tag{3}$$

state and input matrixes are as follows

$$\boldsymbol{A_m} = \begin{bmatrix} 0 & 0 & \vdots & 0 & 0 & 0 \\ 1 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}, \quad \boldsymbol{B_m} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{(N+1)} \tag{4}$$

Volterra linear and nonlinear parameter matrixes are defined by

$$\boldsymbol{H_1} = [h_1(0) \quad h_1(1) \quad h_1(2) \quad \dots \quad h_1(N)] \in \mathbb{R}^{(N+1)}$$

$$\boldsymbol{H_2} = \begin{bmatrix} h_2(0,0) & h_2(0,1) & h_2(0,2) & \cdots & h_2(0,N) \\ 0 & h_2(1,1) & h_2(1,2) & \cdots & h_2(1,N) \\ 0 & 0 & h_2(2,2) & \cdots & h_2(2,N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_2(N,N) \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)} \quad (5)$$

It is assumed that the disturbance signal can be modelled by the following state space equations

$$\begin{aligned} \boldsymbol{x_e}(k+1) &= \boldsymbol{A_e}\boldsymbol{x_e}(k) + \boldsymbol{B_e}\varepsilon(k) \\ d(k) &= \boldsymbol{C_e}\boldsymbol{x_e}(k) + \varepsilon(k) \end{aligned} \quad (6)$$

With state matrix $\boldsymbol{A_e} \in \mathbb{R}^{M\times M}$, input matrix $\boldsymbol{B_e} \in \mathbb{R}^{M}$ and output matrix $\boldsymbol{C_e^T} \in \mathbb{R}^{M}$. Disturbance signal $\varepsilon(k)$ is unknown and must be estimated. With this assumption, we can augment disturbance states and model states as

$$\boldsymbol{X}(k) = \begin{bmatrix} \boldsymbol{x_m}(k) \\ \boldsymbol{x_e}(k) \end{bmatrix} \quad (7)$$

So the augmented state space model becomes

$$\begin{aligned} \boldsymbol{X}(k) &= \boldsymbol{A}\boldsymbol{X}(k-1) + \boldsymbol{B}u(k-d) + \boldsymbol{\Gamma}\varepsilon(k-1) \\ y(k) &= \boldsymbol{C}\boldsymbol{X}(k) + \boldsymbol{X}^T(k)\boldsymbol{H}\boldsymbol{X}(k) + \varepsilon(k) \end{aligned} \quad (8)$$

Where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_m} & \boldsymbol{0}_{(N+1)\times M} \\ \boldsymbol{0_{M\times(N+1)}} & \boldsymbol{A_e} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B_m} \\ \boldsymbol{0_{M\times 1}} \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{0}_{(N+1)\times 1} \\ \boldsymbol{B_e} \end{bmatrix} \quad (9)$$

$$\boldsymbol{C} = [\boldsymbol{H_1} \quad \boldsymbol{C_e}], \qquad \boldsymbol{H} = \begin{bmatrix} \boldsymbol{H_2} & \boldsymbol{0}_{(N+1)\times M} \\ \boldsymbol{0_{M\times(N+1)}} & \boldsymbol{0_{M\times M}} \end{bmatrix}$$

An estimation of $\boldsymbol{X}(k+d)$ is derived by using Eq. (8) recursively. It can be shown that it has two components (Maboodi et al., 2015). The first component is predictable and the second part is unpredictable.

$$\begin{aligned} \boldsymbol{X}(k+d) = \boldsymbol{A^d}\boldsymbol{X}(k) &+ \sum_{i=1}^{d} \boldsymbol{A^{(d-i)}}\boldsymbol{B}u(k-d+i) \\ &+ \boldsymbol{A^{(d-1)}}\boldsymbol{\Gamma}\varepsilon(k) \\ &+ \sum_{i=2}^{d} \boldsymbol{A^{(d-i)}}\boldsymbol{\Gamma}\varepsilon(k-1+i) \end{aligned} \quad (10)$$

Note that the final terms include future noise values. Therefore, estimation of $\boldsymbol{X}(k+d)$ with information up to k is as follows

$$\begin{aligned} \widehat{\boldsymbol{X}}(k+d) = \boldsymbol{A^d}\boldsymbol{X}(k) &+ \sum_{i=1}^{d} \boldsymbol{A^{(d-i)}}\boldsymbol{B}u(k-d+i) \\ &+ \boldsymbol{A^{(d-1)}}\boldsymbol{\Gamma}\varepsilon(k) \end{aligned} \quad (11)$$

So according to equations (8) and (11), the minimum variance predictor can be considered as

$$\hat{y}(k+d|k) = \boldsymbol{C}\widehat{\boldsymbol{X}}(k+d) + \widehat{\boldsymbol{X}}^{\boldsymbol{T}}(k+d)\boldsymbol{H}\widehat{\boldsymbol{X}}(k+d) \quad (12)$$

As augmented states are formed by the past known inputs and the unknown disturbance states, disturbance states must be estimated using an observer as,

$$\begin{aligned} \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k) &= \boldsymbol{A_e}\overline{\boldsymbol{x}}_{\boldsymbol{e}}(k-1) + \boldsymbol{B_e}\hat{\varepsilon}(k-1) \\ \hat{\varepsilon}(k-1) &= y(k-1) - \boldsymbol{H_1}\boldsymbol{x_m}(k-1) \\ &\quad - \boldsymbol{x_m^T}(k-1)\boldsymbol{H_2}\boldsymbol{x_m}(k-1) \\ &\quad - \boldsymbol{C_e}\overline{\boldsymbol{x}}_{\boldsymbol{e}}(k-1) \end{aligned} \quad (13)$$

Hence, the estimated process states are defined as

$$\overline{\boldsymbol{X}}(k) = \begin{bmatrix} \boldsymbol{x_m}(k) \\ \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k) \end{bmatrix} \quad (14)$$

And estimation of $\boldsymbol{X}(k+d)$ with information of observer up to k is

$$\begin{aligned} \widehat{\boldsymbol{X}}(k+d|k) = \boldsymbol{A^d}\overline{\boldsymbol{X}}(k) &+ \sum_{i=1}^{d} \boldsymbol{A^{(d-i)}}\boldsymbol{B}u(k-d+i) \\ &+ \boldsymbol{A^{(d-1)}}\boldsymbol{\Gamma}\big(y(k) - \boldsymbol{C}\overline{\boldsymbol{X}}(k) \\ &+ \overline{\boldsymbol{X}}^{\boldsymbol{T}}(k)\boldsymbol{H}\overline{\boldsymbol{X}}(k)\big) \end{aligned} \quad (15)$$

Eq. (15) can be rewritten as following

$$\widehat{\boldsymbol{X}}(k+d|k) = k_1 u(k) + k_2 \quad (16)$$

Where k_1 and k_2 are defined as

$$\begin{aligned} k_1 &= B \\ k_2 &= \boldsymbol{A^d}\overline{\boldsymbol{X}}(k) + \sum_{i=1}^{d-1} \boldsymbol{A^{(d-i)}}\boldsymbol{B}u(k-d+i) \\ &\quad + \boldsymbol{A^{(d-1)}}\boldsymbol{\Gamma}\big(y(k) - \boldsymbol{C}\overline{\boldsymbol{X}}(k) \\ &\quad + \overline{\boldsymbol{X}}^{\boldsymbol{T}}(k)\boldsymbol{H}\overline{\boldsymbol{X}}(k)\big) \end{aligned} \quad (17)$$

So, by substitution of Eq. (16) in Eq. (12), minimum variance predictor is obtained as follows

$$\hat{y}(k+d|k) = au^2(k) + bu(k) + c \quad (18)$$

Where a, b and c are defined as

$$\begin{aligned} a &= \boldsymbol{k_1^T}\boldsymbol{H}\boldsymbol{K_1} \\ &b \\ c &= \boldsymbol{C}\boldsymbol{k_2} + \boldsymbol{k_2^T}\boldsymbol{H}\boldsymbol{k_2} \end{aligned} \quad (19)$$

3. NMV & GNMV CONTROLLER DESIGN

In the minimum variance control, control signal is designed to minimize the output variance $(\hat{y}(k+d|k) - r)$. So, in the regulation case where $r = 0$, the cost index to be minimized is the variance of the d-step ahead output prediction

$$J_1 = E\left\{\big(y(k+d|k)\big)^2\right\} \quad (20)$$

Where $E\{.\}$ denotes the expectation operator. Hence, the control signal that minimizes Eq. (20) is given by (Maboodi et al., 2015)

$$u(k)=\begin{cases}-\dfrac{b}{2a} & \Delta=b^2-4ac<0\\ \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} & \Delta=b^2-4ac>0\end{cases} \tag{21}$$

According to (Maboodi et al., 2015), selection between two control signals when $\Delta>0$, depends on the type of plant and control strategy. For example, it can be selected using limitations in the slew rate of the actuator or by smallest absolute value. However, in practice, using this method does not gaurantee stability of the closed loop system that is considered in next section.

By considering control action penalty, the nonlinear generalized minimum variance (NGMV) control is introduced (Maboodi et al., 2015). In the NGMV control, the generalized output is defined as

$$\phi(k)=Pe(k)+Qu(k) \tag{22}$$

So, the control signal is obtained by minimizing the following cost defined by the generalized output

$$J_2=E\{\phi(k)^2\} \tag{23}$$

For example, if $P=-1$ and $Q=\lambda q^{-d}$, then the generalized output is defined as

$$\phi(k+d)=y(k+d)+\lambda u(k) \tag{24}$$

And therefore control signal which minimizes the cost function (23) is $u(k)$ in equation (21), replacing b by $b+\lambda$.

4. STABILITY OF NMV & NGMV CONTROLLERS

The minimum variance closed loop system by control signal defined in Eq. (21) is a switching system. Control signal switches by changing the sign of Δ that is a function of the previous inputs. In other words, Δ is a function of states, and so, switching depends on system states. Therefore, the minimum variance closed loop system by control signal defined in Eq. (21) is a state-dependent switching system with the arbitrary switching signal. Stability analysis of switching systems with arbitrary switching signals has been widely studied (Liberzon, Hespanha and Morse, 1999; Margaliot, 2009; Zhai, Xu, Lin and Michel, 2006; Lin and Antsaklis, 2009; Jouili and Benhadj Braiek, 2019). It has been proved that the stability of both systems is the first condition to guarantee asymptotic stability of the switching systems with arbitrary switching signals (Lin and Antsaklis, 2009). Hence, the closed-loop system stability with the three control signals defined in Eq. (21) is subsequently studied.

Due to Eq. (17), (19) and (21), the control signal in k is a nonlinear function of states and output in k, and inputs of $k-1$ to $k-d+1$

$$u(k)=f\big(\overline{\boldsymbol{X}}(k),u(k-1)..,u(k-d+1),y(k)\big) \tag{25}$$

Equations (25) and (8), give the closed-loop system equations in $k+d$ as follow

$$\begin{aligned}\boldsymbol{X}(k+d)=\boldsymbol{A}\boldsymbol{X}(k+d-1)\\ +\boldsymbol{B}f\big(\overline{\boldsymbol{X}}(k),u(k-1)..,u(k-d\\ +1),y(k)\big)+\boldsymbol{\Gamma}\varepsilon(k+d-1)\end{aligned} \tag{26}$$

$$y(k+d)=\boldsymbol{C}\boldsymbol{X}(k)+\boldsymbol{X}^T(k)\boldsymbol{H}\boldsymbol{X}(k)+\varepsilon(k+d)$$

In order to check the stability of the closed-loop system, nonlinear terms of Eq. (26) are linearized around the equilibrium point. Hence, the control signal should be rewritten as a function of the previous step states $\overline{\boldsymbol{X}}(k+d-1)$. To do this, Eq. (14) must be rewritten as follows

$$\begin{aligned}\overline{\boldsymbol{X}}(k+d)&=\begin{bmatrix}\boldsymbol{x_m}(k+d)\\ \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k+d)\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{A_m^d} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A_e^d}\end{bmatrix}\begin{bmatrix}\boldsymbol{x_m}(k)\\ \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k)\end{bmatrix}\\ &+\sum_{i=1}^{d}\begin{bmatrix}\boldsymbol{A_m^{(d-i)}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A_e^{(d-i)}}\end{bmatrix}\begin{bmatrix}\boldsymbol{B_m}\\ \boldsymbol{0}\end{bmatrix}u(k\\ &-d+i)\\ &+\begin{bmatrix}\boldsymbol{A_m^{(d-1)}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A_e^{(d-1)}}\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B_e}\end{bmatrix}\hat{\varepsilon}(k)\\ &+\sum_{i=2}^{d}\begin{bmatrix}\boldsymbol{A_m^{(d-i)}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A_e^{(d-i)}}\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B_e}\end{bmatrix}\varepsilon(k\\ &-1+i)\end{aligned} \tag{27}$$

And if the control signal in the current step k is separated from the rest of terms, we have the following relation for system states:

$$\begin{aligned}&\overline{\boldsymbol{X}}(k+d)=\begin{bmatrix}\boldsymbol{x_m}(k+d)\\ \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k+d)\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{B_m}\\ \boldsymbol{0}\end{bmatrix}u(k)\\ &+\begin{bmatrix}\boldsymbol{A_m^d}\boldsymbol{x_m}(k)+\sum_{i=1}^{d-1}\boldsymbol{A_m^{(d-i)}}\boldsymbol{B}u(k-d+i)\\ \boldsymbol{A_e^d}\overline{\boldsymbol{x}}_{\boldsymbol{e}}(k)+\boldsymbol{A_e^{(d-1)}}\boldsymbol{B_e}\hat{\varepsilon}(k)\end{bmatrix}\\ &+\begin{bmatrix}\boldsymbol{0}\\ \sum_{i=2}^{d}\boldsymbol{A_e^{(d-i)}}\boldsymbol{B_e}\varepsilon(k-1+i)\end{bmatrix}\end{aligned} \tag{28}$$

In this equation, the last term indicates the future information that is unpredictable, so an estimation of $\overline{\boldsymbol{X}}(k+d)$ with information up to k is derived as follows

$$\begin{aligned}&\widehat{\boldsymbol{X}}(k+d)=\begin{bmatrix}\boldsymbol{x_m}(k+d)\\ \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k+d)\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{B_m}\\ \boldsymbol{0}\end{bmatrix}u(k)\\ &+\begin{bmatrix}\boldsymbol{A_m^d}\boldsymbol{x_m}(k)+\sum_{i=1}^{d-1}\boldsymbol{A_m^{(d-i)}}\boldsymbol{B}u(k-d+i)\\ \boldsymbol{A_e^d}\overline{\boldsymbol{x}}_{\boldsymbol{e}}(k)+\boldsymbol{A_e^{(d-1)}}\boldsymbol{B_e}\hat{\varepsilon}(k)\end{bmatrix}\end{aligned} \tag{29}$$

Eq. (29) can be rewritten as

$$\widehat{\boldsymbol{X}}(k+d)=\begin{bmatrix}\boldsymbol{x_m}(k+d)\\ \overline{\boldsymbol{x}}_{\boldsymbol{e}}(k+d)\end{bmatrix}=\boldsymbol{k_1}u(k)+\boldsymbol{k_2} \tag{30}$$

Where k_1 and k_2 are defined

$$\boldsymbol{k_1}=\begin{bmatrix}\boldsymbol{B_m}\\ \boldsymbol{0}\end{bmatrix}$$
$$\boldsymbol{k_2}=\begin{bmatrix}\boldsymbol{A_m^d x_m}(k)+\sum_{i=1}^{d-1}\boldsymbol{A_m^{(d-i)}B}u(k-d+i)\\ \boldsymbol{A_e^d}\overline{\boldsymbol{x}}_{\boldsymbol{e}}(k)+\boldsymbol{A_e^{(d-1)}B_e}\hat{\varepsilon}(k)\end{bmatrix}=\begin{bmatrix}\boldsymbol{k_{21}}\\ \boldsymbol{k_{22}}\end{bmatrix} \tag{31}$$

Therefore, by substitution of Eq.

(30) in Eq. (12), minimum variance predictor can be considered as Eq. (18) where a, b and c are defined as

$$\begin{aligned}a&=\boldsymbol{k_1^T H k_1}=\boldsymbol{B_m^T H_2 B_m}\\ b&=\boldsymbol{Ck_1}+\boldsymbol{k_1^T H k_2}+\boldsymbol{k_2^T H k_1}\\ &=\boldsymbol{H_1 B_m}+\boldsymbol{B_m^T H_2 k_{21}}+\boldsymbol{k_{21}^T H_2 B_m}\\ c&=\boldsymbol{Ck_2}+\boldsymbol{k_2^T H k_2}\\ &=\boldsymbol{H_1 k_{21}}+\boldsymbol{C_d k_{22}}+\boldsymbol{k_{21}^T H_2 k_{21}}\end{aligned} \tag{32}$$

By simplifying the above relations, parameters a, b and c are obtained as follows

$$\begin{aligned}a&=h_2(0,0)\\ b&=h_1(0)+\sum_{i=1}^{N}h_2(0,i)u(k-i)\\ c&=\sum_{i=1}^{N}h_1(i)u(k-i)+\sum_{i=1}^{N}\sum_{j=i}^{N}h_2(i,j)u(k-i)u(k-j)+\boldsymbol{C_d}\left(\boldsymbol{A_e^d}\overline{\boldsymbol{x}}_{\boldsymbol{e}}(k)+\boldsymbol{A_e^{(d-1)}B_e}\hat{\varepsilon}(k)\right)\end{aligned} \tag{33}$$

From Eq. (21) and (33), the control signal in k is defined as a function of control signals of $k-N$ up to $k-1$, $\bar{x}_e(k)$ and $\hat{\varepsilon}(k)$

$$u(k)=f\big(u(k-1),\dots,u(k-N),\bar{x}_e(k),\hat{\varepsilon}(k)\big) \tag{34}$$

An estimation of $\bar{x}_e(k+d-1)$ is derived by Eq. (8) recursively

$$\begin{aligned}\bar{x}_e(k+1)&=\boldsymbol{A_e}\bar{x}_e(k)+\boldsymbol{B_e}\hat{\varepsilon}(k)\\ \bar{x}_e(k+2)&=\boldsymbol{A_e}\bar{x}_e(k+1)+\boldsymbol{B_e}\varepsilon(k+1)\\ &=\boldsymbol{A_e^2}\bar{x}_e(k)+\boldsymbol{A_e B_e}\hat{\varepsilon}(k)+\boldsymbol{B_e}\varepsilon(k+1)\\ &\vdots\\ \bar{x}_e(k+d-1)&=\boldsymbol{A_e^{d-1}}\bar{x}_e(k)+\sum_{i=1}^{d-1}\boldsymbol{A_e^{(d-1-i)}B_e}\varepsilon(k-1+i)\end{aligned} \tag{35}$$

So we have

$$\boldsymbol{A_e^{d-1}}\bar{x}_e(k)=\bar{x}_e(k+d-1)-\sum_{i=2}^{d-1}\boldsymbol{A_e^{(d-1-i)}B_e}\varepsilon(k-1+i)-\boldsymbol{A_e^{(d-2)}B_e}\hat{\varepsilon}(k) \tag{36}$$

Hence, by substituting Eq. (36) in parameter c of Eq. (33), we have

$$c=\sum_{i=1}^{N}h_1(i)u(k-i)+\sum_{i=1}^{N}\sum_{j=i}^{N}h_2(i,j)u(k-i)u(k-j)+\boldsymbol{C_e}\left(\boldsymbol{A_e}\bar{x}_e(k+d-1)-\sum_{i=2}^{d-1}\boldsymbol{A_e^{(d-i)}B_e}\varepsilon(k-1+i)\right) \tag{37}$$

Due to Eq. (21), (33) and (37), minimum variance control signal in step k should be rewritten so that it becomes a function of $\boldsymbol{X}(k+d-1)$ and $\{\varepsilon(k+1)\ \dots\ \varepsilon(k+d-2)\}$

$$u(k)=f\big(\boldsymbol{X}(k+d-1),\varepsilon(k+1),\dots,\varepsilon(k+d-2)\big) \tag{38}$$

It is clear that the MV control signal is not a function of future information; it is only modified to this form in order to be linearized around the equilibrium point. Now, the control in Eq. (26) can be linearized as follows

$$\begin{aligned}\boldsymbol{X}(k+d)&=A\boldsymbol{X}(k+d-1)\\ &+B\big(A_L\boldsymbol{X}(k+d-1)+\Gamma_{L1}\varepsilon(k+1)+\cdots+\Gamma_{L(d-2)}\varepsilon(k+d-2)\big)\\ &+\Gamma\varepsilon(k+d-1)\\ &=(A+BA_L)\boldsymbol{X}(k+d-1)\\ &+B\Gamma_{L1}\,\varepsilon(k+1)+\cdots\\ &+B\Gamma_{L(d-2)}\varepsilon(k+d-2)\\ &+\Gamma\varepsilon(k+d-1)\end{aligned} \tag{39}$$

Linearization matrixes of Eq. (39) are defined as below

$$\begin{aligned}A_L&=\left[\frac{\partial f}{\partial u(k-1)}\ \ \frac{\partial f}{\partial u(k-2)}\ \cdots\ \frac{\partial f}{\partial u(k-N-1)}\ \ \frac{\partial f}{\partial \bar{x}_e(k+d-1)}\right]\\ \Gamma_{Li}&=\frac{\partial f}{\partial \varepsilon(k+i)}\,,\quad i=1,\dots,d-2\end{aligned} \tag{40}$$

According to Eq. (21), In case $\Delta>0$, partial derivations are

$$\frac{\partial f}{\partial u(k-i)} = \frac{1}{2a}\left(-\frac{\partial b}{\partial u(k-i)} \pm \frac{b\,\frac{\partial b}{\partial u(k-i)} - 2a\frac{\partial c}{\partial u(k-i)}}{\sqrt{b^2-4ac}}\right) \quad i = 1 \dots N$$

$$\frac{\partial f}{\partial \bar{x}_e(k+d-1)} = \frac{1}{2a}\left(-\frac{\partial b}{\partial \bar{x}_e(k+d-1)} \pm \frac{b\,\frac{\partial b}{\partial \bar{x}_e(k+d-1)} - 2a\frac{\partial c}{\partial \bar{x}_e(k+d-1)}}{\sqrt{b^2-4ac}}\right) \tag{41}$$

$$\frac{\partial f}{\partial \varepsilon(k+i)} = \frac{1}{2a}\left(-\frac{\partial b}{\partial \varepsilon(k+i)} \pm \frac{b\,\frac{\partial b}{\partial \varepsilon(k+i)} - 2a\frac{\partial c}{\partial \varepsilon(k+i)}}{\sqrt{b^2-4ac}}\right)$$

with

$$\frac{\partial b}{\partial u(k-i)} = h_2(0,i)$$

$$\frac{\partial c}{\partial u(k-i)} = h_1(i) + \sum_{j=i}^{N} h_2(i,j)u(t-j) + \sum_{j=1}^{i} h_2(j,i)u(t-j) \tag{42}$$

$$\frac{\partial b}{\partial \bar{x}_e(k+d-1)} = 0 \ , \quad \frac{\partial c}{\partial \bar{x}_e(k-d+1)} = \boldsymbol{C_e A_e}$$

$$\frac{\partial b}{\partial \varepsilon(k+i)} = 0 \ , \quad \frac{\partial c}{\partial \varepsilon(k+i)} = \boldsymbol{C_e A_e^{d-1-i} B_e}$$

And when $\Delta < 0$, partial derivations are

$$\frac{\partial f}{\partial u(k-i)} = \frac{1}{2a}\left(-\frac{\partial b}{\partial u(k-i)}\right) \quad i = 1 \dots N$$

$$\frac{\partial f}{\partial \bar{x}_e(k+d-1)} = 0 \tag{43}$$

$$\frac{\partial f}{\partial \varepsilon(k+i)} = 0$$

with

$$\frac{\partial b}{\partial u(k-i)} = h_2(0,i) \tag{44}$$

Closed-loop system stability can be determined by checking the eigenvalues of the closed-loop linearized matrix $\boldsymbol{A} + \boldsymbol{BA_L}$. We rename signals $(-b+\sqrt{b^2-4ac})/2a$, $(-b-\sqrt{b^2-2ac})/2a$ and $-b/2a$ respectively as u_1, u_2 and u_0 in the following. If $\boldsymbol{A} + \boldsymbol{BA_{L1}}$, $\boldsymbol{A} + \boldsymbol{BA_{L2}}$ and $\boldsymbol{A} + \boldsymbol{BA_{L0}}$ are closed-loop linearized matrixes when u_1, u_2 and u_0 are respectively applied to the system as control signal, stability of the system can be checked using the following assumptions and theory.

Assumption 1. At least one of the linearized closed loop matrixes $\boldsymbol{A} + \boldsymbol{BA_{L1}}$ and $\boldsymbol{A} + \boldsymbol{BA_{L2}}$ are Hurwitz. Linearized closed loop matrix $\boldsymbol{A} + \boldsymbol{BA_{L0}}$ is Hurwitz.

Assumption 2. The value of Δ is positive in steady state for closed loop systems with Hurwitz matrix in Assumption 1, which means $\Delta_{i\_ss} > 0$ for $i = 1 \ or \ 2, and \ 0$.

Remark 1. The Assumptions 1 and 2 are completely related to the dynamic of the system. They can be examined just after estimating the system with Volterra series model. So, the following theorem is restricted to the systems which satisfy the above assumptions.

Theorem 1. (Sufficient stability condition) Under Assumptions 1 and 2, the closed-loop system (8) with minimum variance control signal (21) is asymptotically stable.

Proof. Suppose that the conditions of theorem1 are satisfied. Assume that Δ is positive in the initial state and so, control signal u_1 (or u_2) is applied to the system and the closed loop system with this control signal is asymptotically stable. After a while, if Δ changes to negative, due to disturbance reactions or transient state, u_0 is applied to the system. After a few sample times, since $\boldsymbol{A} + \boldsymbol{BA_{L0}}$ is a Hurwitz matrix and $\Delta_{0\_ss} > 0$, Δ changes to positive and so u_1 (or u_2) is applied to the system again. This means that if u_0 and u_1 (or u_2) are used as control signal, the closed loop system will be asymptotically stable.■

Remark 2. If the conditions of Theorem 1 are not satisfied, then the generalized output can be defined to satisfy these conditions.

In the simplest form, if the generalized output is defined as in equation (24), for the design of control signal which minimizes the cost function (23), we can use equation (19) and (21) replacing b by $b+\lambda$. By this modification, eigenvalues of the closed loop linearized system matrix will change. Therefore, by the proper design of λ, the stability conditions of Assumption 1 and 2 are satisfied.

5. SIMULATION

Example 1. Consider a nonlinear dynamical system represented by a second-order Volterra series model as:

$$y(t) = 0.2u_{t-3} + 0.3u_{t-4} + u_{t-5} + 0.8u_{t-3}^2 + 2u_{t-4}^2 + 0.5u_{t-5}^2 + 0.8u_{t-3}u_{t-4} - 0.5u_{t-3}u_{t-5} + d(t)$$

With a disturbance

$$d(t) = \frac{\varepsilon(t)}{1 - 1.6q^{-1} + 0.8q^{-2}}$$

This system is a perturbed model of the example used by (Maboodi et al., 2015). The closed-loop system with minimum variance signal control has three equilibrium points for $u_1(k)$, $u_2(k)$ and $u_0(k)$ that is showed in Table 1. Furthermore, eigenvalues of the linearized closed-loop system matrix for these control signals and the value of Δ in the steady state for each equilibrium points are given in this table. The first two equilibrium points for $\Delta > 0$ are unstable and the last one for $\Delta < 0$ is stable.

Table 1. Equilibrium points, Eigenvalues and Δ in steady state for $u_1(k)$, $u_2(k)$ and $u_0(k)$ in the NMV control.

$u(k)$	$\frac{-b+\sqrt{b^2-4ac}}{2a}$	$\frac{-b-\sqrt{b^2-4ac}}{2a}$	$\frac{-b}{2a}$
u_{ss}	0	−0.4167	−0.1053
$Eig.val.$	0 −0.75 + 2.1065i −0.75 − 2.1065i 0.8 + 0.4i 0.8 − 0.4i	0 0.4078 −3.2810 0.8 + 0.4i 0.8 − 0.4i	0 0.3624 −0.8624 0.8 + 0.4i 0.8 − 0.4i
Δ_{ss}	0.04	0.3501	0.3776

If one of the first two signals and the last one are used to control this system, the control signal will have successive switches due to the stability condition of equilibrium points and their steady-state values of Δ. Assuming that u_{0ss} and u_{1ss} are applied to the system, it means that for $\Delta > 0$, u_{1ss} and for $\Delta < 0$, u_{0ss} is used as the control signal. Consider Δ is positive in the initial state, so u_{1ss} is applied to the system. After a while, Δ changes to negative due to unstability of u_{1ss} and so u_{0ss} is applied to the system. Then, Δ changes to positive again because u_{0ss} is stable and Δ_{ss} is positive for u_{0ss}. This switching is repeated in succession and causes unstability. Output, control signal and the type of signal applied to system is shown in Fig. 2. In control signal type chart, value 1 and 0 respectively demonstrate that signals u_{1ss} and u_{0ss} are applied to the system.

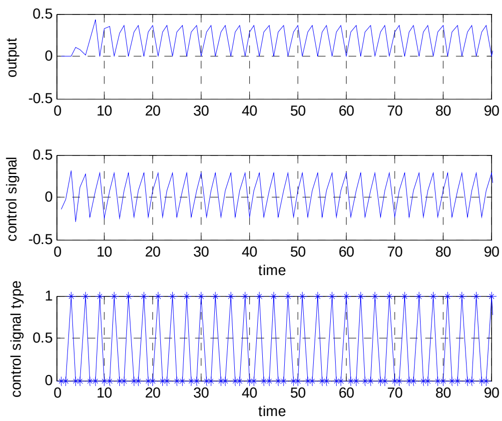

Fig. 2. output, control signal and control signal type for NMV control in Example1.

It should be noted that Assumption 1 is not satisfied and so, the closed loop system is not asymptotically stable. Therefore, according to the Remark 2, generalized output can be defined in such a way that Assumptions 1 and 2 are satisfied. So, in the previous example, the generalized output is considered as equation (24) with $\lambda = 1$. By this modification, generalized minimum variance control has the required conditions in Assumptions 1 and 2. Three equilibrium points, eigenvalues of linearized closed loop system and Δ in steady state for $u_1(k)$, $u_2(k)$ and $u_0(k)$ are shown in Table 2. The second equilibrium point is unstable and the first and last ones are stable. Moreover, the value of Δ in steady state for all equilibrium points are positive. It is noticeable that Assumptions 1 and 2 are established and so, the closed loop system will be asymptotically stable according to Theory 1.

Table 2. Equilibrium points, Eigenvalues and Δ in steady state for $u_1(k)$, $u_2(k)$ and $u_0(k)$ in the NMV control.

$u(k)$	$\frac{-b+\sqrt{b^2-4ac}}{2a}$	$\frac{-b-\sqrt{b^2-4ac}}{2a}$	$\frac{-b}{2a}$
u_{ss}	0	−0.6944	−0.6316
$Eig.val.$	0 −0.125 + 0.9043i −0.125 − 0.9043i 0.8 + 0.4i 0.8 − 0.4i	0 0.2134 −25.6088 0.8 + 0.4i 0.8 − 0.4i	0 0.3624 −0.8624 0.8 + 0.4i 0.8 − 0.4i
Δ_{ss}	1.44	0.0143	0.4574

However, there are two choices for the control signal; applying u_{1ss} and u_{0ss} or applying u_{2ss} and u_{0ss}. If u_{2ss} and u_{0ss} are used to control this system, the control signal will have successive switches due to the stability condition of equilibrium points and their steady state values of Δ like what happened for minimum variance control in this example.

Another choice is applying u_{0ss} and u_{1ss} as the control signal. Assuming that u_{0ss} and u_{1ss} are applied to the system, it means that for $\Delta > 0$, u_{1ss} and for $\Delta < 0$, u_{0ss} is used as control signal. Consider Δ is positive in the initial state, so u_{1ss} is applied to the system. After a while, if Δ changes to negative due to disturbance reactions or transient state, u_{0ss} is applied to the system. Then, Δ changes to positive again because u_{0ss} is stable and Δ_{0ss} is positive and so u_{1ss} is applied to the system again. This means that if u_{0ss} and u_{1ss} are used as control signal, the closed loop system will be stable. Output, control signal and the type of signal applied to system is shown in Fig. 3. In control signal type chart, value 1 and 0 demonstrate that signal u_{1ss} and u_{0ss} are applied to the system respectively.

Example 2. Consider a nonlinear dynamical system represented by a second-order Volterra series model as:

$$\begin{aligned} y(t) = 0.2u_{t-3} + 0.3u_{t-4} + u_{t-5} + 0.8u_{t-3}^2 - 0.7u_{t-4}^2 \\ + 0.5u_{t-5}^2 + 0.8u_{t-3}u_{t-4} - 0.5u_{t-3}u_{t-5} \\ + 0.3u_{t-4}u_{t-5} + d(t) \end{aligned}$$

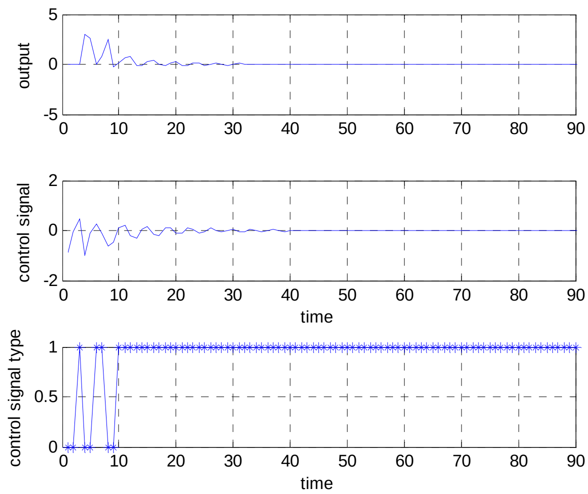

Fig. 3. output, control signal and control signal type for NGMV control in Example1.

With disturbance $d(t)$ considered in Example 1. This system is a perturbed model of the example used by (Maboodi et al., 2015) with additional second order term as $u_{t-4}u_{t-5}$.

Three equilibrium points of closed-loop system with minimum variance signal control for $u_1(k)$, $u_2(k)$ and $u_3(k)$, eigenvalues of the linearized closed-loop system matrix for these control signals and the value of Δ in the steady state for each equilibrium points is showed in Table 13. The first equilibrium points for $\Delta > 0$ is unstable and the two last ones are stable.

Table 3. Equilibrium points, Eigenvalues and Δ in steady state for $u_1(k)$, $u_2(k)$ and $u_0(k)$ in the NMV control.

$u(k)$	$\frac{-b+\sqrt{b^2-4ac}}{2a}$	$\frac{-b-\sqrt{b^2-4ac}}{2a}$	$\frac{-b}{2a}$
u_{ss}	0	-1.25	-0.1053
$Eig.val.$	0 $-0.75+2.1065i$ $-0.75-2.1065i$ $0.8+0.4i$ $0.8-0.4i$	0 0 0.3103 $0.8+0.4$ $0.8-0.4i$	0 0.3624 -0.8624 $0.8+0.4i$ $0.8-0.4i$
Δ_{ss}	0.04	4.7306	0.4627

It is concluded from Table 13 that Assumption 1 and 2 are satisfied in this system, so by choosing control signals with stable Eigen values and positive Δ_{ss} closed loop system will be stabilized. Assuming that u_{0ss} and u_{1ss} are applied to the system, it means that for $\Delta > 0$, u_{1ss} and for $\Delta < 0$, u_{0ss} is used as the control signal. Similar to the first part of Example 1, switching between these to control signals causes instability in closed loop system. Output, control signal and the type of signal applied to system is shown in Fig. 24.

However, in case of applying u_{0ss} and u_{2ss} to the system, it means that for $\Delta > 0$, u_{2ss} and for $\Delta < 0$, u_{0ss} is used as control signal, the closed loop system will be stable. Consider Δ is positive in the initial state, so u_{2ss} is applied to the system. After a while, if Δ changes to negative due to disturbance reactions or transient state, u_{0ss} is applied to the system. Then, Δ changes to positive again, because u_{0ss} is stable and Δ_{0ss} is positive and so u_{2ss} is applied to the system again. This means that if u_{0ss} and u_{2ss} are used as control signal, the closed loop system will be stable. Output, control signal and the type of signal applied to system is shown in Fig. 25.

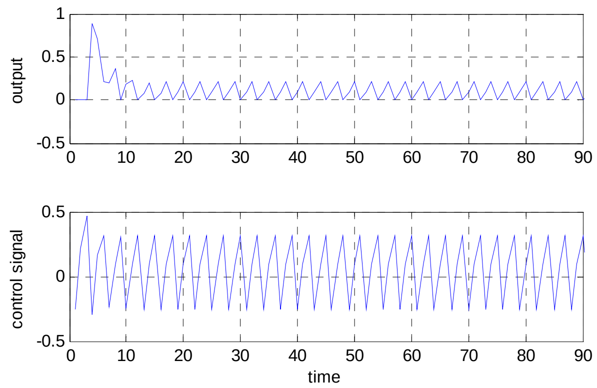

Fig. 4. output, control signal for NMV control with u_{0ss} and u_{1ss} in Example2.

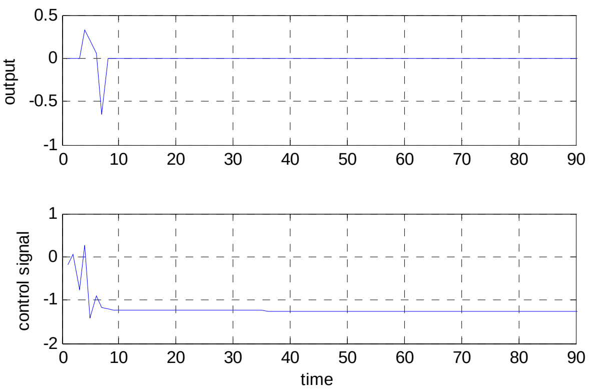

Fig. 5. output, control signal for NMV control with u_{0ss} and u_{2ss} in Example2.

6. CONCLUSION

In this paper, closed-loop system stability of nonlinear minimum variance (NMV) controllers for a second order Volterra series model has been studied. It is shown that the closed-loop system with NMV controller is a state-dependent switching system with the arbitrary switching signal. A sufficient condition for asymptotic stability of closed loop system is introduced that was based on the stability of the linearized subsystems and parameter Δ in the steady state. Moreover, it is noted that if this condition is not met, it can be satisfied by defining a generalized output and applying the NGMV control instead of the NMV control. Simulation results are provided to show the main points of the stability conditions.

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