MODEL REFERENCE ADAPTIVE VS. LEARNING CONTROL FOR THE INVERTED PENDULUM. A COMPARATIVE CASE STUDY

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Abstract: *The Inverted Pendulum system (also called "cart-pole system") is a classic example of a nonlinear and unstable control system. By controlling the force applied to the cart in the horizontal direction, the inverted pendulum can be kept in various unstable equilibrium positions .In this paper we develop a case study where we perform a comparative analysis between a conventional adaptive control technique using the stability theory of Lyapunov (MRAC – Model Reference Adaptive Control) and a fuzzy learning control technique (FMRLC – Fuzzy Model Reference Learning Control). Both control techniques are based on reference models. The term "learning" is used as opposed to "adaptive" to distinguish the two control structures. In particular, the distinction is drawn since the FMRLC, which is also a direct model reference adaptive controller, will tune and to some extent it will remember the values it had tuned in the past, while the conventional adaptive approach will continue to tune the controller parameters.The performances of the proposed control algorithms are evaluated and shown by means of digital simulation.*

Keywords: *Lyapunov, Model Reference, Adaptive Control, FuzzyLearning Control*

1. INTRODUCTION

The control of the inverted pendulum system is one of the most important classical problems of Control Engineering. Due to its characteristics (nonlinear, unstable), this system is widely used to demonstrate and test various control algorithms.

The system consists of a rigid pole attached to a cart with a hinge, a free joint with one degree of freedom. The cart is constrained to move along a linear horizontal direction when a force is exerted on it. If appropriate forces are applied the pole can be kept in various positions from falling over.

This paper uses this simple case study to examine the implementation of two model reference adaptive control algorithms and to compare their control performances. In MRAC, the technical demands and the desired input-output behavior of the closed-loop system are given via the corresponding dynamic of the reference model. Therefore, the basic task is to design such a control, which will ensure the minimal error between the reference model and the plant outputs despite the uncertainties or variations in the plant parameters and working conditions.

The first approach is to design a model reference adaptive controller using the stability theory of Lyapunov. This theory assures that the equilibrium point 0 is asymptotically stable.

The design of controllers, using conventional techniques, for plants with nonlinear dynamics and modeling uncertainties can be often quite difficult. Fuzzy control is a practical alternative for a variety of challenging control applications, since it provides a convenient method for constructing nonlinear controllers via the use of heuristic information. However, some of the problems encountered in practical control problems, such as model uncertainties or the difficulty to choose some of the fuzzy controller's parameters, demand a way to automatically tune the fuzzy controller so that it can adapt to different operating conditions.

Based on a simple fuzzy logic controller we then focus on the design of a Fuzzy Model Reference Learning Controller (FMRLC).

In the end the performances of the two proposed control algorithms are evaluated and shown by means of digital simulation.

2. DYNAMICAL MODEL OF THE PLANT

The dynamic of the inverted pendulum system shown in Figure 1 is described next.
Assuming the pendulum is a uniform rod, its moment of inertia is $I=ml^2/3$ and the nonlinear equations which describe the motion of the inverted pendulum system are:

$$\begin{cases} (M+m)\ddot{X}_p + ml\ddot{\Theta}\cos(\Theta) - ml\dot{\Theta}^2\sin(\Theta) = F \\ mgl\sin(\Theta) - ml^2\ddot{\Theta} - m\ddot{X}_p l\cos(\Theta) = I\ddot{\Theta} \end{cases} \tag{1}$$

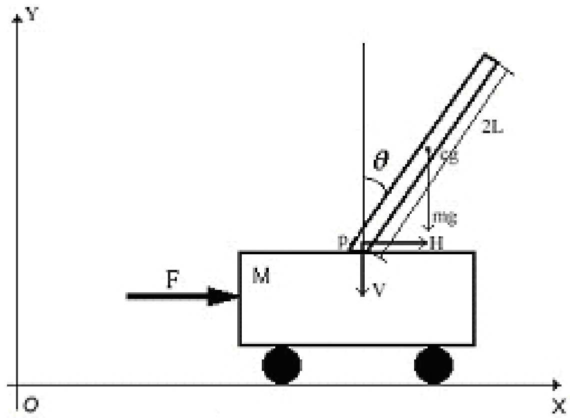

Fig.1. Inverted pendulum on a cart

The plant parameters are given in Table 1.

Table 1: Plant parameters

θ	Angle of the pendulum	
F	Force applied to the cart	
X_p	Position of the pivoting point	
m	0.5 kg	Mass of pendulum
M	1 kg	Mass of cart
l	0.5 m	Distance between the pivot point p and the centre of gravity cg of the pendulum
G	9.81 m/s^2	Gravitational constant
I	0.0833 kg·m^2	Inertia of pendulum

A linear model for the system was developed to help design the MRAC. Equations (1) were linearized about the equilibrium point $\Theta = 0$. For small angles the following approximations were assumed:

$$\sin(\Theta) = 0;\ \cos(\Theta) = 1;\ \frac{d^2\Theta}{dt^2} = 0 \tag{2}$$

The following linear ISO (input state output) model was determined for the inverted pendulum:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{g}{\frac{4}{3}l - \frac{ml}{m+M}}x_1 - \dfrac{1}{(m+M)(\frac{4}{3}l - \frac{ml}{m+M})}u \\ y = x_1 \end{cases} \tag{3}$$

where $x_1 = \Theta$ is the angle of the pendulum, x_2 is the rotational speed of the rod, $u = F$ is the input to the system and $y = x_1$ is the system's output.

3. MODEL REFERENCE ADAPTIVE CONTROL

This section discusses the design of the conventional Model Reference Adaptive Control (MRAC) as applied to the inverted pendulum system.

When the plant parameters and the disturbance are varying slowly, or slower than the dynamic behavior of the plant, then a MRAC control scheme can be used. This adaptive structure offers a superior performance and robustness in time than a classical PID controller [7].

Figure 2 shows the structure of the MRAC scheme designed for the inverted pendulum system.

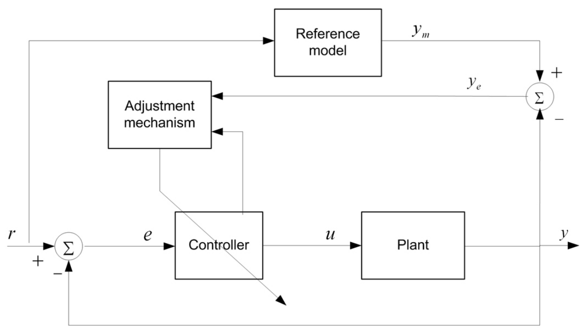

Fig. 2. Model Reference Adaptive Control

The MRAC structure consists of four main parts: the plant, the controller, the reference model and the adjustment mechanism.

The *reference model* is chosen to generate the desired trajectory, y_m, for the plant output y to follow. A standard second order differential equation was chosen as reference model:

$$\ddot{y}_m = -2\xi\omega_n \dot{y}_m - \omega_n^2 y_m + \omega_n^2 r \tag{4}$$

$\omega_n = 3\ rad/sec$ and $\xi = 1$ were chosen to provide a critically damped response.

The following ISO model describes the reference model:

$$\dot{x}_m = A_m x_m + B_m r \tag{5}$$

where $x_m = \begin{bmatrix} y_m & \dot{y}_m \end{bmatrix}^T$ and

$$A_m = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix},\ B_m = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}^T.$$

Due to the specifics of the plant and the reference model (second order model), the feedforward feedback controller for the inverted pendulum has three parameters k_1, k_2, k_3 and the control law takes the following form:

$$u = k_1 r - k_2 y - k_3 \dot{y} \tag{6}$$

For $k_1 = k_2$ the controller becomes a proportional derivative (PD) controller with the derivative component on the feedback loop.

$$u = k_1 e - k_3 \dot{y} \tag{7}$$

Based on equations (3) and (7), the ISO dynamic equation that describes the closed-loop system, which contains both the controller and the plant, is given next:

$$\dot{x} = Ax + Br \tag{8}$$

where $x = \begin{bmatrix} y & \dot{y} \end{bmatrix}^T$ and

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{0.66k_2 + 9.8}{0.5} & \dfrac{0.66}{0.5}k_3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ -\dfrac{0.66}{0.5}\cdot k_1 \end{bmatrix}.$$

The tracking (adaptation) error $y_e = y_m - y$ represents the deviation of the plant output from the desired trajectory. The *adjustment mechanism* uses this adaptation error to adjust the controller's parameters.

Depending on the method used to determine the adjustment mechanism, MRAC assures the stability and convergence of the adaptation error.

The differential equation that describes the adaptation error may be expressed by:

$$\dot{x}_e = A_m x_e + (A_m - A)x + (B_m - B)r \tag{9}$$

where $x_e = \begin{bmatrix} y_e & \dot{y}_e \end{bmatrix}^T$.

The adaptation laws for the controller's parameters are determined using Lyapunov's theory of stability. Other possible approaches include: the MIT rule, the use of input/output analysis for systems, small-gain theorem, passivity theory, etc.

The first major problem in using Lyapunov's theory is how to choose a proper positive definite function $V(t,x_e)$. For our case study we considered the following V function:

$$V = x_e^T P x_e + tr\{(A_m - A)^T \gamma_A (A_m - A)\} + \\ + tr\{(B_m - B)^T \gamma_B (B_m - B)\} \tag{10}$$

According to Lyapunov's theory, the equilibrium point $x_e=0$ is asymptotically stable if [7]:

- V is a positive definite function (P is positive definite);
- $V(0)=0$;
- dV/dt is negative definite.

The second problem is to obtain the derivative of the function V and to make it negative definite.

$$V = x_e^T \left(A_m^T P + P A_m\right) x_e + \\ + 2 \cdot tr\left\{(A_m - A)^T \left(\gamma_A \cdot P \cdot x_e \cdot x^T - \dot{A}(t)\right)\right\} + \\ + 2 \cdot tr\left\{(B_m - B)^T \left(\gamma_B \cdot P \cdot x_e \cdot r^T - \dot{B}(t)\right)\right\} \tag{11}$$

Matrix P ($2x2$) is the symmetric and positive definite solution of the Lyapunov equation (12). We assume that Q is the identity matrix.

$$A_m^T P + P A_m = -Q < 0 \tag{12}$$

The derivative of the Lyapunov function V is negative definite if we choose the following adaptation laws for the variant closed-loop system matrix:

$$\dot{A}(t) = \gamma_A \cdot P \cdot x_e \cdot x^T \\ \dot{B}(t) = \gamma_B \cdot P \cdot x_e \cdot r^T \tag{13}$$

where $\gamma_{A,B}$ refer to the adaptation speed / gains (positive constants).

Solving equation (12), we get the symmetric matrix P of the following form:

$$P = \begin{bmatrix} \frac{1}{\omega_n}\left(\xi + \frac{1+\omega_n^2}{4\xi}\right) & \frac{1}{2\omega_n^2} \\ \frac{1}{2\omega_n^2} & \frac{1}{4\xi\omega_n}\left(1 + \frac{1}{\omega_n^2}\right) \end{bmatrix} \tag{14}$$

Solving equations (13), the adaptation laws for the controller's parameters are:

$$\dot{k}_1 = -\gamma_1 \left(p_{12} \cdot y_e + p_{22} \cdot \dot{y}_e\right) \cdot r \\ \dot{k}_2 = \gamma_2 \left(p_{12} \cdot y_e + p_{22} \cdot \dot{y}_e\right) \cdot y \\ \dot{k}_3 = \gamma_3 \left(p_{12} \cdot y_e + p_{22} \cdot \dot{y}_e\right) \cdot \dot{y} \tag{15}$$

In the adaptation laws (15) some terms were absorbed into the adaptation gains γ_1, γ_2, γ_3 and p_{12}, p_{22} are the second column elements of the matrix P.

4. FUZZY MODEL REFERENCE LEARNING CONTROL

This section discusses the development of the Fuzzy Model Reference Learning Controller (FMRLC) as applied to the inverted pendulum system. The FMRLC algorithm was first introduced in [6].

In order to develop the FMRLC a direct fuzzy controller was first designed.

4.1 Direct fuzzy control

The design of a direct fuzzy controller can be resumed to choosing and processing the inputs and outputs of the controller and designing its four component elements (the rule base, the inference engine, the fuzzification and the defuzzification interfaces) [10].

We consider the inputs to the fuzzy system: the error:

$$e(kT)=r-y \tag{16}$$

and change in error:

$$c(kT) = (e(kT) - e(kT-T)) / T \tag{17}$$

and the output variable: the force applied to the cart:

$$u = F \tag{18}$$

The universe of discourse of the variables (that is, their domain) was normalized to cover a range of [-1, 1] and scaling gains (g_e, g_c, g_u) were used to normalize.

A standard choice for the membership functions was used with five membership functions for the three fuzzy variables (meaning 25 = 5^2 rules in the rule base) and symmetric, 50% overlapping triangular shaped membership functions (Figure

3), meaning that only 4 (=2^2) rules at most can be active at any given time.

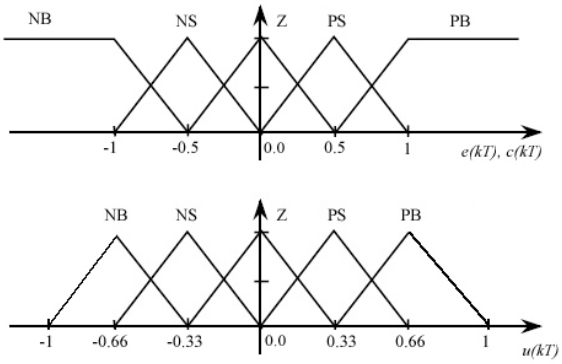

Fig. 3. Membership functions for the fuzzy controller

The fuzzy controller implements a rule base made of a set of IF-THEN type of rules. These rules were determined heuristically based on the knowledge of the plant. An example of IF-THEN rule is the following:

IF e is negative big (NB) and c is negative big (NB) THEN u is positive big (PB)

This rule quantifies the situation where the pendulum is far to the right of the vertical and it is moving clockwise, hence a large force (to the right) is needed to counteract the movement of the pendulum so that it moves toward zero. The resulting rule table is shown in the Table 2.

Table 2: Rule base for the fuzzy controller

"force" u		"change in error" c				
		NB	NS	Z	PS	PB
"error" e	NB	PB	PB	PB	PS	Z
	NS	PB	PB	PS	Z	NS
	Z	PB	PS	Z	NS	NB
	PS	PS	Z	NS	NB	NB
	PB	Z	NS	NB	NB	NB

The *min-max inference engine* was chosen, which for the premises, uses maximum for the OR operator and minimum for the AND operator. The conclusion of each rule, introduced by THEN, is also done by minimum. The final conclusion for the active rules is obtained by the maximum of the considered fuzzy sets.

To obtain the crisp output, the *centre of gravity (COG)* defuzzification method is used. This crisp value is the resulting controller output.

4.2. Adaptive fuzzy control

In this section we design and implement a Fuzzy Model Reference Learning Controller (FMRLC), which will adaptively tune on-line the centers of the output membership functions of the fuzzy controller determined earlier.

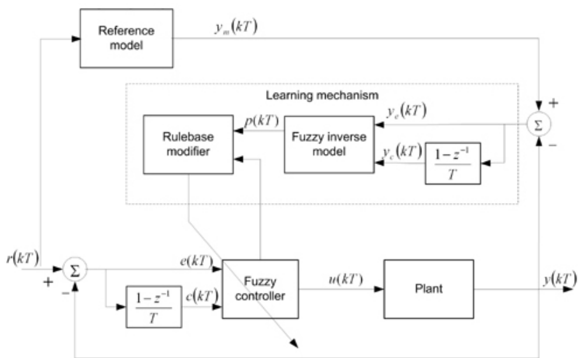

Fig. 4. Fuzzy Model Reference Learning Control [6]

Figure 4 shows the FMRLC as applied to the inverted pendulum system.

The FMRLC uses a *learning mechanism* that a) observes data from a fuzzy control system (i.e. *r(kT)* and *y(kT)*) b) characterizes its current performance, and c) automatically synthesizes and/or adjusts the fuzzy controller so that some pre-specified performance objectives are met [6,7].

In general, the *reference model*, which characterizes the desired performance of the system, can take any form (linear or nonlinear equations, transfer functions, numerical values etc.). However, you want to specify a desirable performance, but at the same time a reasonable one. In the case of the inverted pendulum FMRLC, the same reference model presented in equations (4, 5) was used.

An additional fuzzy system is developed called *"fuzzy inverse model"* which adjusts the centers of the output membership functions of the fuzzy controller, which still controls the process, developed earlier [6, 7]. This fuzzy system acts like a second controller, which updates the rule base of the fuzzy controller by acting upon the output variable (its membership functions centers). The output of the inverse fuzzy model is an adaptation factor *p(kT)* which is used by the rule base modifier to adjust the centers of the output membership functions of the fuzzy controller. The adaptation is stopped when *p(kT)* gets very small and the changes made to the rule base are no longer significant.

The *fuzzy controller* used by the FMRLC structure is the same as the one developed in the previous section.

The *fuzzy inverse model* has a similar structure to that of the controller (the same rule base, membership functions, inference engine, fuzzification and defuzzification interfaces. See section 4.1). The only notable difference between the two fuzzy systems is the normalizing gains values (g_{ye}, g_{yc}, g_p).

The inputs of the fuzzy inverse model are:

$$y_e(kT) = y_m(kT) - y(kT) \quad (19)$$

$$y_c(kT) = (y_e(kT) - y_e(KT\text{-}T)) / T \quad (20)$$

and the output variable is the adaptation factor *p(kT)*.

An example of IF-THEN rule for the fuzzy inverse model is the following:

IF y_e is negative big (NB) and y_c is negative big (NB) THEN p is positive big (PB)

The *rule base modifier* adjusts the centers of the output membership functions in two stages:

1. the active set of rules for the fuzzy controller at time *(k-1)T* is first determined:

$$\mu_i^e\left(e\left(kT-T\right)\right) > 0, i = \overline{1,n}$$
$$\mu_j^c\left(c\left(kT-T\right)\right) > 0, j = \overline{1,m} \quad (21)$$

   The pair *(i, j)* will determine the activated rule. We denoted by *i* and *j* the *i*-th, respectively the *j*-th membership function for the input fuzzy variables error and change in error.

2. the centers of the output membership functions, which were found in the active set of rules determined earlier, are adjusted. The centers of these membership functions (b_l) at time *kT* will have the following value:

$$b_l(kT) = b_l(kT-T) + p(kT) \quad (22)$$

   We denoted by *l* the consequence of the rule introduced by the pair *(i, j)*.

The centers of the output membership functions, which are not found in the active set of rules *(i, j)*, will not be updated. This ensures that only those rules that actually contributed to the current output *y(kT)* were modified. We can easily notice that only local changes are made to the controller's rule base.

For better learning control a larger number of output membership functions (a separate one for each input combination) would be required. This way a larger memory would be available to store information. Since the inverse model updates only the output centers of the rules which apply at that time instant and does not change the outcome of the other rules, a larger number of output membership functions would mean a better capacity to map different working conditions. That is, the controller will remember the adjustments it made in the past for a wider range of specific conditions. This represents an advantage for this method since time consuming re-learning is avoided. At the same time this is one of the characteristics that differences learning control from the more conventional adaptive control.

5. RESULTS. COMPARATIVE ANALISYS

This section presents the results we obtained using the two control schemes for the inverted pendulum problem. Both control structures were tested on the non-linear plant, using the same reference model, to balance the pendulum in the vertical position ($\Theta = 0$) and to follow a square trajectory.

It is known that the performances of the adaptive and intelligent systems are superior to those of the classic control schemes, especially in the case of slowly time varying parameters of the process or in the presence of disturbances. The following figures show interesting results in this direction.

For the first control problem, balancing the pendulum in the vertical position, an angle $\Theta = 0.08rad$ was chosen as the initial condition for the simulation.

The second control problem was to test the capability of the closed-loop adaptive system to adapt or learn (in FMRLC case) a desired trajectory. The desired trajectory proposed for the simulation was a square input between [-π/10, π/10], with 20 seconds period and 50% duty cycle.

Figure 5 shows the response of the MRAC system as compared to the reference model response. The simulation for the equilibrium problem shows that the MRAC scheme is practically impossible to use due to the high amplitude of the response (process output, angle of the inverted pendulum).

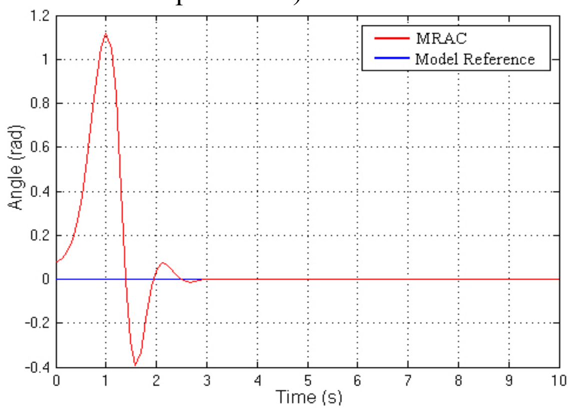

Fig. 5. Pendulum position (y). Equilibrium

The desired trajectory y_m of the second order reference model, the response y of the plant to a square trajectory input and the adaptation error y_e are shown in Figure 6. It is obvious that the plant output tracks the reference output and the adaptation error $y_e(t)$ converges to zero, but in the first pulse the output is too high.

The MRAC adaptation gains used for MRAC simulation were chosen experimentally: γ_1=-90, γ_2=70 and γ_3=0.8.

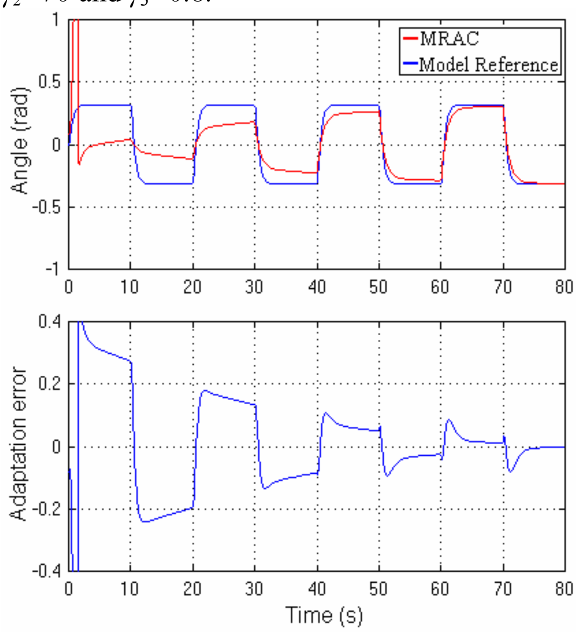

Fig. 6. Pendulum position (y), error (y_e). Square trajectory

Figures 7 and 8 show the evolution of the controller output (u=F) and the evolution of the controller's feedforward feedback parameters.

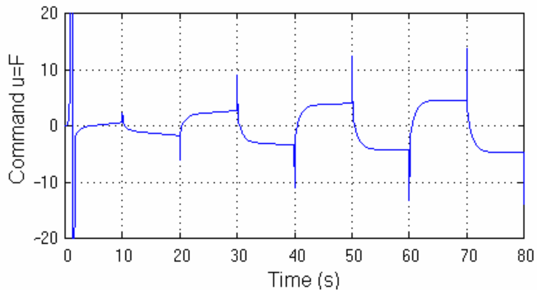

Fig. 7. Plant input (u=F). Square trajectory

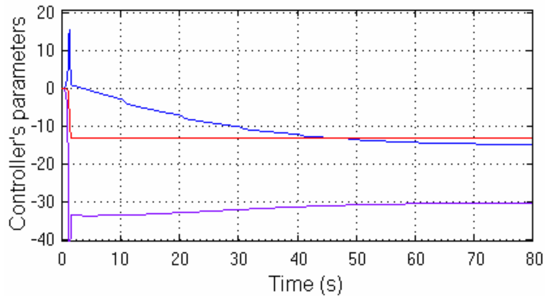

Fig. 8. Controller's parameters k_1, k_2, k_3

The FMRLC control algorithm was implemented with a digital structure, using a sampling time T = 0.001s. The input and output gains for the fuzzy controller and for the fuzzy inverse model were heuristically tuned to: g_e=π/10, g_c=20, g_u=10, g_{ye}=π/10, g_{yc}=0.5 and g_p=1. The same angle Θ = 0.08rad was chosen as the initial condition for the FMRLC simulation.

Figure 9 shows the response of the FMRLC system as compared to the response of the system controlled by the direct fuzzy controller, which served as the base for the FMRLC design. A clear improvement is obvious.

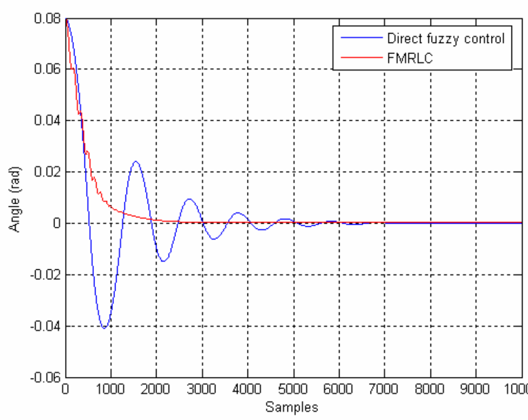

Fig. 9. Pendulum position (y). Equilibrium

To test the FMRLC's capability to learn (adapt) to different working conditions, the square trajectory between [-π/10, π/10] was used. The following figure shows the response of the system for this case.

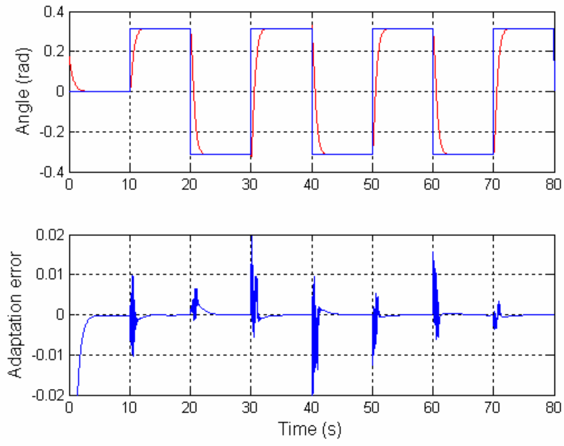

Fig. 10. Pendulum position *(y)*, error *(y_e)*. Square trajectory

Since the output of the system y follows very closely the output of the reference model y_m, the evolution of the error signal, $y_e = y_m\text{-}y$, is also shown in figure 10.

The following figure shows the evolution of the plant input $u=F$ and the evolution of the adaptation factor p.

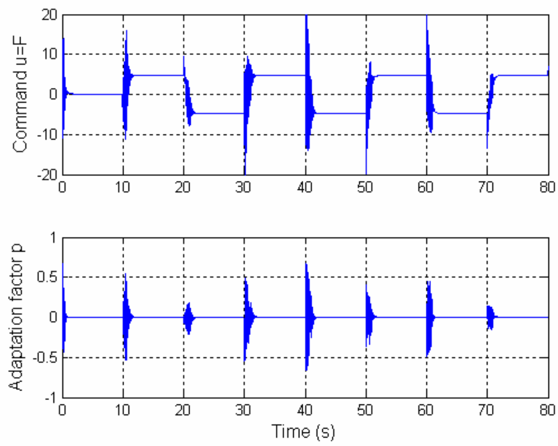

Fig. 11. Plant input *($u=F$)*, adaptation factor *(p)*. Square trajectory

All of the results presented above were obtained using digital simulation. Matlab and Simulink were used for these simulations.

6. CONCLUSION

This paper studied the implementation of two adaptive control techniques as applied to the balancing of the inverted pendulum system.

The MRAC scheme applies to systems with known dynamic structure, linear (second order system for our linearized plant) or nonlinear, but with unknown constants or slow varying parameters. The adaptive controller designed for the inverted pendulum is inherently nonlinear. The MRAC system can handle large variations of the plant parameters with slow varying dynamic response. Otherwise, the stability of the closed-loop system and the convergence of the adaptation error are assured by the Lyapunov theory of stability.

The direct fuzzy controller allowed the use of heuristics (which model the way a human would control the process) via the use of the rule table. Since we generally know the way to balance the pendulum, the heuristics we chose in the design of the fuzzy controller proved very useful. The FMRLC took the controller design a step further by supplying an inductive update method, which produced an adaptive fuzzy controller.

Having analyzed the response of both control schemes it is obvious the FMRLC performs better. Having a superior response time and a faster ability to learn / adapt, the FMRLC proved itself capable to adapt to a wide range of working conditions (not only square trajectory) without the need to modify its parameters. On the other hand the MRAC wasn't able to provide a suitable solution for the balancing of the pendulum, due to the very high overshoot. This shows, that for this case study, the adaptive fuzzy method we investigated has an advantage with respect to the more conventional MRAC method, since it allows more design flexibility, especially in the use of the reference model. Several reference models have been tested successfully with the FMRLC structure, ranging from $y_m=r$ to first or second order dynamical systems. However, there are certain tradeoffs required to achieve this success. These include a greater computational complexity for the intelligent controller and a greater design time required.

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