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L_2-NORM ORDER REDUCTION

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Abstract: *The paper deals with L_2-norm order reduction, i.e. with approximation of a given LTI continuous time system $S = (A, B, C)$ by a lower order system $\hat{S} = (\hat{A}, \hat{B}, \hat{C})$ such that the L_2-norm of the error system is small. The development is made for input balancing but a dual solution for output balancing can be easily obtained. A lot of numerical experiments performed validate the proposed numerical procedures.*

Keywords: *Input balancing, Schwartz form, L_2 optimization, model order reduction.*

1. INTRODUCTION

This paper is a continuation of our previous research works concerning the model order reduction problem, see for example [6].

Consider a (continuous time) linear system

$$(S) \quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases}$$

of order n with m inputs and l outputs which is stable, controllable and observable.

To begin with, let $S = (A, b, c^T)$ be a SISO system, i.e. with $m = l = 1$. The controllability gramian P satisfies

$$AP + PA^T + bb^T = 0 \tag{1}$$

and is given by

$$P = \int_0^\infty e^{tA} bb^T e^{tA^T} \mathrm{d}t. \tag{2}$$

Now, if we denote

$$e^{tA}b = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \quad c^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix},$$

then we have

$$h(t) \overset{\text{def}}{=} c^T e^{tA} b = \sum_{i=1}^{n} c_i g_i(t), \tag{3}$$

i.e. c_i are the coefficients in the expansion of $h(t)$ along the set of the "basis" functions

$g_i(t)$, $i = 1 : n$ generated by (A, b). Moreover, this set is *orthonormal* in $L_2(0, \infty)$ with respect to the following (standard) scalar product

$$(g_i, g_k) = \int_0^\infty g_i(t) g_k(t) \mathrm{d}t$$

if and only if S is "input balanced", i.e. $P = I_n$, see (2). Now, let

$$\hat{h}(t) = \sum_{i=1}^{\hat{n}} \hat{c}_i g_i(t), \qquad (4)$$

be an approximant of order $\hat{n} < n$ for $h(t)$. If $g_i(t)$ in (3) and (4) are orthonormal, then the optimal coefficients in the least squares (LS) sense

$$\min_{\hat{c}} \|h - \hat{h}\|^2 \qquad (5)$$

are given by $\hat{c}_i = c_i$, $i = 1 : \hat{n}$, see [4]. But, to get in (4) *the same* $g_i(t)$ as in (3) we need the whole pair (A, b), therefore from the systemic point of view (4) gives no (real) order reduction.

On the other hand, if we use in (4) *a different* (orthonormal) basis $\hat{g}_i(t)$, $i = 1 : \hat{n}$, then the equality $\hat{c}_i = c_i$, $i = 1 : \hat{n}$ does not generally hold. To sum up, we need a more adequate "systemic" approach in order to obtain a simple and effective order reduction procedure.

Remark 1. The classical least square (LS) approximation of a given function $h(t)$ by (generalized) polynomials (4) is well known. In our case we want to choose a good (adapted) orthonormal basis $\hat{g}_i(t)$ generated by a stable and controllable pair $(\hat{A}, \hat{b})$ of order $\hat{n} < n$ as well as the coefficients $\hat{c}_i$, $i = 1 : \hat{n}$ such that the LS error (5) (reformulated in systemic terms) is as small as possible.

2. INPUT BALANCING

In the multi-input, multi-output (MIMO) case let us solve the Lyapunov equation

$$AP + PA^T + BB^T = 0 \qquad (6)$$

and compute a Cholesky factorization

$$P = RR^T \qquad (7)$$

for the controllability gramian P, where R is a (upper or lower) triangular matrix. (Both jobs are realized at once by using a Hammarling-like algorithm, but this point is no so important at the moment). Now (6) can be written as

$$ARR^T + RR^T A^T + BB^T = 0. \qquad (8)$$

Therefore if we perform the similarity transformation

$$\begin{cases} A \leftarrow \tilde{A} = R^{-1}AR, & B \leftarrow \tilde{B} = R^{-1}B, \\ C \leftarrow \tilde{C} = \quad CR, & \end{cases} \qquad (9)$$

then the linear system $S = (A, B, C)$ is *input balanced* [3], i.e. has $P = I_n$ and the new (updated) system matrices given by (9) satisfy

$$A + A^T + BB^T = 0. \qquad (10)$$

In the single-input case ($m = 1$) this means the corresponding functions $g_i(t)$, $i = 1 : n$, are orthonormal. In the general case the same is true for the rows of the matrix $e^{tA}B$.

Remark 2. The linear transformation R which makes $P = I_n$ is by no means unique, e.g. we may use the (ordered) eigenvalue decomposition

$$P = U \Lambda U^T, \qquad (11)$$

where U is orthogonal and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

with

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0,$$

by letting $R = U \Lambda^{\frac{1}{2}}$ or the symmetric *square root* $R = P^{\frac{1}{2}} = U \Lambda^{\frac{1}{2}} U^T$. In fact, it is easy to see that R is unique up to an orthogonal transformation $R \leftarrow RV$, i.e. input balancing is conserved by orthogonal similarity transformation

$$(A, B, C) \leftarrow (\tilde{A}, \tilde{B}, \tilde{C}) = (V^T AV, V^T B, CV).$$

Especially, we may use the *ordered* decomposition (11) with the aim to concentrate the largest L_2-energy in the first rows of $e^{tA}B$.

Remark 3. In the single-input ($m = 1$) case there exists an almost unique ("canonical") form of the input-balanced pair (A, b), namely the *Schwartz form*, given by

$$A_0 = \begin{bmatrix} -a_1 & -h_2 & \cdots & 0 & 0 \\ h_2 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & -h_n \\ 0 & 0 & \cdots & h_n & 0 \end{bmatrix},$$

$$b_0^T = \begin{bmatrix} h_1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (12)$$

where $h_1 = \pm\sqrt{2a_1}$ with $a_1 > 0$ the second coefficient of the characteristic polynomial of A, i.e.

$$\det(sI_n - A) = s^n + a_1 s^{n-1} + \ldots + a_n,$$

and h_i, $i = 2 : n$ are unique (up to signs) determined by (A, b). Taking in account the previous Remark, the Schwartz form can be easily computed as follows:

1. Perform the input balancing of the system by solving the Lyapunov matrix equation (8) and performing the transformations (9).
2. Reduce the balanced pair (A, b) to upper Hessenberg form by orthogonal transformations $(A, b) \leftarrow (A_0, b_0) = (U_0^T A U_0, U_0^T b)$ (see [5] for the algorithm).

The computed pair (A, b) is in Schwartz form because it must satisfy (10).

Example 1. Let be a single-input pair (A, b) in standard controllability form, with A having the eigenvalue spectrum

$$\lambda(A) = \{-1, -2, -3, -4, -5\},$$

i.e.

$$A = \begin{bmatrix} -15 & -85 & -225 & -274 & -120 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Schwartz form obtained by using the above algorithm is

$$A_0 = \begin{bmatrix} -15.0 & -8.36 & 0 & 0 & 0 \\ 8.36 & 0 & 3.34 & 0 & 0 \\ 0 & -3.46 & 0 & -1.75 & 0 \\ 0 & 0 & 1.75 & 0 & 0.84 \\ 0 & 0 & 0 & -0.84 & 0 \end{bmatrix},$$

$$b_0^T = \begin{bmatrix} -5.47 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

In the multi-input case, by using a similar algorithm (statement 2 is the reduction of the controllable pair (A, B) to the upper block-Hessenberg form - see [5] for the algorithm), one obtains a similar *block Schwartz form*

$$A_0 = \begin{bmatrix} A_1 & -H_2^T & \cdots & 0 & 0 \\ H_2 & A_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & A_{q-1} & -H_q^T \\ 0 & 0 & \cdots & H_q & A_q \end{bmatrix},$$

$$B_0^T = \begin{bmatrix} H_1^T & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (13)$$

where H_1 is an epic $m_1 \times m$ block ($m_1 = \mathrm{rank}B$), A_1 is a $m_1 \times m_1$ block satisfying $A_1 + A_1^T + H_1 H_1^T = 0$, A_i, $i = 2 : q$ are antisymmetric diagonal blocks and H_i, $i = 2 : q$ are blocks of corresponding dimensions.

Example 2. Let be a two-input, stable, controllable pair (A, B) with the same A as in example 1 and

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T.$$

The block Schwartz form obtained by using the above algorithm is

$$A_0 = \begin{bmatrix} -14.78 & -7.97 & -1.77 & 0 & 0 \\ 7.97 & -0.21 & -3.34 & 0 & 0 \\ 1.77 & 3.34 & 0 & -1.79 & 0 \\ 0 & 0 & 1.79 & 0 & -0.85 \\ 0 & 0 & 0 & -0.85 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1.19 & -0.63 & 0 & 0 & 0 \\ 5.30 & 0.14 & 0 & 0 & 0 \end{bmatrix}^T.$$

In the multi-input case the uniqueness property mentioned in the single input case does not hold. For example, by computing first the block-Hessenberg controllable form the pair (A, B) and then performing the input balancing one obtains another block Schwartz form

$$A_0 = \begin{bmatrix} -11.58 & -14.01 & 0 & 0 & 0 \\ 1.93 & -3.41 & -3.78 & 0 & 0 \\ 0 & 3.78 & 0 & -1.79 & 0 \\ 0 & 0 & 1.79 & 0 & -0.85 \\ 0 & 0 & 0 & 0.85 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 1.35 & 0 & 0 & 0 & 0 \\ 4.61 & 2.61 & 0 & 0 & 0 \end{bmatrix}^T .$$

3. THE L_2 PROBLEM

Having the given LTI system $S = (A, B, C)$ in the input-balanced form (9) (eventually with the pair (A, B) in Schwartz form), let us select a reduced order $\hat{n} < n$ and write

$$\begin{cases} A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \end{cases} \qquad (14)$$

where $A_1 \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $B_1 \in \mathbb{R}^{\hat{n}\times m}$ and $C_1 \in \mathbb{R}^{l\times\hat{n}}$. Also define

$$(\hat{A}, \hat{B}) = (A_1, B_1) \qquad (15)$$

and consider the reduced order system

$$(\hat{S}) \quad \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ y = \hat{C}\hat{x}, \end{cases}$$

where $\hat{C}$ is a free $l \times \hat{n}$ real matrix.

Remark 4. The system $\hat{S}$ is stable, controllable and input-balanced because from (10) and (13) by taking leading blocks we get

$$\hat{A} + \hat{A}^T + \hat{B}\hat{B}^T = 0. \qquad (16)$$

In fact we can take $\hat{C} = C_1$ (i.e. construct $\hat{S}$ as a simple input-balanced truncation of S) but this choice is clearly not optimal with respect to the LS criterion (5).

As a matter of facts, we shall state the *sub-optimal L_2-norm order reduction problem* as follows.

Let be given $S = (A, B, C)$ and let us consider $\hat{S} = (\hat{A}, \hat{B}, \hat{C})$, where $(\hat{A}, \hat{B})$ is defined by (15). Choose $\hat{C}$ such that

$$\min_{\hat{C}} \|S - \hat{S}\|, \qquad (17)$$

whith $\|S\|$ denoting the systemic L_2-norm, i.e. if $S = (A, B, C)$ is a stable system, then

$$\|S\|^2 = \operatorname{tr} CPC^T,$$

where P satisfies $AP + PA^T + BB^T = 0$.

4. THE OPTIMAL L_2 SOLUTION

In order to solve (17), let us consider the *error system* $\Sigma = S - \hat{S}$, defined by

$$(\Sigma) \quad \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ \hat{B} \end{bmatrix} u \\ e = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \end{cases} \qquad (18)$$

where (A, B) and $(\hat{A}, \hat{B})$ are as in (9) and (15). The corresponding input gramian Π, partitioned as

$$\Pi = \begin{bmatrix} P & Q \\ Q^T & \hat{P} \end{bmatrix} > 0, \qquad (19)$$

satisfies

$$\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} P & Q \\ Q^T & \hat{P} \end{bmatrix} + \begin{bmatrix} P & Q \\ Q^T & \hat{P} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ 0 & \hat{A}^T \end{bmatrix} + \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \begin{bmatrix} B^T & \hat{B}^T \end{bmatrix} = 0,$$

i.e.

$$AP + PA^T + BB^T = 0,$$
$$\hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = 0$$

where, in according to (10) and (16), we have $P = I_n$ and $\hat{P} = I_{\hat{n}}$. Additionally

$$AQ + Q\hat{A}^T + B\hat{B}^T = 0. \qquad (20)$$

Moreover

$$\|\Sigma\|^2 = \operatorname{tr} \begin{bmatrix} C & -\hat{C} \end{bmatrix} \begin{bmatrix} P & Q \\ Q^T & \hat{P} \end{bmatrix} \begin{bmatrix} C^T \\ -\hat{C}^T \end{bmatrix},$$

hence

$$\|\Sigma\|^2 = \operatorname{tr} (CC^T - CQ\hat{C}^T - \hat{C}Q^TC^T + \hat{C}\hat{C}^T)$$

is minimized for

$$\hat{C}^* = CQ \qquad (21)$$

and the optimal error is determined by the Schur complement of $\hat{P}$ in (19), i.e.

$$\|\Sigma^*\|^2 = \operatorname{tr} C(I_n - QQ^T)C^T. \qquad (22)$$

5. THE COMPUTATIONAL PROCEDURE

By following the above developments, we can present two algorithms to compute a sub-optimal L_2-norm reduced order approximation of a given stable and controllable linear system $S = (A, B, C)$.

The first one is as follows.

1. Perform the input balancing by solving the Lyapunov matrix equation (8) and applying the transformations (9).
2. Choose the reduced order $\hat{n}$ and determine the pair $(\hat{A}, \hat{B})$.
3. Solve the Sylvester matrix equation (20) for the matrix Q.
4. Compute the sub-optimal $\hat{C}^*$ from (21).

On the above algorithm we have to make two remarks. First, our numerical experiments show very clearly that in order to compute the input balanced form (9) the Hammarling-like algorithms for solving positive definite Lyapunov matrix equations [2] are much better than to compute the Cholesky factorization of the computed solution by standard methods. Second, it is difficult to quantify the benefits of the use of Schwartz form instead of unstructured input balanced pairs (A, B).

The main contribution of the paper consists in the following second computational procedure, which is numerically sound and highly efficient.

1. As a preliminary step, we reduce A to the (real) Schur form S and apply the corresponding similarity transformations to matrices B and C

$$\left\{ \begin{array}{ll} A \leftarrow S = U^T A U, & B \leftarrow \tilde{B} = U^T B, \\ C \leftarrow \tilde{C} = CU, & \end{array} \right. \tag{23}$$

   where U is orthogonal and S is (quasi)upper triangular.

2. Next, by using a Hammarling-like procedure (see [2]), we compute the *upper triangular* Cholesky factor R, see (7), (8), and perform the input balancing transformation (9).

   Clearly, because R is upper triangular, the Schur form of A is preserved and, additionally, the pair (A, B) is input balanced.

3. For a given reduced order $\hat{n} < n$, let consider the partition

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

   where $A_2 \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $B_2 \in \mathbb{R}^{\hat{n}\times m}$ and $C_2 \in \mathbb{R}^{l\times\hat{n}}$ and select

$$(\hat{A}, \hat{B}) = (A_2, B_2). \tag{24}$$

   Clearly, by the same argument as before, $(\hat{A}, \hat{B})$ is stable, controllable and input balanced (see Remark 4).

   The reason for which (24) is now the right choice is obvious, namely the corresponding Sylvester equation (20) has the simple apparent solution [1]

$$Q = \begin{bmatrix} 0 \\ I_{\hat{n}} \end{bmatrix}. \tag{25}$$

   Therefore, from (21) we immediately get

$$C = C_2 \tag{26}$$

   and moreover, the L_2-norm (22) of the corresponding (sub-optimal) error system is symply

$$\|\Sigma^*\| = \|C_1\|_F. \tag{27}$$

   The relation (24) toghether with (27) gives directly the reduced order approximation $\hat{S} = (\hat{A}, \hat{B}, \hat{C})$ as a simple tail-truncation of the system $S = (A, B, C)$, brought to the input-balanced Schur form. As a consequence of this remarkable fact we may complete with a final *optimization step* 4.

[1] Indeed, by multiplying (10) on right with Q given bellow and taking into account the (block)-upper triangular structure of A we get (20).

4. Iteratively reorder the Schur form of A by using orthogonal similarity transformations (which preserve both Schur form and balancing property) such the corresponding C_1 has the smallest Frobenius norm.

Three main points concerning the above computational procedure must be stressed.

a) The proposed procedure is entirely based and intensively exploits the computational facilities offered by the (ordered) Schur-form in conjunction with Hammarling algorithm for solving the Lyapunov equation for the controllability gramian.

b) The procedure delivers a very simple expression (26) for the approximation error involved in the order reduction process, which facilitates the development of the final optimization step.

c) The procedure works with A in Schur form, therefore keeps track at every step on the system poles which are retained or removed in the reduction order process.

This last point covers an old and yet unanswered question in system theory, namely which are the essential poles of a given (large order) linear system and which are the best quantitative indices ascertaining their dynamic significance.

6. NUMERICAL EXPERIMENTS

We have tested the presented procedures by performing a lot of numerical experiments which confirm the theoretical expectations. Like in other methods, a problem is the choice of the reduced order $\hat{n}$ of the approximation model. Obviously, the greater $\hat{n}$ lead to better approximations. A good choice can be obtained by increasing $\hat{n}$ and stop when the decreasing of the error system L_2 norm is too small. The experimental study of the procedure based on Schur form did not give a definitely conclusion on the significance of the original system poles. The common opinion that the big time constants are more important than the small ones is not generally true in the context of our problem.

7. CONCLUDING REMARKS

The approximation of a given LTI continuous time system $S = (A, B, C)$ by a lower order system $\hat{S} = (\hat{A}, \hat{B}, \hat{C})$ such that the L_2-norm of the error system is as small as possible is an interesting alternative to other model order reduction methods like truncation of entirely balanced space state models, methods based on cross-gramians etc. Additionally, our main procedure, based on the truncation of the ordered Schur form, delivers a reduced order approximation which preserves a part of the original system poles. This way opens a possibility to find an answer to the question of which poles of a given system are significant and which are not and why.

All the presented solutions are based on the input balancing but a similar dual development based on the output balancing can be easily obtained. The performed numerical experiments validate the presented numerical procedures.

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