Non–minimal Spectral Factorization of a Descriptor System ⋆

Cristian Oară, Cristian Flutur

Faculty of Automatic Control and Computers, University "Politehnica" Bucharest, Splaiul Independentei 313, RO 060 042, Bucharest, ROMANIA, Emails: {oara,flutur}@riccati.pub.ro

Abstract: Given a descriptor (singular) system whose transfer function matrix $W(\lambda)$ is analytic and invertible on a Cauchy contour Γ, we use state–space realizations to construct the spectral factorization of $W(\lambda)$ with respect to Γ in the general case when there is no minimal factorization. Besides the simple algebraic nature of the arguments, we extend the theory of non–minimal Wiener–Hopf factorization of biproper rational matrices to the case of matrices which are polynomial, strictly proper or improper.

Keywords: linear systems, descriptor systems, nonminimal factorization, nonsymmetric factorization, centered realizations.

1. INTRODUCTION

Let $W(\lambda)$ be a square (not necessarily proper) $m \times m$ rational matrix function (rmf) and Γ a Cauchy contour in the complex plane $\mathbb{C}$ such that $W(\lambda)$ is analytic and invertible on Γ, i.e., $W(\lambda)$ has neither poles nor zeros on Γ. Denote by Γ_+ (Γ_-) the interior (exterior) domains of Γ. The problem of writing $W(\lambda)$ as

$$W(\lambda) = W_+(\lambda)W_-(\lambda), \tag{1}$$

where $W_+(\lambda)$ and $W_-(\lambda)$ are analytic and invertible on $\Gamma_- \cup \Gamma$ and $\Gamma_+ \cup \Gamma$, respectively, is called the spectral factorization with respect to Γ. Provided the two spectral factors $W_+(\lambda)$ and $W_-(\lambda)$ exist, (1) is actually a minimal factorization in the sense that

$$\delta(W) = \delta(W_+) + \delta(W_-), \tag{2}$$

where $\delta(W)$ denotes the McMillan degree of the rmf $W(\lambda)$.

The spectral factorization problem (1) is solvable if and only if some complementarity condition is met (see Bart et al. (1980)) – and in this case is called *minimal*. The complementarity condition is written on the basis of a geometric factorization principle expressed in terms of invariant subspaces of the pole and system pencil associated with a minimal state–space realization of $W(\lambda)$. Otherwise, in order to separate the two spectral factors one has to relax the underlying minimality requirement (2) and add certain poles and zeroes in Γ_+ and Γ_- in order to force the fulfillment of the respective complementarity condition. Once the minimality requirement relaxed, the spectral factorization problem is replaced with

$$W(\lambda) = W_+(\lambda)D(\lambda)W_-(\lambda), \tag{3}$$

where the spectral factors $W_-(\lambda)$ and $W_+(\lambda)$ fulfill the same requirements as before and $D(\lambda)$ is a diagonal rmf of the form

$$\operatorname{diag}\left\{\left(\frac{\lambda-\lambda_-}{\lambda-\lambda_+}\right)^{k_1}, \left(\frac{\lambda-\lambda_-}{\lambda-\lambda_+}\right)^{k_2}, \ldots, \left(\frac{\lambda-\lambda_-}{\lambda-\lambda_+}\right)^{k_m}\right\},$$

where λ_+ and λ_- are two arbitrary finite points in Γ_+ and Γ_-, respectively, different from the poles and zeroes of $W(\lambda)$, and $k_1 \le k_2 \le \cdots \le k_m$ are a set of integers called the *Wiener–Hopf factorization indices* (see Bart et al. (1986b, 2008)). Though the factorization indices are unique, the spectral factorization problem (3) has many different solutions corresponding to various possible choices of the additional canceling poles and zeroes, resulting in highly non–unique spectral factors. Fortunate enough, fixing the additional poles and zeroes λ_+ and λ_- recaptures to a large extent the uniqueness of the *non–minimal* factorization (3).

The spectral factorization both in its minimal and non–minimal versions plays important parts as main technical tool in solving robust control, estimation and filtering problems formulated in Krein spaces with definite and indefinite metric, problems in system identification, signal processing, network and circuit theory, to mention just a few (see for example Kimura (1997); Ionescu et al. (1999); Hassibi et al. (1999)). In particular, systems of singular integral equations, vector–valued Wiener–Hopf integral equations, equations involving block Toeplitz and Hankel matrices can be explicitly solved when a spectral factorization of the symbol of the equation is known (see for example Gohberg et al. (1974); Krein (1962)).

Since many fundamental problems in these branches of science can be solved once the factors are known, a wealth of research efforts has been invested in their construction and finding their various properties. Albeit their huge importance, all approaches proposed so far in the non–minimal case fail short in two respects: the restrictive hypotheses ($W(\lambda)$ should be proper and without zeroes at infinity) and the exceedingly cumbersome constructions used (see for example Bart et al. (1986a,b, 2008)).

In this paper we remedy these drawbacks by giving the main steps of a simple and self–contained algebraic derivation of the non–minimal spectral factorization (3) of a rmf

⋆ This work has been supported by the Romanian National University Research Council (CNCSIS) under grant ID 814/2007.

$W(\lambda)$ which may be polynomial, strictly proper or even improper. Based on elementary operations on state–space realizations, we construct both the spectral factors and the Wiener–Hopf factorization indices. The main result is expressed in terms of a special type of realization – called centered – that exhibits the same attractive features and allows for formulas that bear essentially the same elegant simplicity of the proper case.

Our approach is to reduce the factorization problem to finding a minimal McMillan degree invertible rmf that cancels *simultaneously* the poles and zeroes of $W(\lambda)$ in the interior (or in the exterior) of Γ. This is a generalization of the factorization techniques used in Oară et al. (2000, 2009b) where either some poles or some zeroes are cancelled by choosing a suitable left factor.

The paper is organized as follows. Section 2 contains some notation, definitions, and preliminaries on centered realizations. Section 3 states and proves the main result which is based on a technical lemma elaborated in an Appendix. Some concluding remarks are in Section 4.

2. PRELIMINARIES

By $\mathbb{C}$ we denote the complex plane and let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the closed complex plane. I_n will stand for the identity matrix of size $n \times n$. The subscript is dropped if the size is clear from the context. We use $\Lambda(A-\lambda E)$ to denote the set of generalized eigenvalues of the matrix pencil $A-\lambda E$ (finite and infinite). By $W(\lambda)$ we denote a rational matrix function with real or complex coefficients in the variable λ.

Let $W(\lambda)$ be a general $p \times m$ rmf (possibly improper or polynomial). For $W(\lambda)$ we introduce a particular type of realization called centered. Centered realizations have been previously used to solve various problems for singular systems whose transfer matrix function are improper Gohberg et al. (1992); Rakowski (1992); Oară et al. (2000). When associated with singular systems, centered realizations have the main advantage (over the more common generalized state–space realizations) of minimal order equal to the McMillan degree of the underlying system. They also allow to recapture all nice properties of standard state–space realizations. To define a centered realization we fix first a $\lambda_0 \in \overline{\mathbb{C}}$ and further α, β such that $\alpha = 1$, $\beta = 0$, if $\lambda_0 = \infty$, and $\alpha = \lambda_0$, $\beta = 1$, if λ_0 is finite. Throughout the paper for a fixed λ_0 we assume this implicit choice of α and β. Denote also $p(\lambda) := \alpha - \beta\lambda$. A realization centered at λ_0 of $W(\lambda)$ is a representation of the form

$$W(\lambda) = D + C(\lambda E - A)^{-1}B(\alpha - \beta\lambda) =: \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right]_{\lambda_0}, \tag{4}$$

where $A - \lambda E$ is a regular pencil (i.e., it is square and $\det(A-\lambda E) \not\equiv 0$), and the matrices A, E, B, C, D are $n\times n$, $n\times n$, $n\times m$, $p\times n$, $p\times m$, respectively, with real or complex elements. The positive integer n is called the *order* (or the *dimension*) of the realization (4). A realization is called *minimal* if its order is as small as possible. In particular, if $\lambda_0 = \infty$ we simply drop the index λ_0 from the notation introduced in the right–hand side of (4) and get the well–known generalized state–space realization used throughout the control theory of singular systems. Therefore, a generalized state–space (or descriptor) realization is simply a realization centered at $\lambda_0 = \infty$. Centered realizations can be obtained alternatively either by the procedures exposed in Rakowski (1992) (starting from the rmf representation) or by the algorithmic procedure in Section 5 of Oară et al. (2009a) which allows switching back and forth to a generalized state–space realization.

A realization (or the pair $(A - \lambda E, B)$) is called controllable if for all finite λ we have rank $[\, A - \lambda E \;\, B \,] = n$ and rank $[\, E \;\, B \,] = n$ (corresponding to $\lambda = \infty$). Analogously, we say that a realization (4) is observable (or the pair $(C, A - \lambda E)$ is observable) provided the pair $(A^T - \lambda E^T, C^T)$ is controllable. We call the realization (4) *proper* if $\alpha E - \beta A$ is invertible and *normalized* if in addition $\alpha E - \beta A = I$. Notice that $W(\lambda)$ has a proper realization centered at λ_0 only if it has no poles at λ_0. If the realization (4) is proper then $D = W(\lambda_0)$.

Two invertible matrices S and T acting on a realization (4) as

$$\left[\begin{array}{c|c} S(A-\lambda E)T^{-1} & SB \\ \hline CT^{-1} & D \end{array}\right]_{\lambda_0}$$

define a state–space *equivalence* transformation, and they leave unchanged the underlying transfer function matrix.

Remark 1. By a preliminary equivalence, we can assume a proper realization to be always normalized.

We have the following lemma which gives certain properties of centered realizations.

Theorem 2. Let $W(\lambda)$ be a $p \times m$ rmf of McMillan degree n and $\lambda_0 \in \overline{\mathbb{C}}$.

(I) Any realization (4) of $W(\lambda)$ has an order greater or equal to n, with equality possible if and only if the realization is proper.

(II) A proper realization (4) has minimal order n (and it is called minimal) if and only if it is controllable and observable.

(III) Given two proper minimal realizations

$$W(\lambda) = \left[\begin{array}{c|c} A_1-\lambda E_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{\lambda_0} = \left[\begin{array}{c|c} A_2-\lambda E_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{\lambda_0},$$

then $D_1 = D_2$ and there exist two unique invertible matrices S and T that define a state–space equivalence transformation between the two realizations, i.e., $S(A_1 - \lambda E_1)T^{-1} = A_2 - \lambda E_2, SB_1 = B_2, C_1T^{-1} = C_2$.

(IV) Given a centered realization (4) of $W(\lambda)$, if λ_0 is neither a pole nor a zero of $W(\lambda)$, then the following hold. i) rank $(D) = \text{rank}_n(W(\lambda))$; ii) $W(\lambda)$ is invertible if and only if D is invertible; iii) if D is invertible, a centered realization for the inverse is given by $W^{-1}(\lambda)$

$$= \left[\begin{array}{c|c} A-\alpha BD^{-1}C-\lambda(E-\beta BD^{-1}C) & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]_{\lambda_0} \tag{5}$$

and is proper as well. Provided the realization (4) is minimal/controllable/observable then (5) is also minimal/controllable/observable.

The proof is straightforward and is omitted for brevity.

With any realization (4) we associate two pencils that play an important role in the sequel: the *pole pencil* $\mathcal{P}_W(\lambda) = A - \lambda E$ and the *system pencil*

$$\mathcal{S}_W(\lambda) = \begin{bmatrix} A-\lambda E & Bp(\lambda) \\ C & D \end{bmatrix} = \begin{bmatrix} A & \alpha B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & \beta B \\ 0 & 0 \end{bmatrix}. \tag{6}$$

The next theorem shows that for a minimal realization there is a one–to–one correspondence between the poles, zeros, and their partial multiplicities on one side, and the generalized eigenvalues of the associated pole and system pencils on the other side (see Theorems 1 and 2 in Verghese et al. (1979)).

Theorem 3. Let $W(\lambda)$ be given by a minimal realization (4) of order n. Then we have: If $\mu \in \overline{\mathbb{C}}$ is a pole (zero) of $W(\lambda)$ with partial multiplicities $k_1 \geq k_2 \geq \ldots \geq k_g$, then μ is a generalized eigenvalue of $\mathcal{P}_W(\lambda)$ ($\mathcal{S}_W(\lambda)$) with partial multiplicities $s_1 \geq s_2 \geq \ldots \geq s_h$, where

$$\begin{cases} g = h, \quad \text{and} \quad k_i = s_i, \quad i = 1,\ldots,g, \quad \text{if} \quad \mu \neq \lambda_0, \\ g \leq h, \quad \text{and} \quad k_i = s_i - 1, \quad i = 1,\ldots,g, \quad \text{if} \quad \mu = \lambda_0. \end{cases}$$

For our main results we need the following solution to the generalized eigenvalue assignment problem (an extension of Lemma 4.1 in Oară et al. (2000)).

Lemma 4. Assume $(A - \lambda E, B)$ is a controllable pair, with A, $E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, having the nonzero row Kronecker indices (controllability indices) $n_1,\ldots,n_k$, ($\sum_{i=1}^k n_k = n$). Let an arbitrary $\lambda_\bullet \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$, not both zero, such that $\frac{\alpha}{\beta} \notin \Lambda(A - \lambda E)$ and $\frac{\alpha}{\beta} \neq \lambda_\bullet$.

1. There exists a matrix $F \in \mathbb{C}^{m\times n}$ such that

$$\Lambda(A - \lambda E + BF(\alpha - \lambda\beta)) = \{\lambda_\bullet\}. \tag{7}$$

2. There exist an equivalence transformation (S, T), an invertible matrix $V \in \mathbb{C}^{m\times m}$, and the matrix $F \in \mathbb{C}^{m\times n}$ can be chosen such that

$$S(A - \lambda E + BVF(\alpha - \lambda\beta))T^{-1} = \operatorname{diag}\{A_1,\ldots,A_k\} - \lambda I_n, \tag{8}$$

$$SBV = \left[\operatorname{diag}\{b_1,\ldots,b_k\} \vdots 0_{n\times(m-k)}\right], \tag{9}$$

where $A_i \in \mathbb{C}^{n_i\times n_i}$, $b_i \in \mathbb{C}^{n_i\times 1}$ are given explicitly by

$$A_i := \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\gamma_0 & -\gamma_1 & \ldots & -\gamma_{n_i-1} \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{10}$$

$$(\lambda - \lambda_\bullet)^{n_i} = \lambda^{n_i} + \gamma_{n_i-1}\lambda^{n_i-1} + \ldots + \gamma_1\lambda + \gamma_0.$$

Proof. Part 1. is a particular case of Lemma 4.1 in Oară et al. (2000). For part 2, assume $\beta A - \alpha E = I$. This is always possible by performing a preliminary equivalence transformation (see also Remark 1). Further, by the conformal mapping $\lambda = (\alpha z + \beta)/(\beta z - \alpha)$ the point $\lambda_\bullet$ changes to $z_\bullet$ which is obviously finite. Define $A_z - zE_z := \alpha A + \beta E - z(\beta A - \alpha E)$ and $B_z := (\alpha^2 + \beta^2)B$. With the transformed data we have reduced (7) to

$$\Lambda(A_z + B_zF - zE_z) = \{z_\bullet\}, \tag{11}$$

with $E_z = I$, which is actually a standard eigenvalue problem. From the controllability of the pair $(A - \lambda E,\, B)$, we get the controllability of the pair (A_z, B_z). Thus (11) is a standard eigenvalue assignment problem for the controllable pair (A_z, B_z) and has always a solution F which is the solution of the original problem (7) as well. From Section 2.8.1 in Ionescu et al. (1986), it follows that for the pair (A_z, B_z) there exists an equivalence transformation T, an invertible matrix V and a feedback matrix F such that $T(A_z - \lambda I + B_zVF(\alpha - \lambda\beta))T^{-1} = \operatorname{diag}\{A_{z1},\ldots,A_{zk}\} - \lambda I_n$, where A_{zi}, $i = 1,\ldots,k$, are companion matrices with characteristic polinomials $(z - z_0)^{n_i}$, and

$$TB_zV = \left[\operatorname{diag}\{b_{z_1},\ldots,b_{z_k}\} \vdots 0_{n\times(m-k)}\right],$$

where b_{z_i}, have the same form as b_i in (10), $i = 1,\ldots,k$. Mapping back to the original variable λ we get, by using an appropriate update of the transformation matrices S, T, V and F, precisely (8)–(10). ■

The following formula which will be used in the sequel can be easily proved (provided all the intervening matrices have appropriate dimensions)

$$\left[\begin{array}{c|c} A_1 - \lambda E_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]_{\lambda_0} \left[\begin{array}{c|c} A_2 - \lambda E_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]_{\lambda_0} = \left[\begin{array}{cc|c} A_1 - \lambda E_1 & B_1C_2(\alpha - \lambda\beta) & B_1D_2 \\ 0 & A_2 - \lambda E_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]_{\lambda_0}. \tag{12}$$

3. MAIN RESULT

Here follows the precise statement of the main factorization result.

Theorem 5. Let Γ be an arbitrary given Cauchy contour and $W(\lambda)$ an $m \times m$ rmf analytic and invertible on Γ. Denote by Γ_+ and Γ_- the interior and the exterior domains of Γ, respectively. Then $W(\lambda)$ has a factorization

$$W(\lambda) = W_+(\lambda)D(\lambda)W_-(\lambda), \tag{13}$$

where $W_+(\lambda)$ and $W_-(\lambda)$ are analytic and invertible on $\Gamma_- \cup \Gamma$ and $\Gamma_+ \cup \Gamma$, respectively, and $D(\lambda)$ is a diagonal rmf of the form

$$= \operatorname{diag}\left\{\left(\frac{\lambda - \lambda_-}{\lambda - \lambda_+}\right)^{k_1}, \left(\frac{\lambda - \lambda_-}{\lambda - \lambda_+}\right)^{k_2}, \ldots, \left(\frac{\lambda - \lambda_-}{\lambda - \lambda_+}\right)^{k_m}\right\}. \tag{14}$$

Here λ_+ and λ_- are two arbitrary finite points in Γ_+ and Γ_-, respectively, different from the poles and zeroes of $W(\lambda)$, and $k_1 \leq k_2 \leq \cdots \leq k_m$ are a set of integers called *the Wiener–Hopf indices* of the factorization.

As opposed to the spectral factors $W_+(\lambda)$ and $W_-(\lambda)$, the Wiener–Hopf indices are uniquely determined by the original function $W(\lambda)$ and the contour Γ. If $k_1 = k_2 = \ldots = k_m = 0$, the factorization is minimal, otherwise it is non–minimal.

Before proving the theorem, we recall and adapt first a technical tool used in Oară et al. (2000) for the non–canonical coprime factorization with J–inner denominator. This tool will be used further on to state a preparatory lemma used in the proof of the main result.

Assume the analytic and invertible on Γ rmf $W(\lambda)$ is given by the minimal centered realization

$$W(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]_{\lambda_0} \tag{15}$$

where a natural choice of λ_0 is on the Cauchy contour Γ. Since $D = W(\lambda_0)$ is invertible, without restricting generality, assume $D = I$. The inverse of $W(\lambda)$ can be given alternatively by the minimal realizations

$$\begin{aligned} W^{-1}(\lambda) &= \left[\begin{array}{c|c} F-\lambda K & G \\ \hline H & I \end{array}\right]_{\lambda_0} \\ &= \left[\begin{array}{c|c} A-\alpha BC-\lambda(E-\beta BC) & B \\ \hline -C & I \end{array}\right]_{\lambda_0}, \end{aligned} \tag{16}$$

where the first is a notation for an arbitrary realization while the second is constructed from (15) by using the inversion formula (5). Using part (III) of Theorem 2, there is a unique state–space equivalence transformation between the two realization (16) defined by S and T such that

$$S(A-\alpha BC-\lambda(E-\beta BC))T^{-1} = F-\lambda K,$$
$$SB = G, -CT^{-1} = H.$$

This equivalently implies the existence of two nonsingular matrices S and T such that

$$S(A-\lambda E)-(F-\lambda K)T = GC(\alpha-\lambda\beta). \tag{17}$$

With appropriate bases changes, assume the poles given by the generalized eigenvalues of $A-\lambda E$ belonging to Γ_+ and the zeroes given by the generalized eigenvalues of $F-\lambda K$ belonging to Γ_+ are already separated such that

$$\begin{aligned} A-\lambda E &= \begin{bmatrix} A_+-\lambda E_+ & A_\pm-\lambda E_\pm \\ 0 & A_--\lambda E_- \end{bmatrix}, \quad C = [\, C_+ \; C_- \,], \\ F-\lambda K &= \begin{bmatrix} F_--\lambda K_- & F_\pm-\lambda K_\pm \\ 0 & F_+-\lambda K_+ \end{bmatrix}, \quad G = \begin{bmatrix} G_- \\ G_+ \end{bmatrix}, \\ S &= \begin{bmatrix} S_1 & S_2 \\ S_+ & S_3 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & T_2 \\ T_+ & T_3 \end{bmatrix}, \end{aligned} \tag{18}$$

where $A_+-\lambda E_+$ specifies the poles and $F_+-\lambda K_+$ specifies the zeroes in Γ_+, while $A_- - \lambda E_-$ specifies the poles while $F_--\lambda K_-$ specifies the zeroes in Γ_- (see Theorem 3). With (18) in (17) we get from the (2,1)–block entry

$$S_+(A_+-\lambda E_+)-(F_+-\lambda K_+)T_+ = G_+C_+(\alpha-\lambda\beta). \tag{19}$$

Remark 6. If we have an equation (17) with all intervening matrices known and T invertible, then the realizations of both $W(\lambda)$ and $W^{-1}(\lambda)$ are well–defined in terms of these matrices. Indeed, evaluating (17) at $\lambda_0 = \frac{\alpha}{\beta}$, we get

$$S(A-\lambda_0 E)-(F-\lambda_0 K)T = 0 \tag{20}$$

and since $A-\lambda_0 E$ and $F-\lambda_0 K$ are both invertible as λ_0 is neither a pole nor a zero of $W(\lambda)$, it follows that S is invertible as well, $B = S^{-1}G$ and $H = -CT^{-1}$. Therefore, whenever we have an equation of the form (17), with T (and implicitly S) invertible, we may think at an associated rational matrix $W(\lambda)$ with a realization defined in terms of the matrices involved in this equation,

$$W(\lambda) = \left[\begin{array}{c|c} A-\lambda E & S^{-1}G \\ \hline C & I \end{array}\right]_{\lambda_0}, W^{-1}(\lambda) = \left[\begin{array}{c|c} F-\lambda K & G \\ \hline -CT^{-1} & I \end{array}\right]_{\lambda_0}.$$

Remark 7. Given equation (19), we may look at (17) and (18) as to an embedding equation. Alternatively, given (17) and the partition (18), we may look at equation (19) as to a restriction corresponding to the poles and zeroes in Γ_+.

The key to the factorization process are the matrices S_+ and T_+: if they are square and nonsingular then $W(\lambda)$ has a spectral factorization (1) which is minimal, i.e., fulfills (2) (see Bart et al. (1980)). Otherwise, there is only a non–minimal factorization (3) and some poles and zeroes should be added in order to separate in the two spectral factors the poles and zeroes in Γ_+ and Γ_-.

The needed technical tool is formalized as a lemma below whose proof is deferred to an Appendix. The first point is a customization of a result from Oară et al. (2000).

Lemma 8. Suppose (19) is given, where $S_+, T_+ \in \mathbb{R}^{m\times n}$, the pair $(F_+-\lambda K_+, G_+)$ is controllable, the pair $(C_+, A_+-\lambda E_+)$ is observable, and let $r := \operatorname{rank} T_+$.

1. There exists an embedding equation
$$S(A-\lambda E)-(F-\lambda K)T = GC(\alpha-\lambda\beta) \tag{21}$$
of (19), where the matrices in (21) are depicted in (18), with both T and S invertible of minimal dimension $m+n-r$, where $\Lambda(A_- - \lambda E_-)$ and $\Lambda(F_- - \lambda K_-)$ can be arbitrarily specified.
2. Let (21) and
$$\widetilde{S}(\widetilde{A}-\lambda\widetilde{E})-(\widetilde{F}-\lambda\widetilde{K})\widetilde{T} = \widetilde{G}\widetilde{C}(\alpha-\lambda\beta) \tag{22}$$
be two minimal embedding equations of (19), with invertible S, T, $\widetilde{S}$ and $\widetilde{T}$. Denote by $\widehat{W}(\lambda)$ and $\widetilde{W}(\lambda)$ the rmf associated with (21) and (22), respectively (see Remark 6). Then all poles given by $A_+ - \lambda E_+$ and zeroes given by $F_+ - \lambda K_+$ of both $\widehat{W}(\lambda)$ and $\widetilde{W}(\lambda)$ are canceled in $\widehat{W}^{-1}(\lambda)\widetilde{W}(\lambda)$.
3. The two embedding equations (21) and (22) at point 2 can always be constructed such that $\widehat{W}^{-1}(\lambda)\widetilde{W}(\lambda)$ is diagonal of the form (14).

Remark 9. Lemma 8 allows an extension of the results in Oară et al. (2000) and Oară et al. (2009b) in the sense that it characterizes the class of minimal McMillan degree invertible rmf which cancel simultaneously the poles and zeroes in Γ_+ of a rmf $W(\lambda)$. Once $W(\lambda)$ given by a minimal realization (which automatically defines the first equation (17) and the reduced equation (19) corresponding to the poles and zeroes in Γ_+), all rmf $\widetilde{W}(\lambda)$ having minimal McMillan degree that simultaneously cancel in the product $\widetilde{W}(\lambda)^{-1}W(\lambda)$ the poles $A_+ - \lambda E_+$ and zeroes $F_+ - \lambda K_+$ are obtained by constructing all minimal embeddings (22) of (19), with invertible $\widetilde{S}$ and $\widetilde{T}$. Indeed, the embedding equation (22) uniquely defines $\widetilde{W}(\lambda)$ via

$$\widetilde{W}(\lambda) = \left[\begin{array}{c|c} \widetilde{A}-\lambda\widetilde{E} & \widetilde{S}^{-1}\widetilde{G} \\ \hline \widetilde{C} & I \end{array}\right]_{\lambda_0}.$$

Proof.[Theorem 5]. Assume $W(\lambda)$ is given by the minimal centered realization (15). Form equation (17) and restrict it to (19) by splitting the poles and zeroes in the two disjoint regions Γ_+ and Γ_-. Getting a restriction (19) of equation (17) is equivalent to separating first the poles and computing further the maximal left deflating subspace with spectrum in Γ_+ of the regular system pencil. Clearly, the pair $(F_+ - \lambda K_+, G_+)$ is controllable while the pair $(C_+, A_+ - \lambda E_+)$ is observable. Therefore, we are in the position of Lemma 8. If S_+ and T_+ are either non–square or non–invertible, we construct two embedding equations (21) and (22) of minimal dimensions, which define the associated rmf denoted $\widehat{W}(\lambda)$ and $\widetilde{W}(\lambda)$, respectively. Lemma 8 guarantees that the two embedding equations can be constructed such that $\widehat{W}(\lambda)$ has all additional poles and zeroes at a specified point $\lambda_+ \in \Gamma_+$, $\widetilde{W}(\lambda)$

has all additional poles and zeroes at a specified point $\lambda_- \in \Gamma_-$, and $\widetilde{W}^{-1}(\lambda)\widehat{W}(\lambda)$ is diagonal of the form (14). Finally, define in the factorization (13) $W_+(\lambda) := \widehat{W}(\lambda)$, $D(\lambda) := \widehat{W}^{-1}(\lambda)\widetilde{W}(\lambda)$, $W_-(\lambda) = \widetilde{W}^{-1}(\lambda)W(\lambda)$. By construction, $W_+(\lambda)$ is analytic and invertible on $\Gamma_- \cup \Gamma$, $D(\lambda)$ is diagonal of the form (14), while from point 2 of Lemma 8 we get by setting $\widehat{W}(\lambda) := \widetilde{W}(\lambda)$ and $\widetilde{W}(\lambda) := W(\lambda)$ that all poles and zeroes in Γ_+ are canceled in the product $W_-(\lambda) = \widetilde{W}^{-1}(\lambda)W(\lambda)$. Hence $W_-(\lambda)$ is analytic and invertible on $\Gamma_+ \cup \Gamma$. This concludes the proof. ■

4. CONCLUSIONS

We have extended the non–minimal factorization of a transfer function matrix W to the cases in which $W(\lambda)$ is improper, strictly proper, or even polynomial. En route, we have considerably simplified the state–space construction of the spectral factors and Wiener–Hopf indices even in the standard case (when $W(\lambda)$ is biproper and invertible at infinity). Though constructive, our proof should be modified to some extent in order to provide a numerically–sound algorithm. To this end, the nonorthogonal bases transformations used in several instances should be avoided or replaced by orthogonal ones.

The results may be extended in several directions. The most challenging generalization is to the singular case where the rational matrix $W(\lambda)$ is non–invertible or even non-square but it still has no zeroes or poles on the Cauchy contour. In this case the constructive technique should be modified to some extent, although it essentially should remain the same. The central concept will be the left proper deflating subspace (see Ionescu et al. (1999)) with a certain spectrum of the zero pencil of $W(\lambda)$. This proper deflating subspace replaces equation (19) in the singular case while makes the tight connection between the poles and zeroes of $W(\lambda)$ inside and outside the Cauchy contour. Again, the minimality and non–minimality of the factorization (13) will be decided by the invertibility of a certain matrix, a sort of "disconjugacy" of the proper deflating subspace (see Ionescu et al. (1999) for details). For a given rational matrix of dimension $m \times n$ and normal rank r the resulting spectral factor $W_+(\lambda)$ in (13) will have dimension $m \times r$ and will be left invertible, the factor $W_-(\lambda)$ will be $r \times n$ and right invertible, while the $r \times r$ invertible diagonal factor $D(\lambda)$ will have some poles and zeroes in Γ_- and Γ_+. In this non–minimal singular case the freedom in choosing additional poles and zeroes is amplified by the singularity of $W(\lambda)$.

APPENDIX

Remark 10. We notice first that an equivalence transformation on the realization or a change of the input and output bases of $W(\lambda)$ has no effect on the factorization (13). Indeed, an equivalence transformation on the realization leaves $W(\lambda)$ unchanged, while a change of the input or output basis comes up to a pre– or post–multiplication of $W(\lambda)$ with a constant invertible matrix V and a simple update of the right or left spectral factor to $W_-(\lambda)V$ and $VW_+(\lambda)$, respectively.

Proof.[Lemma 8]. 1. Evaluating (19) at λ_0, we get that T_+ and S_+ always have the same rank, i.e., $r := \operatorname{rank} T_+ = \operatorname{rank} S_+$. We look now to construct an equation of form (21), with invertible T and S, which can be written with the partition in (18) in the explicit form

$$\begin{aligned} &\begin{bmatrix} S_1 & S_2 \\ S_+ & S_3 \end{bmatrix} \begin{bmatrix} A_+ - \lambda E_+ & A_\pm - \lambda E_\pm \\ 0 & A_- - \lambda E_- \end{bmatrix} \\ &- \begin{bmatrix} F_- - \lambda K_- & F_\pm - \lambda K_\pm \\ 0 & F_+ - \lambda K_+ \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_+ & T_3 \end{bmatrix} \\ &= \begin{bmatrix} G_- \\ G_+ \end{bmatrix} [\, C_+ \ C_- \,]\, p(\lambda). \end{aligned} \tag{23}$$

First, it is easy to see that a minimal invertible embedding T of T_+ has size $m+n-r$. Moreover, without restricting generality we may assume that

$$T_+ = S_+ = \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}$$

which can be always achieved by a suitable equivalence transformation (see Remark 10). We choose the extension

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_+ & T_3 \end{array}\right] = \left[\begin{array}{cc|c} I_{n-r} & 0 & 0 \\ \hline 0 & I_r & 0 \\ 0 & 0 & I_{m-r} \end{array}\right] = \left[\begin{array}{c|c} S_1 & S_2 \\ \hline S_+ & S_3 \end{array}\right] = S \tag{24}$$

and show how we can determine the remaining unknown matrices in (23). Writing component-wise the embedded equation

$$\begin{aligned} &\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{+_{11}} - \lambda E_{+_{11}} & A_{+_{12}} - \lambda E_{+_{12}} & A^x_{\pm_1} - \lambda E^x_{\pm_1} \\ A_{+_{21}} - \lambda E_{+_{21}} & A_{+_{22}} - \lambda E_{+_{22}} & A^x_{\pm_2} - \lambda E^x_{\pm_2} \\ 0 & 0 & A^x_- - \lambda E^x_- \end{bmatrix} \\ &- \begin{bmatrix} F^x_- - \lambda K^x_- & F^x_{\pm_1} - \lambda K^x_{\pm_1} & F^x_{\pm_2} - \lambda K^x_{\pm_2} \\ 0 & F_{+_{11}} - \lambda K_{+_{11}} & F_{+_{12}} - \lambda K_{+_{12}} \\ 0 & F_{+_{21}} - \lambda K_{+_{21}} & F_{+_{22}} - \lambda K_{+_{22}} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} G^x_- \\ G_{+_1} \\ G_{+_2} \end{bmatrix} [\, C_{+_1} \ C_{+_2} \ C^x_- \,]\, p(\lambda), \end{aligned} \tag{25}$$

where for clarity the unknown matrices have been temporarily marked by an additional upper-script "x", we get

$$A_{+_{11}} - \lambda E_{+_{11}} - (F^x_- - \lambda K^x_-) = G^x_- C_{+_1} p(\lambda), \tag{26}$$
$$A_{+_{12}} - \lambda E_{+_{12}} - (F^x_{\pm_1} - \lambda K^x_{\pm_1}) = G^x_- C_{+_2} p(\lambda), \tag{27}$$
$$A^x_{\pm_1} - \lambda E^x_{\pm_1} - (F^x_{\pm_2} - \lambda K^x_{\pm_2}) = G^x_- C^x_- p(\lambda), \tag{28}$$
$$A_{+_{21}} - \lambda E_{+_{21}} = G_{+_1} C_{+_1} p(\lambda), \tag{29}$$
$$A_{+_{22}} - \lambda E_{+_{22}} - (F_{+_{11}} - \lambda K_{+_{11}}) = G_{+_1} C_{+_2} p(\lambda), \tag{30}$$
$$A^x_{\pm_2} - \lambda E^x_{\pm_2} - (F_{+_{12}} - \lambda K_{+_{12}}) = G_{+_1} C^x_- p(\lambda), \tag{31}$$
$$0 = G_{+_2} C_{+_1} p(\lambda), \tag{32}$$
$$-(F_{+_{21}} - \lambda K_{+_{21}}) = G_{+_2} C_{+_2} p(\lambda), \tag{33}$$
$$A^x_- - \lambda E^x_- - (F_{+_{22}} - \lambda K_{+_{22}}) = G_{+_2} C^x_- p(\lambda). \tag{34}$$

Since the pair $(F_+ - \lambda K_+, G_+)$

$$\begin{aligned} &= \left(\begin{bmatrix} F_{+_{11}} - \lambda K_{+_{11}} & F_{+_{12}} - \lambda K_{+_{12}} \\ F_{+_{21}} - \lambda K_{+_{21}} & F_{+_{22}} - \lambda K_{+_{22}} \end{bmatrix}, \begin{bmatrix} G_{+_1} \\ G_{+_2} \end{bmatrix} \right) \\ &\overset{(33)}{=} \left(\begin{bmatrix} F_{+_{11}} - \lambda K_{+_{11}} & F_{+_{12}} - \lambda K_{+_{12}} \\ -G_{+_2} C_{+_2} p(\lambda) & F_{+_{22}} - \lambda K_{+_{22}} \end{bmatrix}, \begin{bmatrix} G_{+_1} \\ G_{+_2} \end{bmatrix} \right) \end{aligned} \tag{35}$$

is controllable, it follows from (35) that the pair $(F_{+_{22}} - \lambda K_{+_{22}}, G_{+_2})$ is controllable as well. In a completely similar way, since $(C_+, A_+ - \lambda E_+)$ is observable, it follows that $(C_{+_1}, A_{+_{11}} - \lambda E_{+_{11}})$ is observable as well. As we will see later on, these pairs completely specify the Wiener-Hopf indices of the factorization.

We determine the unknown matrices in (26)–(34) as follows. We solve two generalized pole placement problems defined by equations (26) and (34), and determine G_-^x and C_-^x such that the matrix pencils $F_-^x - \lambda K_-^x$ and $A_-^x - \lambda E_-^x$ have their spectrum placed in any desired location. The pole placement problems can be always solved since the hypotheses of Lemma 4 are in force. In particular, G_-^x and C_-^x specify through equations (27) and (31) the pencils $F_{\pm_1}^x - \lambda K_{\pm_1}^x$ and $A_{\pm_2}^x - \lambda E_{\pm_2}^x$. Finally, we may choose freely the $A_{\pm_1}^x - \lambda E_{\pm_1}^x$ and $F_{\pm_2}^x - \lambda K_{\pm_2}^x$ to satisfy equation (28). The remaining equations are identically satisfied since we assumed that (19) holds. This ends the construction of the embedding.

2. Writing (21) and (22) in partitioned form we get (23) and with obvious notation

$$
\begin{bmatrix} \widetilde{S}_1 & \widetilde{S}_2 \\ S_+ & \widetilde{S}_3 \end{bmatrix} \begin{bmatrix} A_+ - \lambda E_+ & \widetilde{A}_\pm - \lambda \widetilde{E}_\pm \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- \end{bmatrix}
- \begin{bmatrix} \widetilde{F}_- - \lambda \widetilde{K}_- & \widetilde{F}_\pm - \lambda \widetilde{K}_\pm \\ 0 & F_+ - \lambda K_+ \end{bmatrix} \begin{bmatrix} \widetilde{T}_1 & \widetilde{T}_2 \\ T_+ & \widetilde{T}_3 \end{bmatrix}
= \begin{bmatrix} \widetilde{G}_- \\ G_+ \end{bmatrix} \begin{bmatrix} C_+ & \widetilde{C}_- \end{bmatrix} p(\lambda), \tag{36}
$$

respectively. Since S, T, $\widetilde{S}$ and $\widetilde{T}$ are invertible, we can define the associated rational matrices

$$
\widehat{W}(\lambda) = \left[\begin{array}{c|c} A - \lambda E & S^{-1}G \\ \hline C & I \end{array}\right]_{\lambda_0}, \widehat{W}^{-1}(\lambda) = \left[\begin{array}{c|c} F - \lambda K & G \\ \hline -CT^{-1} & I \end{array}\right]_{\lambda_0}, \tag{37}
$$

$$
\widetilde{W}(\lambda) = \left[\begin{array}{c|c} \widetilde{A} - \lambda \widetilde{E} & \widetilde{S}^{-1}\widetilde{G} \\ \hline \widetilde{C} & I \end{array}\right]_{\lambda_0}, \widetilde{W}^{-1}(\lambda) = \left[\begin{array}{c|c} \widetilde{F} - \lambda \widetilde{K} & \widetilde{G} \\ \hline -\widetilde{C}\widetilde{T}^{-1} & I \end{array}\right]_{\lambda_0}. \tag{38}
$$

We have

$$
\widehat{W}^{-1}(\lambda)\widetilde{W}(\lambda) = \left[\begin{array}{cc|c} F_- - \lambda K_- & F_\pm - \lambda K_\pm & G_- \\ 0 & F_+ - \lambda K_+ & G_+ \\ \hline H_- & H_+ & I \end{array}\right]_{\lambda_0}
\times \left[\begin{array}{cc|c} A_+ - \lambda E_+ & \widetilde{A}_\pm - \lambda \widetilde{E}_\pm & \widetilde{B}_+ \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & \widetilde{B}_- \\ \hline C_+ & \widetilde{C}_- & I \end{array}\right]_{\lambda_0}
$$

$$
= \left[\begin{array}{cccc|c} F_- - \lambda K_- & F_\pm - \lambda K_\pm & G_- C_+ p(\lambda) & G_- \widetilde{C}_- p(\lambda) & G_- \\ 0 & F_+ - \lambda K_+ & G_+ C_+ p(\lambda) & G_+ \widetilde{C}_- p(\lambda) & G_+ \\ 0 & 0 & A_+ - \lambda E_+ & \widetilde{A}_\pm - \lambda \widetilde{E}_\pm & \widetilde{B}_+ \\ 0 & 0 & 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & \widetilde{B}_- \\ \hline H_- & H_+ & C_+ & \widetilde{C}_- & I \end{array}\right]_{\lambda_0}, \tag{39}
$$

where we have denoted $[\, H_- \ H_+ \,] := -CT^{-1}$ and $\begin{bmatrix} \widetilde{B}_+ \\ \widetilde{B}_- \end{bmatrix} := \widetilde{S}^{-1}\widetilde{G}$. Applying on the realization in the right–hand side of (39) the equivalence transformation defined by the matrices

$$
\begin{bmatrix} I & 0 & -S_1 & 0 \\ 0 & I & -S_+ & -\widetilde{S}_3 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 & T_1 & 0 \\ 0 & I & T_+ & \widetilde{T}_3 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},
$$

and using the equalities (23) and (36) we get $\widehat{W}^{-1}(\lambda)\widetilde{W}(\lambda)$

$$
= \left[\begin{array}{cccc|c} F_- - \lambda K_- & F_\pm - \lambda K_\pm & 0 & X_1 - \lambda X_2 & X_3 \\ 0 & F_+ - \lambda K_+ & 0 & 0 & 0 \\ 0 & 0 & A_+ - \lambda E_+ & \widetilde{A}_\pm - \lambda \widetilde{E}_\pm & \widetilde{B}_+ \\ 0 & 0 & 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & \widetilde{B}_- \\ \hline H_- & H_+ & 0 & H_+\widetilde{T}_3 + \widetilde{C}_- & I \end{array}\right]_{\lambda_0}
$$

$$
= \left[\begin{array}{cc|c} F_- - \lambda K_- & X_1 - \lambda X_2 & X_3 \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & \widetilde{B}_- \\ \hline H_- & H_+\widetilde{T}_3 + \widetilde{C}_- & I \end{array}\right]_{\lambda_0}, \tag{40}
$$

where $X_1 - \lambda X_2 := (F_\pm - \lambda K_\pm)\widetilde{T}_3 - S_1(\widetilde{A}_\pm - \lambda \widetilde{E}_\pm) + G_- \widetilde{C}_- p(\lambda)$, $X_3 := G_- - S_1 \widetilde{B}_+$ and the last equality follows by removing the uncontrollable and unobservable parts. Hence the conclusion.

3. Construct first two different embeddings (21) and (22) of equation (19), having the same invertible $T = \widetilde{T}$ and the same invertible $S = \widetilde{S}$ given by (24). With these two embeddings associate through (37) the rmfs $\widehat{W}(\lambda)$ and $\widetilde{W}(\lambda)$. Using point 2 of this lemma, we have $D(\lambda) :=$

$$
\widehat{W}^{-1}(\lambda)\widetilde{W}(\lambda) = \left[\begin{array}{cc|c} F_- - \lambda K_- & X_1 - \lambda X_2 & X_3 \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & G_{+_2} \\ \hline -C_{+_1} & \widetilde{C}_- - C_- & I \end{array}\right]_{\lambda_0}, \tag{41}
$$

where now $X_1 - \lambda X_2 = G_- \widetilde{C}_- p(\lambda) + F_{\pm_2} - \lambda K_{\pm_2} - (\widetilde{A}_{\pm_1} - \lambda \widetilde{E}_{\pm_1})$, $X_3 = G_- - \widetilde{G}_-$, and we have used (40) in which we have replaced $T = \widetilde{T} = S = \widetilde{S}$ by the explicit expressions in (24).

From the construction made at point 1. of this lemma, we have an additional freedom in choosing the pencils in the left–hand side of equation (28). With the actual notation, we have the freedom to specify the pencils in the left–hand side of $A_{\pm_1} - \lambda E_{\pm_1} - (F_{\pm 2} - \lambda K_{\pm 2}) = G_- C_- p(\lambda)$ and $\widetilde{A}_{\pm_1} - \lambda \widetilde{E}_{\pm_1} - (\widetilde{F}_{\pm_2} - \lambda \widetilde{K}_{\pm_2}) = \widetilde{G}_- \widetilde{C}_- p(\lambda)$. We choose $F_{\pm_2} - \lambda K_{\pm_2}$ and $\widetilde{A}_{\pm_1} - \lambda \widetilde{E}_{\pm_1}$ such that the (1,2) element in the realization (41) is zero, i.e., $X_1 - \lambda X_2$

$$
= G_- \widetilde{C}_- p(\lambda) + F_{\pm_2} - \lambda K_{\pm_2} - (\widetilde{A}_{\pm_1} - \lambda \widetilde{E}_{\pm_1}) = 0. \tag{42}
$$

We show further that the feedback matrices G_- and C_- can always be chosen to simultaneously fulfil the following properties:

$$
(a)\, \mathrm{Im}\,(G_-^*) \subset \mathrm{Im}\,(C_{+_1}); \quad (b)\, \mathrm{Im}\,(C_-) \subset \mathrm{Im}\,(G_{+_2}^*);
\quad (c)\, G_- C_- = 0. \tag{43}
$$

Since G_-^* is an $m \times (n-r)$ matrix and $\mathbb{C}^m = \mathrm{Ker}\,(C_{+1}^*) \oplus \mathrm{Im}\,(C_{+_1})$, we can write

$$
G_-^* = N L_2 + C_{+1} L_1, \tag{44}
$$

where N spans the null space (kernel) of C_{+1}^* and L_1 and L_2 are two appropriate matrices. In particular, it follows $G_- C_{+1} = (L_1^* C_{+1}^* + L_2^* N^*) C_{+1} = L_1^* C_{+1}^* C_{+1}$ and therefore in the generalized eigenvalue assignment problem $A_{+_{11}} - \lambda E_{+_{11}} - (F_- - \lambda K_-) = G_- C_{+_1} p(\lambda)$ we can simply replace G_-^* with $C_{+1} L_1$ proving in this way (a). Part (b) follows analogously. From (a) and (b) there exist matrices M_1 and M_2 such that $G_-^* = C_{+_1} M_1$ and $C_- = G_{+_2}^* M_2$ and we get $G_- C_- = M_1^* C_{+1}^* G_{+2}^* M_2 = 0$ where for the last equality we have used (32).

We finally show that $D(\lambda)$ can be constructed diagonal. From (b) in (43) we can always choose C_- and $\widetilde{C}_-$ in

$\mathrm{Im}\,(G^*_{+_2})$ while from (a) we can always chose G^*_- and $\widetilde{G}^*_-$ in $\mathrm{Im}\,(C_{+_1})$ from where it follows $\Delta G_- \Delta C_- = 0$, where

$$\Delta C_- := \widetilde{C}_- - C_-, \quad \Delta G_- := G_- - \widetilde{G}_-. \tag{45}$$

We also have

$$\begin{aligned} \begin{bmatrix} \Delta G_- \\ G_{+2} \end{bmatrix} &\overset{(43)(a)}{=} \begin{bmatrix} PC^*_{+1} \\ G_{+2} \end{bmatrix}, \\ [\, -C_{+1} \;\; \Delta C_- \,] &\overset{(43)(b)}{=} [\, -C_{+1} \;\; -G^*_{+2} Q \,], \end{aligned} \tag{46}$$

for two appropriate matrices P and Q. Let V be a unitary matrix that compresses $[\, C_{+1} \;\; G^*_{+2} \,]$ by rows such that

$$V\,[\, C_{+1} \;\; G^*_{+2} \,] = \begin{bmatrix} C_{+11} & (G_{+2c})^* \\ C_{+1c} & 0 \\ 0 & 0 \end{bmatrix} \tag{47}$$

where $(G_{+2c})^*$ and C_{+1c} have full row rank. We have $VD(\lambda)V^*$

$$\begin{aligned}
&\overset{(41),(42),(45)}{=} V \left[\begin{array}{cc|c} F_- - \lambda K_- & 0 & \Delta G_- \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & G_{+_2} \\ \hline -C_{+_1} & \Delta C_- & I \end{array} \right]_{\lambda_0} V^* \\
&\overset{(46),(47)}{=} \left[\begin{array}{cc|ccc} F_- - \lambda K_- & 0 & PC^*_{+11} & PC_{+1c} & 0 \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & G_{+2c} & 0 & 0 \\ \hline -C_{+11} & -(G_{+2c})^* Q & I & 0 & 0 \\ -C_{+1c} & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array} \right]_{\lambda_0} \\
&= \left[\begin{array}{cc|ccc} F_- - \lambda K_- & 0 & 0 & PC_{+1c} & 0 \\ 0 & \widetilde{A}_- - \lambda \widetilde{E}_- & G_{+2c} & 0 & 0 \\ \hline 0 & -G^*_{+2c} Q & I & 0 & 0 \\ -C_{+1c} & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array} \right]_{\lambda_0} \qquad (48) \\
&=: \begin{bmatrix} D_1(\lambda) & 0 & 0 \\ 0 & D_2(\lambda) & 0 \\ 0 & 0 & I \end{bmatrix},
\end{aligned}$$

where the equality in (48) follows because $C_{+11} = 0$ which in turn follows from $0 \overset{(32)}{=} G_{+2} C_{+1} \overset{(47)}{=} G_{+2c} C_{+11}$ and the full column rank of G_{+2c}.

The last step is to show that $D_1(\lambda)$ and $D_2(\lambda)$ can themselves be diagonalized and put in the form (14), from where, by a mere permutation, we get (13). We illustrate in detail only for

$$D_1(\lambda) = I + \Delta C^c_- (\lambda \widetilde{E}_- - \widetilde{A}_-)^{-1} G^c_{+_2} p(\lambda), \tag{49}$$

where $\Delta C^c_- = \widetilde{C}^c_- - C^c_- := -G^*_{+2c} Q$, since the rest follows by duality. Recall first that $D_1^{-1}(\lambda) = I - \Delta C^c_- (\lambda E_- - A_-)^{-1} G^c_{+_2} p(\lambda)$, $\lambda \widetilde{E}_- - \widetilde{A}_-$ and $\lambda E_- - A_-$ are constructed by solving the generalized eigenvalue problems in (34), i.e.,

$$A_- - \lambda E_- = (F_{+_{22}} - \lambda K_{+_{22}}) + G^c_{+_2} C^c_- p(\lambda), \tag{50}$$

$$\widetilde{A}_- - \lambda \widetilde{E}_- = (F_{+_{22}} - \lambda K_{+_{22}}) + G^c_{+_2} \widetilde{C}^c_- p(\lambda), \tag{51}$$

respectively, and the pair $(F_{+_{22}} - \lambda K_{+_{22}}, G^c_{+_2})$ is controllable. According to point 2 of Lemma 4, choose C^c_- in (50) (after applying the transformation matrices S, T and V) such that $A_- - \lambda E_-$ is in the form (8), G_{+_2} is in the form (9), and $\lambda_\bullet := \lambda_-$. Proceed similarly for $\widetilde{C}^c_-$ in (51) such that $\widetilde{A}_- - \lambda \widetilde{E}_-$ is in the form (8), G_{+_2} is in the form (9), and $\lambda_\bullet := \lambda_+$. By construction, we get (49) in the diagonal form $D_1(\lambda) = \mathrm{diag}\,\{ \left(\frac{\lambda - \lambda_-}{\lambda - \lambda_+} \right)^{n_1}, \ldots, \left(\frac{\lambda - \lambda_-}{\lambda - \lambda_+} \right)^{n_k} \}$, where $n_1, \ldots, n_k$ are the nonzero Kronecker indices of the pair $(F_{+_{22}} - \lambda K_{+_{22}}, G^c_{+_2})$.

Dual arguments apply to the pair $(C_{+_1}, A_{+_{11}} - \lambda E_{+_{11}})$ or, equivalently, to the pair $(C^c_{+_1}, A_{+_{11}} - \lambda E_{+_{11}})$. This ends the whole construction. ■

REFERENCES

H. Bart, I. Gohberg, M. Kaashoek, and P. Van Dooren. Factorization of transfer functions. *SIAM J. Control Optim.*, 18(6):675–696, November 1980.

H. Bart, I. Gohberg, M. A. Kaashoek. *Minimal Factorization of Matrix and Operator Functions.* OT 1, Birkhauser, Basel, 1986.

H. Bart, I. Gohberg, M. A. Kaashoek. Explicit Wiener–Hopf Factorization and Realization. *Constructive Methods of Wiener–Hopf Factorization*, OT 21, Birkhauser, Basel, 1986.

H. Bart, I. Gohberg, M. A. Kaashoek, A. Ran. *Factorization of Matrix and Operator Functions: The State Space Method.* OT 178, Birkhauser, Basel, 2008.

I. Gohberg, I. A. Feldman. Convolution equations and projection methods for their solution. *Transl. Math. Monographs*, Vol. 41, Providence, R.I., Amer. Math. Soc., 1974.

I. Gohberg, M.A. Kaashoek, A. C. M. Ran. Factorizations of and extensions to J–unitary rational matrix functions on the unit circle. *Int. Eq. Op. Th.*, 15, (1992), pp. 262–300.

M. G. Krein. Integral equations on the half line with a kernel depending on the difference of the arguments. *Amer. Math. Soc. Transl.*, (2) 22, 163-288 (1962).

V. Ionescu, C. Popeea. *Conducerea structurală a sistemelor liniare (in romanian).* Ed. Tehnica, (1986).

V. Ionescu, C. Oară, M. Weiss. *Generalized Riccati Theory and Robust Control: A Popov Function Approach.* John Wiley & Sons, New York, (1999).

B. Hassibi, A. Sayed, T. Kailath. *Indefinite-Quadratic Estimation and Control: A Unified Approach to H^2 and H^∞ Theories.* SIAM, Philadelphia, (1999).

H. Kimura. *Chain–Scattering Approach to H^∞–Control.* Birkhauser, Boston, (1997).

C. Oară, A. Varga. Minimal degree coprime factorization of rational matrices. *SIAM J. Matrix Anal. Appl.*, 21 (1), (1999), pp. 245–278.

C. Oară, A. Varga. Computation of general inner–outer and spectral factorizations. *IEEE Trans. Auto. Contr.*, AC 45(12), (2000), 2307–2325.

C. Oară, S. Sabău. All doubly coprime factorizations of a general rational matrix. *Automatica*, 45 (8), (2009), pp. 1960–1964.

C. Oară, R. Andrei. Zero Cancellation for General Rational Matrix Functions. *Lin. Alg. Appl.*, Vol. 431 (11), 2009, pp. 2000-2023.

M. Rakowski. Minimal factorizations of rational matrix functions. *IEEE Trans. Circ. Syst. I, Fund. Theory and Applic.*, 39(6), (1992), pp. 440-445.

G. Verghese, P. Van Dooren, T. Kailath. Properties of the system matrix of a generalized state-space system. *Int. J. Control*, 30, (1979), pp. 235–243.

P. Van Dooren. Factorization of a rational matrix : the singular case. *Integr. Eq. Operator Th.*, 7:704–741, October 1984.