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FOUR LECTURES ON STABILITY[1]

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Abstract: *Starting with the basic notions about Liapunov and input/output stability there are presented those elements that represent the minimal knowledge of Stability Theory and its methods.A particular attention is paid to absolute stability of feedback systems and connected framework (Liapunov functions, frequency domain inequalities, hyperstability and dissipativity).*
Acknowledgement: In 1990 the Romanian Society for Automation and Industry Applied Information Processing emerged as an initiative of some outstanding control engineers active both in Education and Research. Systems and Control had been a long date field of interest in Romania however a dedicated academic and application oriented society was missing for, mildly said, non-academic reasons. Neither 1990 nor the following 1991 did not show very stimulating for normal daily research. For this reason the idea of Professor Ioan Dumitrache, first (and actual) President of the Society to organize a cycle of lectures on some basic (but freely chosen) topics of Systems and Control was highly stimulating for a new start. The cycle of lectures took place during the Academic Year 1991/1992 at "Politehnica" Technical University of Bucharest, Department of Automatic Process Control. These lecture notes represent author's contribution to that worth remembering event.

Keywords: *Stability, Absolute stability, Hyperstability.*

1. LECTURE THREE. THE FREQUENCY DOMAIN CRITERION FOR ABSOLUTE STABILITY

The frequency domain method, elaborated by *V.M. Popov* between 1958-1962, represents a completely different treatment of the absolute stability in comparison with the method of the Liapunov function; the reader is sent to [1], [2] as basic references: the first one is the paper where the method is presented for the first time and the second is the paper that spread it throughout the world.

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The method of Popov applies also to the structure of Fig. 1 but with the linear block described by a Volterra integral equation. Worth mentioning that [1] is elaborated directly for systems described by integral equations for which at that time (1958) the stability concept was only foreseen. The fact that the integral equations occur in a natural way from the differential ones *via* the formula of variation of constants, also that the results of Popov obtained further and formulated for integral equations may be applied e.g. to time lag or distributed parameter systems seem to have been understood later [3], [4], [5].

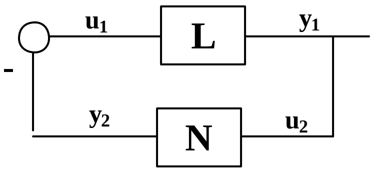


Fig.1 Feedback structure associated to the problem of absolute stability

3.1 Quadratic restrictions, integral index and generalized feedback

In order to illustrate the method of Popov as well as to understand the general aspects that will be found again in the next developments of Popov's theory, we shall consider a specific case, that of a system with a pole at the origin (also called *the simplest critical case* or *the indirect control system*). It is the integro-differential system

$$y(t) = r(t) - \int_0^t h(t-t)\, j\left(y(t)\right) dt + gx(t) \qquad (1)$$
$$\dot{x} = -j\left(y\right)$$

It can be immediately seen that (1) corresponds to a feedback structure as in Fig. 1 where the linear block is described by

$$y(t) = r(t) - \int_0^t h(t-t)\, u_1(t)\, dt + gx(t) \qquad (2)$$
$$\dot{x} = u_1(t)$$

and the nonlinear block has the usual form

$$y_2 = j\,(u_2) \qquad (3)$$

together with the negative feedback connection relations

$$u_1 = -y_2\,, \quad u_2 = y_1 \qquad (4)$$

The nonlinear function belongs to the class defined by the sector inequalities

$$0 < \frac{j\,(y)}{y} < k \le +\infty \qquad (5)$$

which lead to the following inequalities

$$j\,(y)\left(ky - j\,(y)\right) > 0\,, \quad \int_0^y j\,(s)ds > 0 \qquad (6)$$

Remark that (3) and (4) allow to define the input signal of (2) as a nonlinear feedback $u_1 = -j\,(y_1)$; inequalities (6) will impose the following *quadratic type restrictions* on the input/output pairs of the linear block (2)

$$\begin{aligned} & u_1(u_1 + ky_1) < 0 \\ & \int_0^t u_1(t)\dot{y}_1(t)dt = -\int_0^t j\left(y_1(t)\right)\dot{y}_1(t)dt \\ & = \int_0^{y_1(0)} j\,(s)ds - \int_0^{y_1(t)} j\,(s)ds < \int_0^{y_1(0)} j\,(s)ds \end{aligned} \qquad (7)$$

Following V.M.Popov we associate to the integral block (2) the following integral index

$$h(t_0, t_1) = \int_{t_0}^{t_1} \left[a_0 u_1(t)\left(u_1(t) + ky_1(t)\right) + a_1 u_1(t)\dot{y}_1(t)\right] dt \qquad (8)$$

where $a_0 \ge 0, a_1 \ge 0$ are parameters to be specified. Remark that if $u_1(t) = -j\left(y_1(t)\right)$, where $y_1(t)$ is a solution of the integro-differential system (1) then the following inequality follows from (7)

$$h(t_0, t_1) < a_1 \int_0^{y_1(t_0)} j\,(s)ds \qquad (9)$$

which defines *a restriction on the input/output pairs* of the block (2). As it has been pointed out, this restriction issued from restrictions (7), one of them being "static" or local and the other being "dynamic" or integral. The restrictions on the input/output pairs were called by Popov *generalized feedback* [6].

In his study of the absolute stability Popov considered objects of the type (2) + (8) that is *dynamic blocks + integral index*. While the construction of the integral index from the conditions satisfied by the nonlinear block is a rational one, this was less obvious in the pioneering work of Popov; for this reason the method had been called *of the* a priori *integral indices*. The understanding of the rational way of constructing the integral index came with a paper of *Yakubovich* from 1967 [7].

Still two remarks are necessary:

1^0 In the structure of block (2) ?(t) incorporates the effect of the initial conditions; therefore, by using the classical definition we find that the the transfer function of the block follows from

$$\tilde{y}_1(s) = \tilde{h}(s)\tilde{u}_1(s) + g\tilde{x}(s)$$

$$s\tilde{x}(s) = \tilde{u}_1(s)$$

and reads

$$H(s) = \tilde{h}(s) + \frac{g}{s} \qquad (10)$$

where $\tilde{h}(s)$ is the Laplace transform of the kernel $h(t)$.

2^o The local restrictions on the input/output pairs correspond in the Liapunov function method to the S-procedure.

1.2 The frequency domain condition and the basic inequality

We consider the system *integral block + integral index* defined by

$$y(t) = r(t) - \int_0^t h(t-t)u(t)\,dt + gx(t)$$

$$\dot{x} = u(t) \qquad (11)$$

$$h(0,T) = \int_0^T u(t)\left[a_0\big(u(t) + ky(t)\big) + a_1\dot{y}_1(t)\right]dt \qquad (12)$$

and define the following input signal

$$u_T(t) = \begin{cases} -j\big(y(t)\big) & \text{if } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad (13)$$

where $y(t)$ is a solution of (1) corresponding together with $x(t)$, to the initial condition $x(0)$. Let $\big(y_T(t), x_T(t)\big)$ be the solution of (11) corresponding to $u_T(t)$ and to the same $x(0)$. From (7), (12), (13) the following inequality follows

$$h(0,T) = \int_{-\infty}^{\infty} u_T(t)\left[a_0\big(u_T(t) + ky_T(t)\big) + a_1\dot{y}_T(t)\right]dt$$

$$< a_1\int_0^{y(0)} j(s)ds - a_1\int_0^{y(T)} j(s)ds$$

Defining the convolution

$$w_T(t) = \int_0^t h(t-t)u_T(t)\,dt \qquad (14)$$

the following equalities may be written

$$y_T(t) = w_T(t) + r(t) + gx_T(t)$$

$$\dot{y}_T(t) = \dot{w}_T(t) + \dot{r}(t) + g\dot{x}_T(t)$$

From here we deduce the equality

$$h(0,T) = \frac{1}{2}g\,k\left[x^2(T) - x^2(0)\right]$$

$$+ \int_0^T \dot{x}(t)\left[a_0 kr(t) + a_1\dot{r}(t)\right]dt$$

$$+ \int_{-\infty}^{\infty} u_T(t)\left[(a_0 + a_1 g)u_T(t) + a_0 kw_T(t) + a_1\dot{w}_T(t)\right]dt \qquad (15)$$

After an integration by parts and taking into account the inequality satisfied by $h(0,T)$ we deduce

$$\frac{1}{2}gkx^2(T) + x(T)\big(a_0 kr(T) + a_1\dot{r}(T)\big) + \int_0^{y(T)} j(s)ds$$

$$+ \int_{-\infty}^{\infty} u_T(t)\left[(a_0 + a_1 g)u_T(t)\right.$$

$$\left. + a_0 kw_T(t) + a_1\dot{w}_T(t)\right]dt$$

$$< \frac{1}{2} gkx^2(0) + x(0)\left(a_0 kr(0) + a_1 \dot{r}(0)\right)$$

$$+ \int_0^{y(0)} j(s)ds + \int_0^T x(t)\left(a_0 k\dot{r}(t) + a_1 \ddot{r}(t)\right)dt$$

In the following we shall first make some assumptions about the functions representing the output of the linear block "for zero input" and "for zero (ground) state" i.e. $r(t)$ and $h(t)$ respectively.

We shall assume that

a) $r(t)$ is C^1 and absolutely integrable on $(0,\infty)$ together with its first and second derivatives:

$$r \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+),\ \dot{r} \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+),$$

$$\ddot{r} \in L^1(\mathbb{R}_+)$$

from here we have that

$$\sup_{t>0} |r(t)| < \infty,\ \sup_{t>0} |\dot{r}(t)| < \infty;$$

b) $h(t) \equiv 0$ for $t < 0$ (the system is causal - non-anticipative) and $h \in L^1(\mathbb{R}^+)$ i.e. the block is BIBO stable; $\dot{h}$ has the same property.

The first assumption - on r and its derivatives - allows obtaining from the above inequality

$$\frac{1}{2} gkx^2(T) - g_0 \sup_{0 \le t \le T} |x(t)| < \frac{1}{2} gkx^2(T) +$$

$$|x(0)|\left(a_0 k|r(0)| + a_1 |\dot{r}(0)|\right) + \int_0^{y(0)} j(s)ds$$

$$- \int_{-\infty}^{\infty} u_T(t)\left[(a_0 + a_1 g)u_T(t) + a_0 k w_T(t) + a_1 \dot{w}_T(t)\right]dt \tag{16}$$

where $g_0 > 0$.

The assumptions on $h(t)$ allow to estimate the integral from (16) using the Fourier transform and the theorem of Plancherel. The properties of $h(t)$ show that its Fourier transform is obtained as restriction to the imaginary axis of the corresponding Laplace transform and this is valid for $u_T(t)$ also. We have

$$\int_{-\infty}^{\infty} u_T(t)\left[(a_0 + a_1 g)u_T(t) + a_0 k w_T(t) + a_1 \dot{w}_T(t)\right]dt =$$

$$\frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} \left[a_0 + a_1 g + a_0 k\tilde{h}(jw) + a_1 jw\tilde{h}(jw)\right] |\tilde{u}_T(jw)|^2 dw$$

If the following frequency domain inequality

$$\mathrm{Re}\left[a_0 + a_1 g + a_0 k\tilde{h}(jw) + a_1 jw\tilde{h}(jw)\right] \ge 0$$

which may be immediately rewritten as

$$\mathrm{Re}\left[a_0 + a_0 kH(jw) + a_1 jwH(jw)\right] \ge 0 \tag{17}$$

where $H(s)$ is the transfer function (10), holds for all real w, then the integral of (16) is nonnegative. Therefore we deduce from (16)

$$\frac{1}{2} gkx^2(T) - g_0 \sup_{0 \le t \le T} |x(t)|$$

$$- \Phi\left(|x(0)|, |y(0)|, |r(0)|, |\dot{r}(0)|\right) < 0 \tag{18}$$

where

$$\lim_{\substack{r_i \to 0,\\ i=1,4}} \Phi(r_1, r_2, r_3, r_4) = 0$$

From here, using the properties of the second order trinomial we find

$$|x_i(t)| < \Phi_1\left(|x(0)|, |y(0)|, |r(0)|, |\dot{r}(0)|\right) \tag{19}$$

where Φ_1 has the same properties as Φ. Inequality (19) contains in fact the stability property of (1) since (19) together with the assumptions on h and r allow to obtain a similar inequality for $y(t)$. Indeed from (1) we deduce

$$y(t) = r(t) - \int_0^t h(t-t)\dot{x}(t)dt + gx(t)$$

and, after integrating by parts

$$y(t) = r(t) + h(t)x(0) + \left(g + h(0)\right)x(t)$$

$$+ \int_0^t \dot{h}(t-t)x(t)dt$$

From the above equality we deduce

$$|y(t)| \le |r(t)| + |h(t)||x(0)| + |g + h(0)||x(t)|$$

$$+ \sup_{t>0} |x(t)| \int_0^\infty \dot{h}(t) dt$$

and taking into account the properties of $r(t)$; $h(t)$, $\dot{h}(t)$ and $x(t)$ an estimate which is specific for stability is obtained.

1.3 The frequency domain condition of V.M.Popov

It has been seen that the constant sign of the integral containing *the motions with zero (ground) initial state* is ensured by a frequency domain inequality e.g. (17) deduced from the theorem of Plancherel. If we take in (17) $a_0 = 1/k$, $a_1 = q$ then we find Popov's condition as it is known from the textbooks

$$\frac{1}{k} + \mathrm{Re}(1 + jwq)H(jw) \ge 0 \tag{20}$$

where $q \ge 0$ is an arbitrary parameter. If this parameter can be chosen such that (20) holds, then the basic inequality (18) holds. Remark also that if we compare (12) with (17) then denoting by $W(u, y, \dot{y})$ the quadratic form of (12) or, generally speaking, the quadratic form obtained by considering the quadratic restrictions that define the generalized feedback Popov-Yakubovich, inequality (20) becomes

$$W(1, H(jw), jwH(jw)) \ge 0 \tag{21}$$

In (21) the quadratic form was extended from real to complex values of the variables preserving its Hermitian property.

Formula (21) is valid in the general case, for any types of quadratic restrictions; recently it has been extended to the case of nonstationary blocks where the frequency domain characteristic has been replaced by a *Toeplitz operator*. Remark also that (21) when applied to other quadratic restrictions may lead to other frequency domain stability criteria.

3.4 Types of linear blocks that lead to integral blocks

The integral blocks introduced previously represent the generalization of some blocks that occur in systems with distributed parameters or in those systems whose parameters are "less lumped" than of the blocks described by ordinary differential equations; but even these may be given an integral form. Consider the linear block

$$\dot{x} = Ax + bu(t)$$

$$y = c^* x$$

and apply the formula of variations of constants to find

$$y(t) = c^* e^{At} x_0 + \int_0^t c^* e^{A(t-t)} bu(t) dt$$

hence $r(t) = c^* e^{At} x_0$, $h(t) = c^* e^{At} b$.

Consider now the linear block with time lag in the state

$$\begin{aligned} &\dot{x} = A_0 x(t) + A_1 x(t-q) + bu(t) \\ &y = c^* x, \quad x(t) \equiv x_0(t), \quad -q \le t \le 0 \end{aligned} \tag{22}$$

Let $X(t)$ be the state transition matrix for (22) defined (see [9]) by

$$\dot{X} = A_0 X(t) + A_1 X(t-q),$$

$$X(t) \equiv 0, \quad t < 0; \quad X(0) = I$$

Then the formula of variations of constants will give

$$x(t) = X(t) x_0(0) + \int_{-q}^{0} X(t - q - t) A_1 x_0(t) dt$$

$$+ \int_0^t X(t-t) bu(t) dt$$

$$y(t) = c^* X(t) x_0(0) + \int_{-q}^{0} c^* X(t - q - t) A_1 x_0(t) dt$$

$$+ \int_0^t c^* X(t-t) bu(t) dt$$

Therefore

$$r(t) = c^* X(t) x_0(0) + \int_{-q}^{0} c^* X(t - q - t) A_1 x_0(t) dt \ ,$$

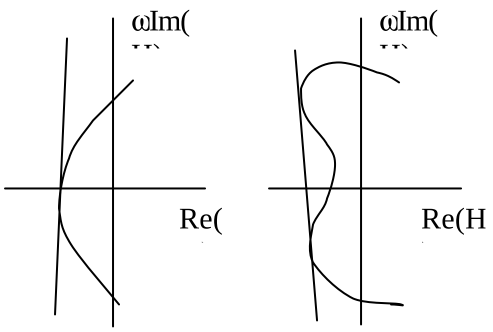
$$h(t) = c^* X(t) b$$

Similar representations may be obtained for the blocks having time lags in their output or/in their input also for blocks described by partial differential equations [8], [9].

1.5 Conjectures of Aizerman and Kalman

The pioneering period of the absolute stability has also two interesting connections to linearized systems. Indeed absolute stability means asymptotic stability for all functions $j(s)$ satisfying $k_1 < j(s)/s < k_2$ hence also for *linear functions* $j(s) = ks, k_1 < k < k_2$. For the linear case it is possible to compute the stability sector (k_{1H}, k_{2H}) using some method (the Hurwitz criterion, for instance). Obviously *the absolute stability sector cannot be larger than the Routh-Hurwitz sector.* M.A. Aizerman [10] formulated *the conjecture on the identity of the two sectors* but a *by now celebrated conter-example* of V.A. Pliss [11] showed that *this conjecture was not valid in general.*
R.E.Kalman [12] formulated *another conjecture: the linear stability sector would coincide with the sector of the derivative restrictions*: $k_1' < j' < k_2'$. This conjecture too *is, generally speaking, not valid*.

Nevertheless sector comparison may be used to check how closed the sufficient stability conditions are to the necessary ones. The geometric form of the frequency domain stability inequality is the most suitable for this.



2a 2b

Fig. 2 The Popov criterion and the conjecture of Aizerman

As it may be seen in Fig. 2 the case 2a corresponds to an affirmative answer to the conjecture of Aizerman while the case 2b to a negative one.

1.6 The connection between the method of Liapunov and the frequency domain method of Popov

Since his early papers Popov was interested in the connection of his method with the method of the Liapunov function. It was Popov himself who was able to prove [2], [13], [14] that *the existence of a Liapunov function of the type "quadratic form plus integral of the nonlinearity" implies the fulfillment of the frequency domain condition*. Later Yakubovich [15] and Kalman [16] were able *to prove converse thus obtaining* the perfect equivalence *of the two methods.* Further development lead to the *positivity theory* [17], better known in its concentrated form as *Yakubovich-Kalman-Popov lemma*. We shall give below, without proof, a form of the lemma which is useful to our purpose.

Lemma 1 *Consider the 5-tuple (A, b, ?, l, M) where A and M are n x n matrices, M = M* (is Hermitian), b and l are n-dimensional vectors and ? a real scalar. Assume that the eigenvalues of A satisfy* $l_j + \overline{l}_k \neq 0$ *for any j and k from the finite set of indices defining the uncontrollable spectrum of A (controllability defined in connection with (A, b)). Then the following are equivalent*
1° *The following frequency domain inequality holds for all real w*

$$k + b^*\left(-jwI - A^*\right)^{-1} l + l^*\left(jwI - A\right)^{-1} b$$
$$+ b^*\left(-jwI - A^*\right)^{-1} M\left(jwI - A\right)^{-1} b \geq 0 \qquad (23)$$

2° *There exist a scalar g, a vector w and a Hermitian matrix P such that*

$$PA + A^* P = M - ww^*$$
$$-Pb + l = gw \qquad (24)$$
$$k = |g|^2$$

and if (A, b, ?, l, M) are real then g, w, P can be chosen real.

Let now $x(t)$ and $u(t)$ be a pair of functions which verify the differential equation

$$\dot{x} = Ax + bu(t)$$

We multiply equalities (24) with these functions (according to the dimension)

$$x^*(t)\left(PA + A^*P\right)x(t) =$$

$$= x^*(t)Mx(t) - x^*(t)ww^*x(t),$$

$$x^*(t)(-Pb + l)u(t) = x^*(t)gwu(t),$$

$$k|u(t)|^2 = \overline{u}(t)\overline{g}gu(t).$$

Adding these equalities and integrating from 0 to t we obtain, after an integration by parts

$$\int_0^t \left[k|u(t)|^2 + \mathrm{Re}\left(x^*(t)lu(t)\right) + x^*(t)Mx(t)\right]dt$$

$$= x^*(t)Px(t) - x^*(0)Px(0)$$

$$+ \int_0^t \left|gu(t) + w^*x(t)\right|^2 dt \,. \tag{25}$$

We come now back to the integral index (8) where we assume u_1 and y_1 to be determined from

$$\dot{x} = Ax + bu_1, \quad y_1 = c^*x$$

hence

$$\dot{y}_1 = c^*\left(Ax + bu_1\right)$$

We deduce

$$h(0,t) =$$

$$\int_0^t \left[a_0u^2(t) + a_0ku(t)c^*x(t) + \right.$$

$$\left. a_1u(t)\left(c^*Ax(t) + c^*bu(t)\right)\right]dt$$

and therefore, in this case

$$k = a_0 + \frac{1}{2}a_1\left(c^*b + b^*c\right),$$

$$l = \frac{1}{2}\left(a_0kI + a_1A^*\right)c, \quad M = 0$$

The frequency domain inequality (23) will be in this case

$$a_0 + \frac{1}{2}a_1\left(c^*b + b^*c\right)$$

$$+ \frac{1}{2}b^*\left(-jwI - A^*\right)^{-1}\left(a_0kI + a_1A^*\right)c$$

$$+ \frac{1}{2}c^*\left(a_0kI + a_1A\right)\left(jwI - A\right)^{-1}b$$

$$= a_0 + \mathrm{Re}\left(a_0k + a_1jw\right)H(jw) \geq 0$$

that is exactly the condition of Popov. The fulfillment of this condition leads to the existence of P, g, w such that

$$PA + A^*P = -ww^*$$

$$-Pb + \frac{1}{2}\left(a_0kI + a_1A^*\right)c = gw$$

$$a_0 + \frac{1}{2}a_1\left(c^*b + b^*c\right) = g^2$$

and remark that if we choose $a_0 = 1$; $a_1 = b$ the equations of A.I. Lurie (see *Lecture two. The Liapunov function and the absolute stability*, eq. (27))

$$PA + A^*P = -ww^*$$

$$-Pb + \frac{1}{2}\left(kI + bA^*\right)c = gw$$

$$1 + \frac{1}{2}b\left(b^*c + c^*b\right) = |g|^2$$

are found. Therefore the frequency condition of Popov ensures the existence of a Liapunov function of the form

$$V(x) = x^*Px + \int_0^{c^*x} j(s)ds$$

and satisfying (see *Lecture two. The Liapunov function and the absolute stability*, eq. (26)):

$$V(x(t)) = V(x(0)) - \int_0^t \left|-gj\left(c^*x(t)\right) + w^*x(t)\right|^2 dt$$

$$-\int_0^t j\left(c^*x(t)\right)\left(kc^*x(t) - j\left(c^*x(t)\right)\right)dt \,.$$

In this way the problem formulated there is solved by the Yakubovich-Kalman-Popov lemma.

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