OFF-LINE METHOD FOR IMPROVING ROBUSTNESS OF MODEL PREDICTIVE CONTROL LAWS

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Abstract: *This paper presents a unified off-line state-space methodology for enhancing the robustness of MPC controlled systems using the Youla parametrization. An initial stabilizing MPC controller is firstly designed in the state-space representation and further robustified off-line via LMI techniques to find the best trade-off between stabi-lity robustness and nominal performance. This robustification method, that represents the original contribution of the paper, is finally applied to the velocity control of an induction machine to reduce the influence of additive and multiplicative unstructured uncertainties, while respecting a time-domain template for disturbance rejection.*

Keywords: *MPC, Youla Parameter, Robust Control, LMI, Linear Control Systems.*

1. INTRODUCTION

Model Predictive Control strategies are widely accepted in process industry. Starting with a controller design based on the system model, questions about robustness towards unstructured uncertainties or different disturbances acting on the system always arise in industrial environment. Many methods proposed in the literature [7], [11] have difficulties to efficiently handle the required trade-off between robust stability and nominal performance. Most of them also consider the transfer function formalism [6], [1], an extension to the multivariable case being more complicated.

The contribution of this paper is to present a unified off-line method enhancing robustness of an initial stabilizing model predictive controller towards unstructured uncertainties, while respecting an output time-domain template. In order to simplify the generalization to the MIMO case, this technique is fully based on the state-space formalism. The procedure begins with the design of an initial stabilizing MPC (Model Predictive Control) law that is further robustified via Youla parametrization using LMI (Linear Matrix Inequality) tools.

This paper is organized as follows. Section 2 contains a brief overview of MPC in the state-space representation. Section 3 restates the background material required to formulate the robustification strategy using Youla parametrization. The design of the MPC robustified controller is described in Section 4 and is further applied to the velocity control of an induction machine in Section 5. Some conclusions and future trends are presented in Section 6.

2. DESIGN OF THE MPC LAW

This Section presents the state-space procedure leading to the elaboration of the MPC law. Consider the following single-input single-output discrete time LTI system:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}u(k) \\ y(k) = \mathbf{C}\,\mathbf{x}(k) \end{cases} \tag{1}$$

where $\mathbf{x}$ describes the system state, $u(k)$ and $y(k)$ are respectively the control input and the system output, as shown in Fig. 1.

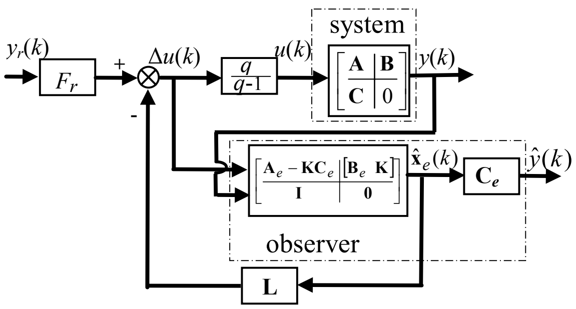

Fig. 1. Block diagram of MPC in state-space description.

An integral action on the control signal has been added to cancel the steady-state error:

$$u(k) = u(k-1) + \Delta u(k) \tag{2}$$

leading to the extended state-space form:

$$\begin{cases} \mathbf{x}_e(k+1) = \mathbf{A}_e\,\mathbf{x}_e(k) + \mathbf{B}_e \Delta u(k) \\ y(k) = \mathbf{C}_e\,\mathbf{x}_e(k) \end{cases} \tag{3}$$

with $\mathbf{x}_e(k) = \left[\mathbf{x}^T(k)\ u(k-1)\right]^T$, $\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & 1 \end{bmatrix}$, $\mathbf{B}_e = \left[\mathbf{B}^T\ 1\right]^T$, $\mathbf{C}_e = [\mathbf{C}\ 0]$.

In order to obtain the control signal $\Delta u(k)$, the following quadratic criterion is minimized:

$$J = \sum_{i=N_1}^{N_2} \left\| \hat{y}(k+i) - y_r(k+i) \right\|^2_{Q_J(i)} + \sum_{i=0}^{N_u-1} \left\| \Delta u(k+i) \right\|^2_{R_J(i)} \tag{4}$$

assuming that the future control increments $\Delta u(k+i)$ are zero for $i \geq N_u$. N_1 and N_2 describe respectively the lower and upper values of the output prediction horizons, N_u characterizes the control horizon and y_r is the setpoint. Using the state estimate $\hat{\mathbf{x}}_e(k)$ derived from the observer:

$$\hat{\mathbf{x}}_e(k+1) = \mathbf{A}_e\,\hat{\mathbf{x}}_e(k) + \mathbf{B}_e \Delta u(k) + \mathbf{K}[y(k) - \mathbf{C}_e\,\hat{\mathbf{x}}_e(k)] \tag{5}$$

the predicted output can be written as:

$$\hat{y}(k+i) = \mathbf{C}\mathbf{A}^i\,\hat{\mathbf{x}}(k) + \sum_{j=0}^{i-1} \mathbf{C}\mathbf{A}^{i-j-1}\mathbf{B}u(k+j) \tag{6}$$

with $u(k+j) = u(k-1) + \sum_{m=0}^{j} \Delta u(k+m)$.

The observer gain $\mathbf{K}$ is designed through a classical method of eigenvectors, arbitrarily placing the eigenvalues of $\mathbf{A}_e - \mathbf{K}\mathbf{C}_e$ in a stable region, as in [10]. However this design aspect is not crucial since the convex robustification method should lead to an optimal set of these eigenvalues. Moreover the input/output transfer function is not influenced by the eigenvalues placement used to find $\mathbf{K}$ [2].

Using the matrix formalism described in [9], [4], the objective function can be rewritten as:

$$J = \left\| \mathbf{Y}(k) - \mathbf{Y}_r(k) \right\|^2_{\mathbf{Q}_J} + \left\| \Delta \mathbf{U}(k) \right\|^2_{\mathbf{R}_J} \tag{7}$$

with the following matrix forms:

$$\mathbf{Q}_J = diag\left(Q_J(N_1), Q_J(N_1+1), \cdots, Q_J(N_2)\right)$$
$$\mathbf{R}_J = diag\left(R_J(0), R_J(1), \cdots, R_J(N_u-1)\right)$$
$$\mathbf{Y}(k) = \left[\hat{y}(k+N_1)\ \cdots\ \hat{y}(k+N_2)\right]^{\mathrm{T}}$$
$$\mathbf{Y}_r(k) = \left[y_r(k+N_1)\ \cdots\ y_r(k+N_2)\right]^{\mathrm{T}}$$
$$\Delta\mathbf{U}(k) = \left[\Delta u(k)\ \cdots\ \Delta u(k+N_u-1)\right]^{\mathrm{T}}$$

The objective function (7) can be rewritten as:

$$J = \left\| \Phi_\Delta \Delta\mathbf{U}(k) - \Theta(k) \right\|^2_{\mathbf{Q}_J} + \left\| \Delta\mathbf{U}(k) \right\|^2_{\mathbf{R}_J} \tag{8}$$

where the following notations are used:

$$\mathbf{Y}(k) = \Psi\,\hat{\mathbf{x}}(k) + \Phi\,u(k-1) + \Phi_\Delta \Delta\mathbf{U}(k) \tag{9}$$

$$\Theta(k) = \mathbf{Y}_r(k) - \Psi\,\hat{\mathbf{x}}(k) - \Phi\,\hat{u}(k-1) \tag{10}$$

with $\Sigma_i = \mathbf{C}\sum_{j=0}^{i}\mathbf{A}^{i-j}\mathbf{B}$, $\Phi = \left[\mathbf{\Sigma}^{\mathrm{T}}_{N_1-1}\ \cdots\ \mathbf{\Sigma}^{\mathrm{T}}_{N_2-1}\right]^{\mathrm{T}}$, $\mathbf{\Sigma}^{\mathrm{T}}_i = (\mathbf{\Sigma}_i)^{\mathrm{T}}$, $\Psi = \left[(\mathbf{C}\mathbf{A}^{N_1})^{\mathrm{T}}\ \cdots\ (\mathbf{C}\mathbf{A}^{N_2})^{\mathrm{T}}\right]^{\mathrm{T}}$,

$$\Phi_\Delta = \begin{bmatrix} \Sigma_{N_1-1} & \cdots & \Sigma_0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \Sigma_{N_2-1} & \cdots & \Sigma_{N_2-N_1} & \Sigma_{N_2-N_1-1} & \cdots & \Sigma_{N_2-N_u} \end{bmatrix}.$$

The analytical minimization of (8) provides:

$$\Delta\mathbf{U}(k) = (\mathbf{R}_J + \Phi_\Delta^{\mathrm{T}}\mathbf{Q}_J\Phi_\Delta)^{-1}\,\Phi_\Delta^{\mathrm{T}}\,\mathbf{Q}_J\,\Theta(k) \qquad (11)$$

According to the receding horizon principle used in predictive control, only the first component of the future control sequence $\Delta u(k) = \mathbf{e}_1^{\mathrm{T}}\Delta\mathbf{U}(k)$ is applied to the system, with $\mathbf{e}_1^{\mathrm{T}} = [1\ 0\ \cdots\ 0]$. Using the notation $\boldsymbol{\mu} = \mathbf{e}_1^{\mathrm{T}}\,(\mathbf{R}_J + \Phi_\Delta^{\mathrm{T}}\mathbf{Q}_J\,\Phi_\Delta)^{-1}\Phi_\Delta^{\mathrm{T}}\,\mathbf{Q}_J$, the control signal can be written as:

$$\Delta u(k) = \boldsymbol{\mu}\,\Theta(k) \qquad (12)$$

As shown in Fig. 1, the control signal has the following form:

$$\Delta u(k) = F_r y_r(k) - \mathbf{L}\,\hat{\mathbf{x}}_e(k) \qquad (13)$$

It appears from the cost function minimization that the control gain $\mathbf{L} = [\mathbf{L}_1\ L_2]$ and the setpoint filter F_r can be expressed as following:

$$\mathbf{L}_1 = \boldsymbol{\mu}\Psi\,,\ L_2 = \boldsymbol{\mu}\Phi\,,\ F_r = \sum_{j=1}^{N_2-N_1+1}\boldsymbol{\mu}(j) \qquad (14)$$

3. IMPROVED ROBUSTNESS USING YOULA PARAMETRIZATION

This part presents a method that improves the robustness of the previous Model Predictive Control using the Youla-Kučera parameter, also called Q parameter. As proved in [2] and [8], any stabilizing controller can be represented by a particular state-space feedback controller coupled with an observer and a Q parameter. This Section presents the most important steps leading to the Youla parameter that robustifies the previous MPC law.

3.1 Stabilizing Controller

Giving an initial stabilizing controller, the Youla parametrization permits to construct the class of stabilizing controllers. Auxiliary input u' and output y' with a zero transfer between them ($T_{22_{zw}} = 0$ in Fig. 2) are first added. Next step is to connect a stable Q parameter between y' and u'. As $T_{22_{zw}} = 0$ the closed-loop stability is guaranteed if Q parameter is stable. In this way, the closed-loop function between w and z is linearly parametrized by the Q parameter, allowing convex specification as detailed in [2]:

$$\mathbf{T}_{zw} = \mathbf{T}_{11_{zw}} + \mathbf{T}_{12_{zw}}\,Q\,\mathbf{T}_{21_{zw}} \qquad (15)$$

where $\mathbf{T}_{11_{zw}}$, $\mathbf{T}_{12_{zw}}$, $\mathbf{T}_{21_{zw}}$ depends on the input w and output z considered.

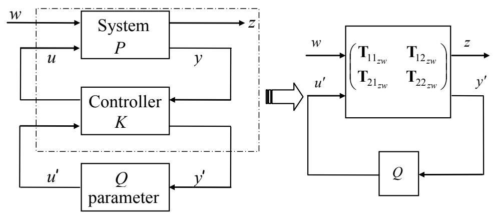

Fig. 2. Class of stabilizing controllers obtained with the Youla parametrization.

3.2 Robustness under frequency constraints

Practical applications always deal with neglected dynamics and possible disturbances, so that robustness towards unstructured uncertainties must be considered, as in Fig. 3.

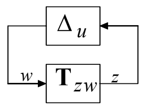

Fig. 3. Unstructured uncertainty.

According to the small gain theorem [12], [8], robustness under unstructured uncertainties Δ_u is maximized formulating a H_∞ norm minimization:

$$\min_{Q\in\Re H_\infty}\left\|\mathbf{T}_{zw}\right\|_\infty \qquad (16)$$

where the transfer $\mathbf{T}_{zw}$ also contains the weighting functions included to achieve the desired robustness specifications.

The following theorem formulates the previous H_∞ norm minimization.

Theorem ([3],[5]): A discrete time system given by the state-space representation $(\mathbf{A}_{cl},\mathbf{B}_{cl},\mathbf{C}_{cl},\mathbf{D}_{cl})$ is stable and admits a H_∞ norm lower than γ if and only if:

$$\exists\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \succ 0 \,/ \begin{bmatrix} -\mathbf{X}_1^{-1} & \mathbf{A}_{cl} & \mathbf{B}_{cl} & \mathbf{0} \\ \mathbf{A}_{cl}^{\mathrm{T}} & -\mathbf{X}_1 & \mathbf{0} & \mathbf{C}_{cl}^{\mathrm{T}} \\ \mathbf{B}_{cl}^{\mathrm{T}} & \mathbf{0} & -\gamma\mathbf{I} & \mathbf{D}_{cl}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_{cl} & \mathbf{D}_{cl} & -\gamma\mathbf{I} \end{bmatrix} \prec 0 \qquad (17)$$

As shown in [5], this expression can be transformed into a LMI, whose variables are $\mathbf{X}_1$, γ and the Q parameter included in the closed-loop matrices, so that the optimization problem is formulated as the minimization of γ under this LMI constraint.

3.2.1 Robustness towards additive unstructured uncertainties

Assuming that model uncertainties can be represented as additive disturbances [12], robust stability requirements towards this type of uncertainties are guaranteed by minimizing the H_∞ norm of the transfer from this disturbance to the control signal.

3.2.2 Robustness towards multiplicative unstructured uncertainties

Moreover system models contain modelling errors that can be described by multiplicative disturbances. Using the small gain theorem to achieve robust stability leads to the H_∞ norm minimization of the complementary sensitivity function [12], meaning that another LMI is added to the previous γ minimization.

3.3 Time domain constraints

This subsection considers the addition of nominal performance specifications, so that the output y influenced by a disturbance should remain in a certain time-domain template. This response y is influenced by the choice of the Q parameter which guarantees that the template will be satisfied. Choosing $z = y$ and $w = b$ as the internal disturbance in (15) assesses that the closed-loop is linear in the Q parameter, which allows convex specification for time-domain limitations [2]. The constraints restricted to the first $N_t + 1$ samples can be expressed as:

$$y_{\min}(k) \le y(k) \le y_{\max}(k), \forall k = \overline{0, N_t} \tag{18}$$

Imposing this time-domain template will add another LMI to the previous H_∞ norm minimization leading to a compromise between robust stability and nominal performance which can be handled searching the optimal Q parameter.

4. OFF-LINE ROBUSTIFICATION OF MPC

The robustification strategy developed in Section 3 is now applied to the initial MPC designed in Section 2. This initial stabilizing controller is robustified via Q parametrization, considering first the maximization of stability robustness towards additive unstructured uncertainties representing the H_∞ norm minimization of $\mathbf{T}_{z_u b}$ (Fig. 4). Next step is to guarantee the stability robustness towards multiplicative unstructured uncertainties equivalent to the H_∞ norm minimization of $\mathbf{T}_{z_y y'_r}$ (Fig. 4). A time-domain template is further added in order to guarantee nominal performance. These robustification problems will be solved using LMI tools.

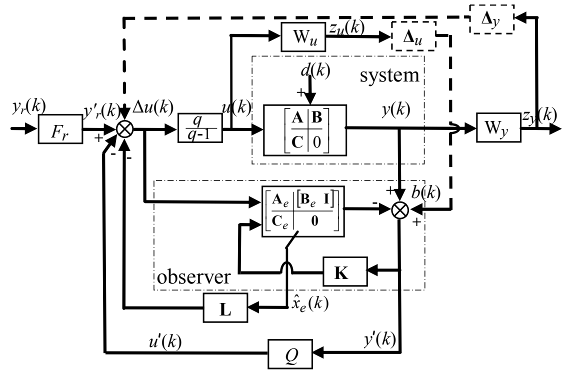

Fig. 4. Stabilizing MPC with Youla parametrization.

4.1 Stabilizing Controller

Let us consider the discrete time LTI system (3) in the state-space representation, including an integral action. Considering the additional input u' and output y' [2], the control law (13) applied to this system becomes (Fig. 4):

$$\Delta u(k) = F_r y_r(k) - \mathbf{L}\,\hat{\mathbf{x}}_e(k) - u'(k) \tag{19}$$

with the observer:

$$\begin{aligned}\hat{\mathbf{x}}_e(k+1) = \mathbf{A}_e\,\hat{\mathbf{x}}_e(k) + \mathbf{B}_e \Delta u(k) + \\ + \mathbf{K}[y(k) - \mathbf{C}_e\,\hat{\mathbf{x}}_e(k) + b(k)]\end{aligned} \tag{20}$$

where $\mathbf{L}$, F_r and $\mathbf{K}$ matrices are calculated as detailed in Section 2.

The initial state-space form (3) is then extended by adding the prediction error:

$$\boldsymbol{\varepsilon}(k) = \mathbf{x}_e(k) - \hat{\mathbf{x}}_e(k) \tag{21}$$

in order to calculate the closed-loop transfer:

$$\begin{bmatrix} \mathbf{x}_e(k+1) \\ \boldsymbol{\varepsilon}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_3 \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(k) \\ \boldsymbol{\varepsilon}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{B}_e \\ -\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} b(k) \\ u'(k) \end{bmatrix} \tag{22}$$

$$y'(k) = \begin{bmatrix} \mathbf{0} & \mathbf{C}_e \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(k) \\ \boldsymbol{\varepsilon}(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} b(k) \\ u'(k) \end{bmatrix} \tag{23}$$

with $\mathbf{A}_1 = \mathbf{A}_e - \mathbf{B}_e\mathbf{L}$, $\mathbf{A}_2 = \mathbf{A}_e - \mathbf{K}\mathbf{C}_e$, $\mathbf{A}_3 = \mathbf{B}_e\mathbf{L}$.

As described in Section 3.1, the Youla parameter can be connected to robustify the initial stabilizing controller, since the transfer from $y'(k)$ to $u'(k)$ is 0 (without measurement noise, the output y' depends only on $\boldsymbol{\varepsilon}(k)$, which is independent from $\mathbf{x}_e(k)$ and $u'(k)$).

4.2 Robustness under frequency constraints

4.2.1 Robustness towards additive unstructured uncertainties

As mentioned before, robust stability problem under additive unstructured uncertainties can be viewed as the H_∞ norm minimization of the term $\mathbf{T}_{z_u b} = \mathbf{T}_{ub} W_u$, where the weighting W_u reflects the frequency range in which model uncertainties are more significant. Including the state-space representation of W_u:

$$\begin{cases} \mathbf{x}_w(k+1) = \mathbf{A}_w \mathbf{x}_w(k) + \mathbf{B}_w u(k) \\ z_u(k) = \mathbf{C}_w \mathbf{x}_w(k) + \mathbf{D}_w u(k) \end{cases} \tag{24}$$

a new extended state-space can be emphasized, permitting the addition of the Youla parameter:

$$\begin{bmatrix} \overline{\mathbf{x}}_1(k+1) \\ \boldsymbol{\varepsilon}(k+1) \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_3 \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \overline{\mathbf{x}}_1(k) \\ \boldsymbol{\varepsilon}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\overline{\mathbf{B}}_{u_1'} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} b(k) \\ u'(k) \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} z_u(k) \\ y'(k) \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2 \\ \mathbf{0} & \mathbf{C}_e \end{bmatrix} \begin{bmatrix} \overline{\mathbf{x}}_1(k) \\ \boldsymbol{\varepsilon}(k) \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{D}_w \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b(k) \\ u'(k) \end{bmatrix} \tag{26}$$

with $\overline{\mathbf{x}}_1(k) = \begin{bmatrix} \mathbf{x}^T(k) & u(k-1) & \mathbf{x}_w^T(k) \end{bmatrix}^T$, $\overline{\mathbf{C}}_2 = \mathbf{D}_w \mathbf{L}$,

$$\overline{\mathbf{A}}_1 = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{L}_1 & \mathbf{B} - \mathbf{B}L_2 & \mathbf{0} \\ -\mathbf{L}_1 & 1 - L_2 & \mathbf{0} \\ -\mathbf{B}_w \mathbf{L}_1 & \mathbf{B}_w(1 - L_2) & \mathbf{A}_w \end{bmatrix}, \quad \overline{\mathbf{A}}_3 = \begin{bmatrix} \mathbf{B}\mathbf{L} \\ \mathbf{L} \\ \mathbf{B}_w \mathbf{L} \end{bmatrix},$$

$$\overline{\mathbf{B}}_{u_1'} = \begin{bmatrix} \mathbf{B}^T & 1 & \mathbf{B}_w^T \end{bmatrix}^T, \quad \overline{\mathbf{C}}_1 = \begin{bmatrix} -\mathbf{D}_w \mathbf{L}_1 & \mathbf{D}_w(1 - L_2) & \mathbf{C}_w \end{bmatrix}.$$

As described in Section 3.2, a Youla parameter is added for robustification purposes. Since $Q \in \Re H_\infty$, a sub-optimal solution to this convex optimization problem considers a finite-dimensional subspace generated by an orthonormal base of discrete stable transfer functions such as a polynomial or FIR (Finite Impulse Response) filter:

$$Q = \sum_{j=0}^{n_Q} q_j q^{-j} \tag{27}$$

To obtain the state-space representation of this Q parameter, a fixed pair $(\mathbf{A}_Q, \mathbf{B}_Q)$ is used; only the variable pair $(\mathbf{C}_Q, \mathbf{D}_Q)$ should be designed following the procedure detailed in [5]:

$$\begin{cases} \mathbf{x}_Q(k+1) = \mathbf{A}_Q \mathbf{x}_Q(k) + \mathbf{B}_Q y'(k) \\ u'(k) = \mathbf{C}_Q \mathbf{x}_Q(k) + \mathbf{D}_Q y'(k) \end{cases} \tag{28}$$

where $\mathbf{A}_Q = \begin{bmatrix} \mathbf{0}_{1,n_Q-1} & 0 \\ \mathbf{I}_{n_Q-1} & \mathbf{0}_{n_Q-1,1} \end{bmatrix}$, $\mathbf{B}_Q = \begin{bmatrix} 1 \\ \mathbf{0}_{n_Q-1,1} \end{bmatrix}$,

$\mathbf{C}_Q = [q_1 \ \cdots \ q_{n_Q}]$, $\mathbf{D}_Q = q_0$.

Adding the Youla parameter leads to the following closed-loop state-space description of the transfer between $b(k)$ and $z_u(k)$:

$$\begin{cases} \mathbf{x}_{cl}(k+1) = \mathbf{A}_{cl} \mathbf{x}_{cl}(k) + \mathbf{B}_{cl} b(k) \\ z_u(k) = \mathbf{C}_{cl} \mathbf{x}_{cl}(k) + \mathbf{D}_{cl} b(k) \end{cases} \tag{29}$$

where $\mathbf{x}_{cl} = \begin{bmatrix} \overline{\mathbf{x}}_1^T & \boldsymbol{\varepsilon}^T & \mathbf{x}_Q^T \end{bmatrix}^T$,

$$\mathbf{A}_{cl} = \begin{bmatrix} \overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_3 - \overline{\mathbf{B}}_{u_1'} \mathbf{D}_Q \mathbf{C}_e & -\overline{\mathbf{B}}_{u_1'} \mathbf{C}_Q \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_Q \mathbf{C}_e & \mathbf{A}_Q \end{bmatrix}, \mathbf{B}_{cl} = \begin{bmatrix} -\mathbf{D}_Q \overline{\mathbf{B}}_{u_1'} \\ -\mathbf{K} \\ \mathbf{B}_Q \end{bmatrix},$$

$$\mathbf{C}_{cl} = \begin{bmatrix} \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_2 - \mathbf{D}_w \mathbf{D}_Q \mathbf{C}_e & -\mathbf{D}_w \mathbf{C}_Q \end{bmatrix}, \ \mathbf{D}_{cl} = -\mathbf{D}_w \mathbf{D}_Q.$$

This state-space formulation $(\mathbf{A}_{cl}, \mathbf{B}_{cl}, \mathbf{C}_{cl}, \mathbf{D}_{cl})$ is the crucial point of the robustification method. As stated in the Theorem from Section 3, the stability robustness is guaranteed if the expression (17) is verified. The first step to transform (17) into a LMI consists in multiplying it to the right and to the left with positive definite matrices $\boldsymbol{\Pi} = \mathrm{diag}(\mathbf{X}_1, \mathbf{I}, \mathbf{I}, \mathbf{I})$ and $\boldsymbol{\Pi}^T$ as described in [5]. Thus the following inequality is derived:

$$\begin{bmatrix} -\mathbf{X}_1 & \mathbf{X}_1 \mathbf{A}_{cl} & \mathbf{X}_1 \mathbf{B}_{cl} & \mathbf{0} \\ \mathbf{A}_{cl}^{\mathrm{T}} \mathbf{X}_1 & -\mathbf{X}_1 & \mathbf{0} & \mathbf{C}_{cl}^{\mathrm{T}} \\ \mathbf{B}_{cl}^{\mathrm{T}} \mathbf{X}_1 & \mathbf{0} & -\gamma_1 \mathbf{I} & \mathbf{D}_{cl}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{C}_{cl} & \mathbf{D}_{cl} & -\gamma_1 \mathbf{I} \end{bmatrix} \prec 0 \tag{30}$$

which is not yet a LMI because terms such as $\mathbf{X}_1 \mathbf{A}_{cl}$ and $\mathbf{X}_1 \mathbf{B}_{cl}$ are not linear in the Lyapunov variable $\mathbf{X}_1$ and in the coefficients of the Youla parameter $\mathbf{C}_Q$ and $\mathbf{D}_Q$ (included in the state-space matrices). To overcome this problem, the bijective substitution (31) is introduced as in [5].

$$\begin{cases} \mathbf{R}^{n \times n} \to \mathbf{R}^{n \times n} \\ \mathbf{X}_1 = \begin{bmatrix} \mathbf{W}_1 & \mathbf{Z}_1 \\ \mathbf{Z}_1^{\mathrm{T}} & \mathbf{Y}_1 \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{R}_1 & \mathbf{S}_1 \\ \mathbf{S}_1^{\mathrm{T}} & \mathbf{T}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{W}_1^{-1} & -\mathbf{W}_1^{-1} \mathbf{Z}_1 \\ -\mathbf{Z}_1^{\mathrm{T}} \mathbf{W}_1^{-1} & \mathbf{Y}_1 - \mathbf{Z}_1^{\mathrm{T}} \mathbf{W}_1^{-1} \mathbf{Z}_1 \end{bmatrix} \end{cases} \tag{31}$$

For linearity reasons of future expressions, the following partitions are defined:

$$\mathbf{S}_1 = \begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\end{bmatrix},\ \mathbf{T}_1 = \begin{bmatrix}\mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{12}^{\mathrm{T}} & \mathbf{T}_{22}\end{bmatrix} \tag{32}$$

After technical manipulations such as the multiplication of (30) on the right with $\mathbf{\Gamma} = \mathrm{diag}\left(\begin{bmatrix}\mathbf{R}_1 & \mathbf{0} \\ \mathbf{S}_1^{\mathrm{T}} & \mathbf{I}\end{bmatrix}, \begin{bmatrix}\mathbf{R}_1 & \mathbf{0} \\ \mathbf{S}_1^{\mathrm{T}} & \mathbf{I}\end{bmatrix}, \mathbf{I}, \mathbf{I}\right)$ and on the left with $\mathbf{\Gamma}^T$, the following LMI form is obtained:

$$\left[\begin{array}{ccc|ccc|c|c}
-\mathbf{R}_1 & \mathbf{0} & \mathbf{0} & \overline{\mathbf{A}}_1\mathbf{R}_1 & t_1 & t_4 & t_7 & \mathbf{0} \\
* & -\mathbf{T}_{11} & -\mathbf{T}_{12} & \mathbf{0} & t_2 & t_5 & t_8 & \mathbf{0} \\
* & * & -\mathbf{T}_{22} & \mathbf{0} & t_3 & t_6 & t_9 & \mathbf{0} \\ \hline
* & * & * & -\mathbf{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & t_{10} \\
* & * & * & * & -\mathbf{T}_{11} & -\mathbf{T}_{12} & \mathbf{0} & t_{11} \\
* & * & * & * & * & -\mathbf{T}_{22} & \mathbf{0} & t_{12} \\ \hline
* & * & * & * & * & * & -\gamma_1 & -t_{13} \\ \hline
* & * & * & * & * & * & * & -\gamma_1
\end{array}\right] \prec 0 \tag{33}$$

with $t_1 = \overline{\mathbf{A}}_1\mathbf{S}_{11} - \mathbf{S}_{11}\mathbf{A}_2 - \mathbf{S}_{12}\mathbf{B}_Q\mathbf{C}_e + \overline{\mathbf{A}}_3 - \overline{\mathbf{B}}_{u_1'}\mathbf{D}_Q\mathbf{C}_e$,
$t_2 = \mathbf{T}_{11}\mathbf{A}_2 + \mathbf{T}_{12}\mathbf{B}_Q\mathbf{C}_e$, $t_3 = \mathbf{T}_{12}^{\mathrm{T}}\mathbf{A}_2 + \mathbf{T}_{22}\mathbf{B}_Q\mathbf{C}_e$,
$t_4 = \overline{\mathbf{A}}_1\mathbf{S}_{12} - \mathbf{S}_{12}\mathbf{A}_Q - \overline{\mathbf{B}}_{u_1'}\mathbf{C}_Q$, $t_5 = \mathbf{T}_{12}\mathbf{A}_Q$, $t_6 = \mathbf{T}_{22}\mathbf{A}_Q$,
$t_7 = -\overline{\mathbf{B}}_{u_1'}\mathbf{D}_Q + \mathbf{S}_{11}\mathbf{K} - \mathbf{S}_{12}\mathbf{B}_Q$, $t_8 = -\mathbf{T}_{11}\mathbf{K} + \mathbf{T}_{12}\mathbf{B}_Q$,
$t_9 = -\mathbf{T}_{12}^{\mathrm{T}}\mathbf{K} + \mathbf{T}_{22}\mathbf{B}_Q$, $t_{10} = \mathbf{R}_1\overline{\mathbf{C}}_1^{\mathrm{T}}$, $t_{13} = \mathbf{D}_Q\mathbf{D}_w^{\mathrm{T}}$,
$t_{11} = \mathbf{S}_{11}^{\mathrm{T}}\overline{\mathbf{C}}_1^{\mathrm{T}} + \overline{\mathbf{C}}_2^{\mathrm{T}} - \mathbf{C}_e^T\mathbf{D}_Q\mathbf{D}_w^{\mathrm{T}}$, $t_{12} = \mathbf{S}_{12}^{\mathrm{T}}\overline{\mathbf{C}}_1^{\mathrm{T}} - \mathbf{C}_Q^{\mathrm{T}}\mathbf{D}_w^{\mathrm{T}}$.

In this way, the stability robustness problem towards additive unstructured uncertainties can be written as follows:

$$\min_{LMI_1} \gamma_1 \tag{34}$$

where LMI_1 is the inequality (33).

4.2.4 Robustness towards multiplicative unstructured uncertainties

The aim of this part is to robustify the control law towards multiplicative unstructured uncertainties that may come from modelling errors. This can be equivalently solved minimizing the H_∞ norm of the complementary sensitivity function $\mathbf{T}_{z_y y_r'} = \mathbf{T}_{yy_r'}W_y$, where the weighting W_y has the following state-space description:

$$\begin{cases}\widetilde{\mathbf{x}}_w(k+1) = \widetilde{\mathbf{A}}_w\widetilde{\mathbf{x}}_w(k) + \widetilde{\mathbf{B}}_w y(k) \\ z_y(k) = \widetilde{\mathbf{C}}_w\widetilde{\mathbf{x}}_w(k) + \widetilde{\mathbf{D}}_w y(k)\end{cases} \tag{35}$$

Similar to the previous developments, after including the weighting W_y, the prediction error $\boldsymbol{\varepsilon}(k)$ and the Youla parameter, the following state-space representation is found:

$$\begin{cases}\widetilde{\mathbf{x}}_{cl}(k+1) = \widetilde{\mathbf{A}}_{cl}\widetilde{\mathbf{x}}_{cl}(k) + \widetilde{\mathbf{B}}_{cl}b(k) \\ z_y(k) = \widetilde{\mathbf{C}}_{cl}\widetilde{\mathbf{x}}_{cl}(k)\end{cases} \tag{36}$$

where:

$$\widetilde{\mathbf{x}}_{cl} = \begin{bmatrix}\widetilde{\overline{\mathbf{x}}}_1 \\ \boldsymbol{\varepsilon} \\ \mathbf{x}_Q\end{bmatrix}, \widetilde{\mathbf{A}}_{cl} = \begin{bmatrix}\widetilde{\overline{\mathbf{A}}}_1 & \widetilde{\overline{\mathbf{A}}}_3 - \widetilde{\overline{\mathbf{B}}}_{u_1'}\mathbf{D}_Q\mathbf{C}_e & -\widetilde{\overline{\mathbf{B}}}_{u_1'}\mathbf{C}_Q \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_Q\mathbf{C}_e & \mathbf{A}_Q\end{bmatrix},$$

$$\widetilde{\mathbf{B}}_{cl} = \begin{bmatrix}-\mathbf{D}_Q\widetilde{\overline{\mathbf{B}}}_{u_1'}^{\mathrm{T}} & -\mathbf{K}^{\mathrm{T}} & \mathbf{B}_Q^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},\ \widetilde{\mathbf{C}}_{cl} = \begin{bmatrix}\widetilde{\mathbf{C}}_1 & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

and denoting $\widetilde{\overline{\mathbf{x}}}_1(k) = \begin{bmatrix}\mathbf{x}^T(k) & u(k-1) & \widetilde{\mathbf{x}}_w^T(k)\end{bmatrix}^T$,

$$\widetilde{\overline{\mathbf{A}}}_1 = \begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{L}_1 & \mathbf{B} - \mathbf{B}L_2 & \mathbf{0} \\ -\mathbf{L}_1 & 1 - L_2 & \mathbf{0} \\ \widetilde{\mathbf{B}}_w\mathbf{C} & \mathbf{0} & \widetilde{\mathbf{A}}_w\end{bmatrix}, \widetilde{\overline{\mathbf{A}}}_3 = \begin{bmatrix}\mathbf{BL} \\ \mathbf{L} \\ \mathbf{0}\end{bmatrix},$$

$$\widetilde{\overline{\mathbf{B}}}_{u_1'} = \begin{bmatrix}\mathbf{B}^T & 1 & \mathbf{0}\end{bmatrix}^T,\ \widetilde{\mathbf{C}}_1 = \begin{bmatrix}\widetilde{\mathbf{D}}_w\mathbf{C} & 0 & \widetilde{\mathbf{C}}_w\end{bmatrix}.$$

To guarantee the robust stability towards multiplicative unstructured uncertainties, following the previous procedure leads to a second LMI:

$$\left[\begin{array}{ccc|ccc|c|c}
-\widetilde{\mathbf{R}}_1 & \mathbf{0} & \mathbf{0} & \widetilde{\overline{\mathbf{A}}}_1\widetilde{\mathbf{R}}_1 & \widetilde{t}_1 & \widetilde{t}_4 & \widetilde{t}_7 & \mathbf{0} \\
* & -\widetilde{\mathbf{T}}_{11} & -\widetilde{\mathbf{T}}_{12} & \mathbf{0} & \widetilde{t}_2 & \widetilde{t}_5 & \widetilde{t}_8 & \mathbf{0} \\
* & * & -\widetilde{\mathbf{T}}_{22} & \mathbf{0} & \widetilde{t}_3 & \widetilde{t}_6 & \widetilde{t}_9 & \mathbf{0} \\ \hline
* & * & * & -\widetilde{\mathbf{R}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{R}}_1\widetilde{\mathbf{C}}_1^T \\
* & * & * & * & -\widetilde{\mathbf{T}}_{11} & -\widetilde{\mathbf{T}}_{12} & \mathbf{0} & \widetilde{\mathbf{S}}_{11}^T\widetilde{\mathbf{C}}_1^T \\
* & * & * & * & * & -\widetilde{\mathbf{T}}_{22} & \mathbf{0} & \widetilde{\mathbf{S}}_{12}^T\widetilde{\mathbf{C}}_1^T \\ \hline
* & * & * & * & * & * & -\gamma_2 & 0 \\ \hline
* & * & * & * & * & * & * & -\gamma_2
\end{array}\right] \prec 0 \tag{37}$$

where the terms $\widetilde{t}_1$ to $\widetilde{t}_9$ are respectively the equivalent of t_1 to t_9 using the corresponding “tilde” coefficients.

In conclusion, robust stability problem towards additive and multiplicative unstructured uncertainties is transformed into the next optimization problem:

$$\min_{LMI1,\, LMI2} c^T\gamma \tag{38}$$

where $c^T\gamma$ is the objective function, $c^T = [c_1\ c_2]$, $\gamma = [\gamma_1\ \gamma_2]^T$, LMI_1 and LMI_2 are respectively the LMIs (33) and (37).

4.3 Time domain constraints

In this part nominal performance specifications are added as an output time-domain template, as specified in Section 3. It means that the output response to an internal disturbance must remain inside a specified template. As shown in Section 3.3, the time-domain response of the system including the internal disturbance is linear in the Youla parameter:

$$y(k) = s_1(k) + \sum_{i=0}^{n_Q} q_i s_2(k) q^{-i} \quad (39)$$

where $s_1(k)$ and $s_2(k)$ denote the coefficients obtained from the output response to the disturbance $d(k)$ (Fig. 4): $s_1(k) = T_{11_{yd}} d(k)$, $s_2(k) = T_{12_{yd}} T_{21_{yd}} d(k)$, where $T_{11_{yd}}$, $T_{12_{yd}}$ and $T_{21_{yd}}$ represent in fact the transfers from d to y, u' to y and d to y' respectively (Fig. 2). The matrix form of the response restricted to the first $N_t + 1$ samples is the following:

$$[y(0)\ y(1)\ \cdots\ y(N_t)]^{\mathrm{T}} = \mathbf{a}[\mathbf{D}_Q\ \ \mathbf{C}_Q]^{\mathrm{T}} + \mathbf{b} \quad (40)$$

with the notation

$$\mathbf{a} = \left[\begin{array}{c|cccc} s_2(0) & 0 & \cdots & 0 \\ s_2(1) & s_2(0) & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ s_2(N_t) & s_2(N_t-1) & \cdots & s_2(N_t - n_Q) \end{array}\right], \mathbf{b} = \begin{bmatrix} s_1(0) \\ s_1(1) \\ \vdots \\ s_1(N_t) \end{bmatrix}.$$

Denoting $\mathbf{a} = [\mathbf{a}_D \mid \mathbf{a}_C]$, (40) can be rewritten as:

$$[y(0)\ y(1)\ \cdots\ y(N_t)]^{\mathrm{T}} = \mathbf{a}_D \mathbf{D}_Q + \mathbf{a}_C \mathbf{C}_Q^{\mathrm{T}} + \mathbf{b} \quad (41)$$

Imposing the time-domain limitations (18) from Section 3.3, these time-domain constraints can be transformed into the following set of LMIs:

$$\boldsymbol{\alpha}_D \mathbf{D}_Q + \boldsymbol{\alpha}_C \mathbf{C}_Q^{\mathrm{T}} + \boldsymbol{\beta} \le 0 \quad (42)$$

with $\boldsymbol{\alpha}_D = \begin{bmatrix} \mathbf{a}_D \\ -\mathbf{a}_D \end{bmatrix}, \boldsymbol{\alpha}_C = \begin{bmatrix} \mathbf{a}_C \\ -\mathbf{a}_C \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mathbf{b} - \mathbf{y}_{\max} \\ -\mathbf{b} + \mathbf{y}_{\min} \end{bmatrix}.$

Adding the time-domain constraints leads to the addition of (42) to the H_∞ norm minimization problem (38) as follows:

$$\min_{LMI_1,\, LMI_2,\, LMI_3} c^T \gamma \quad (43)$$

where LMI_3 is the matrix inequality (42). Solving (43) leads to a polynomial Youla parameter that guarantees both robust stability and nominal performance requirements.

5. APPLICATION TO AN INDUCTION MACHINE

This section presents the main results obtained when applying the previous robustification methodology to the velocity control of an induction machine. Starting from the identified transfer function between the torque Γ_m and the velocity Ω_m for a sampling period $T_e = 5\text{ms}$:

$$\frac{\Omega_m(k)}{\Gamma_m(k)} = \frac{1.344\,q^{-1} + 3.024\,q^{-2}}{1 - 0.98\,q^{-1} - 0.02\,q^{-2}} \quad (44)$$

a 2 dimension state-space representation is designed. Adding the integral action leads to an extended model used to design an initial MPC controller in state-space description with the following tuning parameters $N_1 = 1$, $N_2 = 8$, $N_u = 1$, $\mathbf{Q}_J = \mathbf{I}_{N_2 - N_1 + 1}$ and $\mathbf{R}_J = 200\mathbf{I}_{N_u}$.

Since the effect of additive uncertainties is more significant at high frequency, the weighting $W_u(q^{-1}) = (1 - 0.3\,q^{-1})/0.7$ is considered. Respectively, the following weighting has been chosen $W_y(q^{-1}) = (1 - 0.1q^{-1})/0.9$ for the complementary sensitivity function minimization.

Adding the output time-domain template as shown in Fig. 6 (meaning that the output response for an input unitary step disturbance should respect this template), only the first $N_t = 38$ response coefficients are used in the computation of $s_1(k)$ and $s_2(k)$, as mentioned in Section 4.3.

The optimization problem with $c^T = [0.5\ 0.5]$ and a Youla parameter as a polynomial with a chosen degree of 35 was solved using LMI tools. As shown in Fig. 5 the influence of an additive unstructured uncertainty (corresponding to a high frequency neglected dynamic of the system) is decreased, as the H_∞ norm is reduced. Fig. 7 shows that robust stability towards a multiplicative unstructured uncertainty is also improved, as the H_∞ norm of the complementary sensitivity function is decreased. After robustification, the nominal performances are satisfied as well since the system response to a step disturbance remains within the specified template of Fig. 6.

Comparing the results shown in the last three figures, it can be easily observed that, solving the global minimization problem (43), a compromise between robust stability and nominal performance has been achieved. Indeed, solving only the first LMI (33) leads to the smallest H_∞ norm of the transfer $\mathbf{T}_{ub}$, but the response to a step disturbance does not respect the imposed template. Adding the time-domain constraints decreases the robust stability, keeping it within an acceptable domain, and also satisfies the specified time-domain template. Fig. 5 and 7 also show the compromise between robust stability toward additive and multiplicative unstructured uncertainties; if, on the one hand, a better minimization norm of $\mathbf{T}_{ub}$ is reached, then on the other hand, the complementary sensitivity function is increased, with a loss of robustness for this aspect.

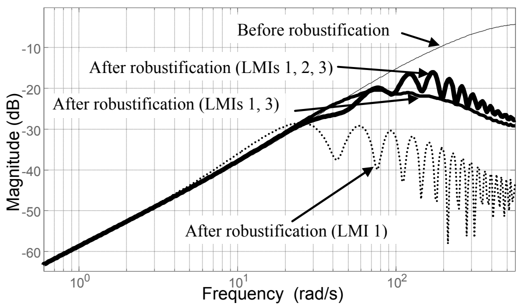

Fig. 5. Bode diagram of the transfer $\mathbf{T}_{ub}$ before and after robustification.

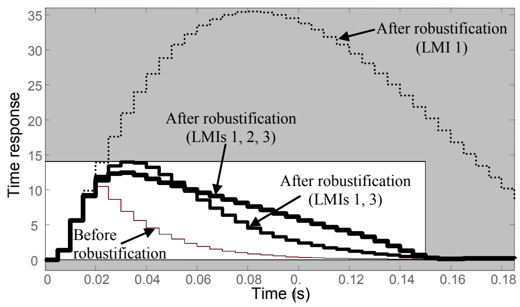

Fig. 6. Time-domain output template with response to disturbance before and after robustification.

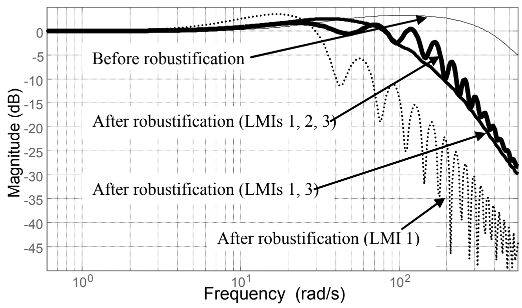

Fig. 7. Bode diagram of the complementary sensitivity function before and after robustification.

6. CONCLUSION

A unified off-line technique to improve robustness via Youla parametrization for MPC structures was presented in this paper. It combines the robust stability of an MPC law towards additive and multiplicative uncertainties with nominal performance specifications, performed as a convex optimization problem solved with dedicated LMIs. The main contribution is the way of managing the compromise between stability robustness and nominal performance.

The major advantage of this work is that the state-space formulation using LMI tools to design a robust MPC controller can be easily extended to MIMO systems, which is not the case of transfer function approaches.

Another perspective is to find the equivalent reduced order representation of the Youla parameter which robustifies the control law.

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