MODEL ORDER REDUCTION WITH DISIPATIVITY PRESERVATION

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Abstract: *The paper presents a computational oriented overview on the order reduction problem for passive (positive real) linear systems. The numerical difficulties associated with the standard order reduction procedure, which involves the stabilizing solutions of two Riccati equations, are discussed. Some new approximate methods are proposed, which are specific to the discrete time setting, but may be easily extended to cope with continuous time models by using a simple bilinear transformation. Numerical results conclude the presentation.*

Keywords: *model reduction, truncated balanced realization, passivity, Riccati equations, square root methods.*

1. INTRODUCTION

Given a stable, controllable and observable, n-order linear system $\mathbf{S} = (A, B, C, D)$, the general problem of model order reduction is to find a stable, controllable and observable linear system $\mathbf{S}_r = (A_r, B_r, C_r, D_r)$, of reduced order $r < n$ (desirable $r \ll n$) such that $\mathbf{S}_r$ is close, in a definite sense, to $\mathbf{S}$. One classical way to obtain order reduction with stability preservation is by balancing the controllability and observability gramians and then truncating the balanced system.

Our paper deals with the model order reduction problem in the case when an additional goal is desired, namely the preservation of some form of system disipativity. To be concise, in the sequel we shall consider only passive or positive real linear systems. In such a case the controllability and observability gramians are replaced by the stabilizing solutions of the two algebraic Riccati equations, derived from the well-known positive real lemma (see [1], [3], [7]; if contractive linear systems are involved then the bounded real lemma is invoked). Needless to say, a lot of numerical problems arise on this way and their surpassing will be our main concern in the sequel.

The paper is organized as follows. In section 2 the main computational procedure is presented in the discrete time setting, together with some comments motivating our work. Order reduction by direct (oblique) projection and a compensated truncation procedure, able to cope with very small singular values, are proposed in sections 3 and 4. In section 5 we replace the (computationally expensive) discrete algebraic Riccati equations by their recurrent counterparts, which are solved by an efficient square root-like procedure in section 6. The continuous time case is briefly considered in section 7; in fact, all the above mentioned computational procedures apply as such in this setting, possible after performing a bilinear transformation of the system data. A few numerical results conclude our presentation together with some remarks concerning related topics and directions for future work.

2. ORDER REDUCTION BY DARE BALANCING. DISCRETE CASE

Let $\mathbf{S} = (A, B, C, D)$ be a minimal strictly positive-real discrete time linear system with $R = D + D^T > 0$. According to the positive real lemma, there exist stabilizing, symmetric positive definite solutions P and Q to the following discrete time algebraic Riccati equations (DARE)

$$P = A^T PA+ \tag{1}$$

$$+(C^T - A^T PB)(R - B^T PB)^{-1}(C - B^T PA)$$

and, by duality,

$$Q = AQA^T+ \tag{2}$$

$$+(B - AQC^T)(R - CQC^T)^{-1}(B^T - CQA).$$

The main procedure for model order reduction with passivity preservation consists in balancing the two Riccati solutions P and Q (i.e. making them diagonal and equal by choosing an adequate state space basis) and then, in truncating the resulting (balanced) system. The computational procedure goes as follows:

1. Solve the Riccati equations (1) and (2) for P and Q.
2. Compute the Cholesky factors L and M of P and Q, i.e. $P = L^T L$, $Q = MM^T$.
3. Compute the singular value decomposition $LM = U\Sigma V^T$, where U, V are orthogonal matrices and Σ contains on its diagonal the singular values $\sigma_i, i = 1 : n$ in descending order.
4. Compute the balancing transformation
$$T = \Sigma^{-\frac{1}{2}} U^T L, \qquad T^{-1} = MV\Sigma^{-\frac{1}{2}}.$$
5. Balance the system
$$\begin{array}{ll} A \leftarrow TAT^{-1}, & B \leftarrow TB, \\ C \leftarrow CT^{-1}, & D \leftarrow D. \end{array}$$
6. Select the reduced order r such that a sensible gap $\sigma_r > \sigma_{r+1}$ is ensured and truncate the balanced system, i.e.
$$\begin{array}{ll} A_r = A(1:r, 1:r), & B_r = B(1:r, :), \\ C_r = C(:, 1:r), & D_r = D. \end{array}$$

It can be shown that such a procedure theoretically preserves the system stability and passivity [5]. Moreover, if P and Q are the computed minimal (i.e. stabilizing) solutions to (1) and (2), then $Q^+ = P^{-1}$ and $P^+ = Q^{-1}$ are the corresponding maximal (antistabilizing) solutions to (2) and (1), respectively. As a consequence, both inequalities

$$P < P^+ = Q^{-1}, \qquad Q < Q^+ = P^{-1}$$

imply

$$QP < I_n,$$

hence the singular values σ_i, $i = 1 : n$ computed at step 3 (i.e. the diagonal entries of Σ) are always bounded by unity, i.e. $0 < \sigma_i < 1$, $i = 1 : n$. In fact, σ_{r+1} (see step 6) is a specific chordal distance between the two systems $\mathbf{S}$ and $\mathbf{S_r}$.

However, our numerical experience reveals bad numerical conditioning for both Riccati equations which are solved for P and Q at step 1; therefore at step 2 the Cholesky factors L and M may be computed highly inaccurate and/or unreliable. Moreover, the small singular values introduce unacceptable errors in computing the balancing transformation T at step 4. Last but not least, in order to obtain the truncated (reduced order) system it seems to be not necessary to balance the whole system (A, B, C, D). (As we shall see, the same is true even when a compensated reduced order model, which preserves the steady state gain of the original system, is desired). In the sequel we propose some numerical tricks to overcome all the above mentioned numerical difficulties.

3. ORDER REDUCTION BY PROJECTION

By referring to the step 3 of the numerical procedure, let us consider the SVD $LM = U^T \Sigma V$ partitioned as

$$LM = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix},$$

where Σ_1 contains the principal $r = n_1$ singular values, and define

$$T_1 = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} T = \Sigma_1{}^{-\frac{1}{2}} U_1^T L,$$

$$S_1 = T^{-1} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} = M V_1 \Sigma_1{}^{-\frac{1}{2}}.$$

The truncated reduced order system is directly computed as

$$\begin{array}{ll} A_r = T_1 A S_1, & B_r = T_1 B, \\ C_r = C S_1, & D_r = D, \end{array} \tag{3}$$

where $T_1 S_1 = I_{n_1}$ and $\Pi = S_1 T_1$ is an oblique projector, i.e. $\Pi^2 = \Pi$. Clearly, in computing T_1 and S_1 only the large singular values (contained in Σ_1) are involved, hence the above projection procedure seems to be less sensitive to round-off errors than the original balancing transformation T.

4. COMPENSATED ORDER REDUCTION

In some applications it is desirable to preserve the gain $G = D + C(\alpha I - A)^{-1} B$ of the original system, computed for some given complex number $|\alpha| \geq 1$. (The standard value $\alpha = 1$ corresponds to the steady state case). To this aim we shall use the transformation $T = U^T L$, which avoids dividing by small singular values (see step 4) and is "almost" balancing, i.e. makes P and Q diagonal but not equal, more precisely

$$\tilde{P} = T^{-T} P T^{-1} = I_n, \quad \tilde{Q} = T Q T^T = \Sigma^2.$$

If the corresponding "almost" balanced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (TAT^{-1}, TB, CT^{-1}, D)$ is partitioned as

$$\begin{array}{ll} \tilde{A} = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_2 \end{bmatrix}, & \tilde{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix}, \\ \tilde{C} = \begin{bmatrix} C_r & C_2 \end{bmatrix}, & \tilde{D} = D, \end{array}$$

then the compensated reduced order system is obtained as usual, i.e.

$$\begin{bmatrix} A_{rc} & B_{rc} \\ C_{rc} & D_{rc} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} + \tag{4}$$

$$+ \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix} (\alpha I - A_2)^{-1} \begin{bmatrix} A_{21} & B_2 \end{bmatrix}.$$

Of course, we can use also the transformation $T = V^T M^{-1}$, which makes $\hat{P} = T^{-T} P T^{-1} = \Sigma^2$ and $\hat{Q} = T Q T^T = I_n$.

5. APPROXIMATE BALANCING

To bypass the numerical difficulties associated with DARE (1), in the sequel we shall consider its recurrent counterpart, i.e. the Riccati equation

$$P(i+1) = A^T P(i) A + (C^T - \tag{5}$$

$$-A^T P(i) B)(R - B^T P(i) B)^{-1} (C - B^T P(i) A),$$

with initial condition $P(0) = 0$. It can be shown that $P(i) \to P$ when $i \to \infty$, therefore the solution $P(k)$ to (5) at time k is a good approximation for the stabilizing solution P to (1) if k is sufficiently large.

In order to obtain a computable expression for $P(k)$, let us consider the positive-real discrete time linear system $\mathbf{S} = (A, B, C, D)$ and denote by

$$y = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(k-1) \\ y(k) \end{bmatrix}, \qquad u = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(k-1) \\ u(k) \end{bmatrix},$$

the vectors describing the history of $\mathbf{S}$ over the time horizon $1:k$.

In a condensed form, the corresponding input-output relation for $\mathbf{S}$ is

$$y = Sx + Tu,$$

where $x(1) = x$ is the initial state vector and the matrices S and T are given by

$$S = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-2} \\ CA^{k-1} \end{bmatrix},$$

$$T = \begin{bmatrix} D & & & & \\ CB & D & & & \\ \vdots & \ddots & \ddots & & \\ CA^{k-3}B & CA^{k-4}B & \cdots & D & \\ CA^{k-2}B & CA^{k-3}B & \cdots & CB & D \end{bmatrix}.$$

According to the Caratheodory criterion, the block-Toeplitz symmetric matrix $T + T^T$ is positive definite if and only if the system $\mathbf{S} = (A, B, C, D)$ is strictly positive real. As it is well known, this property has a variational interpretation (see [1] for its continuous time formulation). To be specific, the quadratic cost (energy) functional

$$J(u) = -y^T u - u^T y = \tag{6}$$

$$= -x^T S^T u - u^T Sx - u^T (T^T + T)u,$$

is maximized for $u^* = -(T^T + T)^{-1} Sx$ and the maximum value is

$$J(u^*) = x^T P(k) x$$

where

$$P(k) = S^T (T^T + T)^{-1} S \tag{7}$$

is the desired approximate solution to (1).

A corresponding formula holds for the solution $Q(k)$ at time k of the recurrent Riccati equation

$$Q(i+1) = AQ(i)A^T + (B - AQ(i)C^T)(R- \tag{8}$$

$$-CQ(i)C^T)^{-1}(B^T - CQ(i)A^T),$$

with $Q(0) = 0$, by referring to the dual system $\mathbf{S}_D = (A^T, C^T, B^T, D^T)$.

Therefore all the above numerical procedures can be applied by using the actual approximate matrices $P(k)$ and $Q(k)$ instead of the true DARE solutions P and Q. For example, by referring to (7), the Cholesky factor L_k such that $P(k) = L_k^T L_k$ can be obtained as follows.

a) First, a triangular matrix N is computed, satisfying $(T + T^T)^{-1} = N^T N$. (Here, a fast method based on the W^{2}R (Wittle-Wiggins-Robinson) algorithm is recommended [6], which exploits the underlying block Toeplitz structure of $T + T^T$).

b) Next, the matrix product NS is reduced to the upper triangular form by orthogonal transformations, i.e. $L_k = U^T(NS)$, where U^T is a sequence of Householder reflectors.

A similar (dual) computational scheme delivers a triangular matrix M_k such that $Q(k) = M_k M_k^T$. Therefore, at the step 3 of the computational procedure the SVD for $L_k M_k$ can be considered instead of the "true" matrix product LM.

6. SQUARE ROOT APPROXIMATE BALANCING

By using (7) and the corresponding formula for $Q(k)$, we must manipulate the large scale matrices S and T in order to compute the Cholesky factors L_k and M_k as explained above. This can be done recurrently by treating (5) and (8) in a square root manner.

By considering $P(0) = 0$ as an initialization in (5), we get $P(1) = C^T R^{-1} C = L_1^T L_1$, where $L_1 = D_1^{-1} C$ and D_1 satisfies $R = D_1 D_1^T$. Assume inductively that at step i an upper triangular matrix L_i is known such that $P(i) = L_i^T L_i$. Denote $X = L_i A$, $Y = L_i B$ and compute D_i such that

$$R - Y^T Y = D_i D_i^T$$

by using a specific down-dating Cholesky procedure, already available in LAPACK and MATLAB. Then, the next Cholesky factor L_{i+1} for P_{i+1} is obtained by performing the

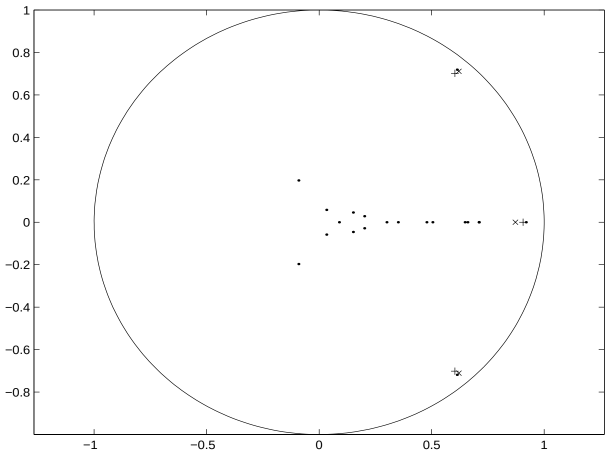

Fig. 1. Pole distribution of original system ('.') (of order 20), approximate balanced truncated model ('+') and compensated approximate balanced truncated model ('x') (of order 3).

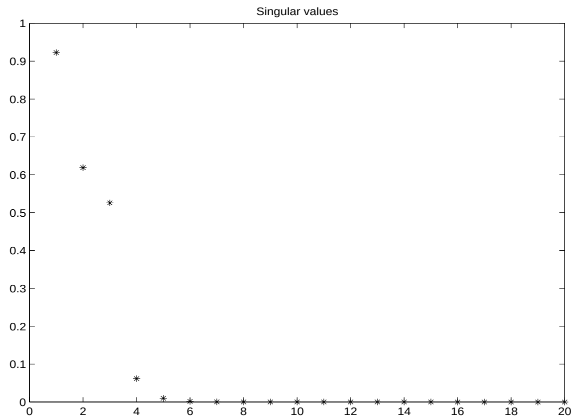

Fig. 2. The singular values of the matrix $L_k M_k$, $k = 20$ at the step 3 of the approximate balancing procedure for the example considered (see the values in table 1).

following QR factorization

$$\left[\begin{array}{c} L_{i+1} \\ 0 \end{array} \right] = U^T \left[\begin{array}{c} D_i^{-1}(C - Y^T X) \\ X \end{array} \right],$$

where U is an adequate orthogonal matrix. The corresponding Cholesky factor M_i such that $Q(i) = M_i M_i^T$ (see (8)) can be computed by duality. By the same argument as in the previous section, at the step 3 of the computational procedure, the SVD for $L_i M_i$ can be performed for various $i \geq 1$, until the principal singular values separate well from the remaining ones and the corresponding gap becomes evident.

$9.2273168221729540e-001$
$6.1870990170483486e-001$
$5.2577408264296710e-001$
$6.1675208191709546e-002$
$9.9272258899499068e-003$
$1.6158056398923116e-003$
$3.0698682861478229e-004$
$1.3735394512943311e-005$
$2.0405955897113167e-006$
$5.8706188987959169e-008$
$3.5120834418515865e-009$
$1.1502970105795745e-010$
$6.6115433548094739e-013$
$5.3933754958146852e-013$
$1.8908831143198570e-015$
$1.1997958540293385e-019$
$1.0935442664272020e-019$
$4.0314097462844509e-022$
$8.9194120188316738e-025$
$1.4546378540634608e-027$

Table 1. The singular values of the matrix $L_{20} M_{20}$ at the step 3 of the approximate balancing procedure for the example considered.

7. ORDER REDUCTION BY CARE BALANCING. CONTINUOUS CASE

Let $S = (A, B, C, D)$ be a minimal, positive-real (continuous time) system satisfying $R = D + D^T > 0$. According to the positive-real lemma, there exist stabilizing, symmetric positive semidefinite solutions P and Q to the following algebraic Riccati equations (CARE)

$$0 = A^T P + PA + \qquad (9)$$

$$+(C^T - PB)R^{-1}(C - B^T P)$$

and, by duality,

$$0 = AQ + QA^T + \qquad (10)$$

$$+(B - QC^T)R^{-1}(B^T - CQ).$$

In this case the main computational procedure remain valid, by simply replacing (1), (2) with (9), (10). Also, the computational improvements presented in sections 3 and 4 are applicable as such. (Of course, in the actual continuous time setting, to obtain the

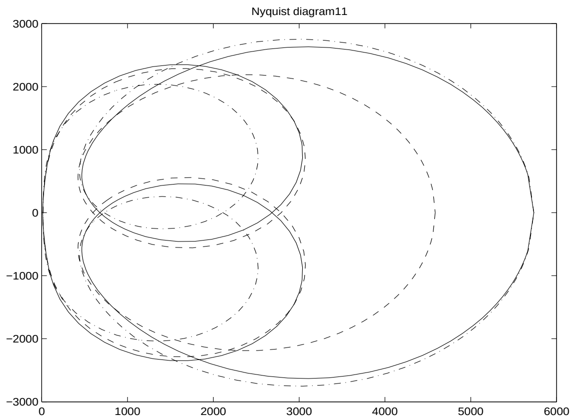

Fig. 3. Nyquist diagram for the original system (solid line) (of order 20), for the approximate balanced truncated model (dashed line) and for the compensated approximate balanced truncated model (dash-dot line)(of order 3).

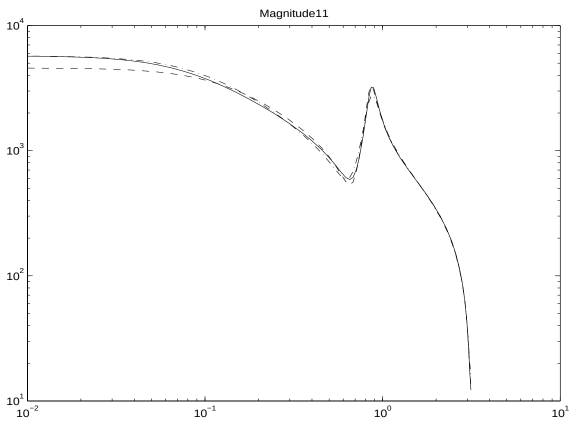

Fig. 4. Bode magnitude diagram for the original system (solid line) (of order 20), for the approximate balanced truncated model (dash-dot line) and for the compensated approximate balanced truncated model (dashed line) (of order 3).

compensated order reduction, $\mathrm{Re}\,\alpha \geq 0$ and steady state corresponds to $\alpha = 0$, see(4)), Eventually, the idea of approximate balancing by recursive (square root) computation may be used if the given continuous time system $\mathbf{S} = (A, B, C, D)$ is converted to a discrete one, by using a well known bilinear transformation.

8. NUMERICAL RESULTS

We have performed a lot of numerical experiments to test all the proposed computational procedures. As an example we present in a synthetic form some numerical results con-

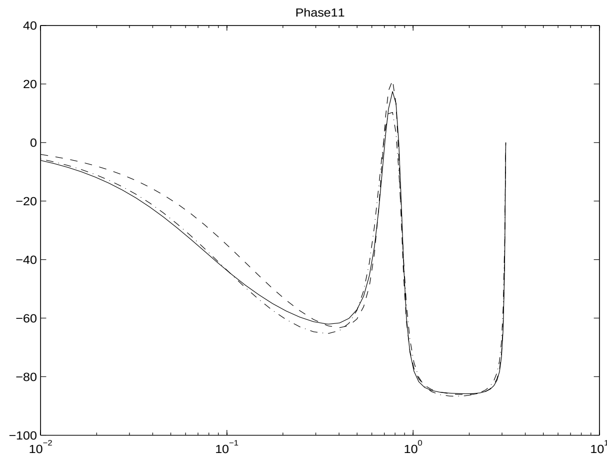

Fig. 5. Bode phase diagram for the original system (solid line) (of order 20), for the approximate balanced truncated model (dashed line) and for the compensated approximate balanced truncated model (dash-dot line) (of order 3).

cerning the model order reduction problem with passivity preservation for a randomly generated SISO discrete time positive real system of order 20. The corresponding pole-distribution, Nyquist and Bode diagrams as well as step response are presented in figures 3, 4 and 5, and figure 6 respectively.

The decision concerning the selection of the reduced order $r = n_1$ at step 6 of the computational procedure is based on the singular values of the matrix product $L_{20}M_{20}$ (see section 6), which are given in the table 1 (see, also, figure 2). (Essentially the same numbers are obtained by using the Cholesky factors of the Riccati solutions P and Q, refined by additional Newton corrections to make them numerically positive definite). The large number of very small singular values motivates our effort to develop sound numerical procedures, able to cope with such difficult situations as described above.

9. CONCLUDING REMARKS

The numerical results confirm our computational oriented approach to the positive real order reduction problem, both for continuous and discrete time linear systems. A different approach, based on rational interpolation along the lines developed e.g. in [2] and [4], is under author's investigation and the cor-

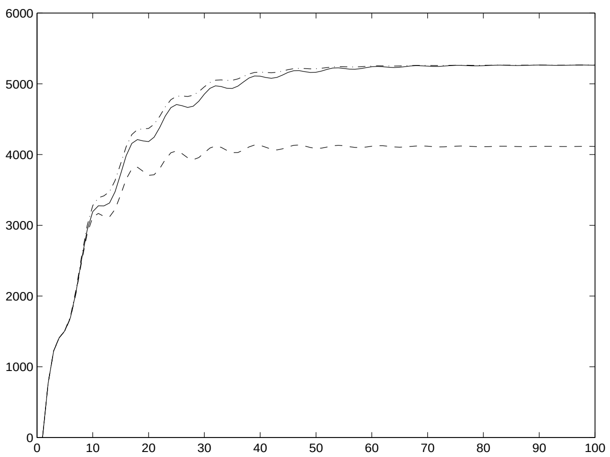

Fig. 6. Step response for the original system (solid line) (of order 20), for the approximate balanced truncated model (dashed line) and for compensated approximate balanced truncated model (dash-dot line) (of order 3).

responding results will be reported in some future works, which are in preparation.

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