

A New Robust Pole Placement Stabilization for a Class of Time-Varying Polytopic Uncertain Switched Nonlinear Systems under Arbitrary Switching

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Abstract: This paper is concerned with the problem of robust stabilization via state feedback control for a class of both continuous and discrete-time switched nonlinear systems with polytopic time-varying uncertainty. These studied systems are modeled by differentials or differences equations. Therefore, a transformation of the systems representation under the arrow form is performed. Subsequently, by using a constructed common Lyapunov function and applying the Kotelyanski lemma associated with the M – matrix **properties**. A new robust pole placement stabilization is proposed. These obtained results provide a solution to one of the basic problems for switched nonlinear systems which ensures asymptotic stability under arbitrary switching. Compared with the existing results of uncertain switched nonlinear systems, these proposed conditions are formulated in terms of the time-varying polytopic uncertain parameters and they allow us to avoid searching a common Lyapunov function which is a difficult **matter**. Finally, **an application** to stabilize a shunt DC motor with uncertain models under variable mechanical loads is performed to illustrate the effectiveness of the theoretical results.

Keywords: Switched nonlinear systems, polytopic time-varying uncertainty, common Lyapunov function, Kotelyanski lemma, state feedback control, pole placement, asymptotic stability, arbitrary switching.

1. INTRODUCTION

As one of the important class of hybrid system, switched dynamic systems can be modeled by a family of continuous or discrete-time subsystems and a rule, called a switching signal that determines the switching manner between the subsystems rule. Mathematically, these subsystems are usually described by a collection of indexed differential or difference equations. The main motivation for studying switched system comes from the fact that many practical systems evolve with switches and hybrid behaviors such as chemical processes (Putyrski *et al.*, 2011), transportation systems (Hamdouch *et al.*, 2007), power systems and power electronics (Sengupta *et al.*, 2009), communication networks (Alnowibet *et al.*, 2006), constrained robotics and robot manufacture (Back *et al.*, 1993), computer disk drives (Gollu *et al.*, 1989), and automated highways (Varaiya, 1993).

Stability and stabilization are two fundamental and important research issues in the control community. Several methods have been developed for solving these problems of switched systems (Vu *et al.*, 2005; Araghi, *et al.*, 2013; Zhao *et al.*, 2012; Hespanha *et al.*, 1999; Zhao *et al.*, 2008), such as the common Lyapunov function approach (Vu *et al.*, 2005) which is mainly investigated for stability under arbitrary switching, the average dwell time (Zhao *et al.*, 2012), and the multiple Lyapunov functions method (Hespanha *et al.*, 1999; Zhao *et al.*, 2008) for studying stability under controlled switching. Hence, stability and stabilization under arbitrary switching which are considered in this work remain more

performed when practical systems are involved. Indeed, the unique practically applicable approach to this problem is based on the construction of a common Lyapunov function for all the subsystems. Therefore, this method is usually very difficult to apply even for switched linear systems. However, it becomes more complicated when switched nonlinear systems are involved and relatively few results have been reported in this context (Yu *et al.*, 2011; Yu *et al.*, 2012; Dayawansa *et al.*, 1999; Mancilla *et al.*, 2000; Liberzon, 2004). So far, some attempts are presented to construct a general Lyapunov function for switched nonlinear systems, by using the Lyapunov converse theorems in (Dayawansa *et al.*, 1999; Mancilla *et al.*, 2000) and by recourse to some nilpotent Lie algebras (Liberzon, 2004).

On the other hand, from the practical viewpoint, it is of great importance to investigate uncertain switched systems. Norm-bounded and polytopic uncertainties are two commonly adopted schemes, and the later one has been proved more general offering solution form any practical applications (Zhang *et al.*, 2007; Daafouz *et al.*, 2002; Zhang *et al.*, 2008). In our investigation, considering switched nonlinear systems, we adopt time-varying polytopic uncertainties which depict strong practical significance of this work.

Based on the above discussion, switched nonlinear systems with polytopic uncertainties are worth studying. Up to now, due primarily to the complexity of this problem, the available results on stability analysis and stabilisation of these systems are limited (Chiang *et al.*, 2014; Shiwei, *et al.*, 2013;

Weiming *et al.*, 2014; Orani *et al.*, 2011; Wang *et al.*, 2009), which motivates this study for us.

The purpose of this paper aims to solve the problem of state feedback stabilization for a class of both continuous and discrete-time switched nonlinear systems under arbitrary switching and subject to time-varying polytopic uncertainties. Indeed, by transforming the representation of these considered systems into the arrow form matrix (Kermani *et al.*, 2014 a; Kermani *et al.*, 2014 b; Elmadssia *et al.*, 2013; Zhang *et al.*, 2008; Borne *et al.*, 1993; Kermani *et al.*, 2012 a; Kermani *et al.*, 2012 b; Benrejeb *et al.*, 2008; Benrejeb *et al.*, 2006; Borne *et al.*, 2003; Borne *et al.*, 2007; Benrejeb *et al.*, 1982; Borne, 1987; Hahn, 1967; Grujc *et al.*, 1987; Borne *et al.*, 1972; Borne *et al.*, 2008) employing a suitable constructed common Lyapunov function and applying the Kotelyanski lemma (Kotelyanski, 1952) combined with the M – matrix **properties** (Robert, 1966; Gantmacher, 1966), a new robust pole placement stabilization under arbitrary switching is deduced.

In contrast with some existing results on switched nonlinear systems with polytopic uncertainties, the contributions of this paper is **twofold**: First, the new pole placement design is given to guarantee the closed loop system asymptotic stability under arbitrary switching and may overcome the conservatism of searching a common Lyapunov function. Second, the new stabilization conditions are formulated in terms of the time-varying polytopic uncertain parameters.

The rest of this paper is organized as follows. In section 2, we present the problem formulation and some preliminaries. Then, we given the main results on stabilization of the considered continuous-time switched nonlinear systems. The research problem formulations and the main results of the studied discrete-time switched are given in section 3. Two examples which modeled a shunt DC motor with uncertain models under variable mechanical loads are **provided** to show the effectiveness of the proposed approach. The conclusions are summarized in section 5.

Notations: The following notation will be used in this paper, $\mathfrak{R}^n$ denotes the n dimensional Euclidean space, I_n is the identity matrix with appropriate dimensions, $\|\cdot\|$ denotes Euclidean vector norm. For any $u = (u_i)_{1 \leq i \leq n}$, $v = (v_i)_{1 \leq i \leq n} \in \mathfrak{R}^n$ We define the scalar product of the vector

u and v as: $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$. **Denote by $\lambda(M)$ the set of**

eigenvalues of matrix M , M^T its transpose and M^{-1} its inverse and if $M = (m_{i,j})_{1 \leq i, j \leq n}$. We denote

$M^* = (m_{i,j}^*)_{1 \leq i, j \leq n}$ with $m_{i,j}^* = m_{i,j}$ if $i = j$ and $m_{i,j}^* = |m_{i,j}|$ if $i \neq j$ and $|M| = |m_{i,j}|$, $\forall i, j$.

$\underline{N} = \{1, 2, \dots, N\}$.

2. CONTINUOUS-TIME SWITCHED SYSTEMS

2.1 Problem statement and preliminaries

Consider a class of a continuous-time uncertain switched nonlinear systems of the form:

$$\dot{x}(t) = A_{\sigma(t)}(\cdot)x(t) + B_{\sigma(t)}(\cdot)u(t) \quad (1)$$

where $x(t) \in \mathfrak{R}^n$ is the state, $u(t) \in \mathfrak{R}$ is the control input, $A_{\sigma(t)}(\cdot)$ and $B_{\sigma(t)}(\cdot)$ are matrices have nonlinear elements with appropriate dimensions and $\sigma(t):$

$\mathfrak{R}^+ \rightarrow \underline{N} = \{1, 2, \dots, N\}$ is the switching signal assumed to be available in real time. Therefore, the continuous-time switched system is composed of N subsystems which are expressed as:

$$\dot{x}(t) = A_i(\cdot)x(t) + B_i(\cdot)u(t), \quad i \in \underline{N} \quad (2)$$

where $A_i(\cdot)$ and $B_i(\cdot)$ are matrices with appropriate dimensions.

The system matrices are subject to polytopic uncertainties

which can be modeled as: $A_i(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(t) A_{il}(\cdot)$ and

$B_i(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(t) B_{il}(\cdot)$, where $A_{il}(\cdot) (l=1, \dots, N_l)$ and

$B_{il}(\cdot) (l=1, \dots, N_l)$ are the vertex matrices denoting the extreme points of the polytope $A_i(\cdot)$ and $B_i(\cdot)$, N_l is the number of the extreme points and the weighting factors $\mu_{il}(t) (l=1, \dots, N_l)$ are time-variant uncertainties belong to:

$$\mu_i(t): \sum_{l=1}^{N_l} \mu_{il}(t) = 1, \mu_{il}(t) \geq 0 \quad (3)$$

when the controllers are switched between the subsystems, the state feedback controllers are formed as:

$$u(t) = -K_{\sigma(t)}(\cdot)x(t) \quad (4)$$

where $K_i(\cdot)$ are the nonlinear controller gains.

Now, we first briefly review some preliminaries.

Kotelyanski lemma. (Kotelyanski, 1952) The real parts of the eigenvalues of matrix A , with non-negative off-diagonal elements, are less than a real number μ if and only if all those of matrix M ; $M = \mu I_n - A$ are positive, with I_n the n identity matrix.

The following definitions and remarks will be used in the sequel.

Definition 1. The matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is called an M – matrix, if the following conditions are met:

- The principal minors of A are all positive:

$$(A) \begin{pmatrix} 1 & 2 & \dots & j \\ 1 & 2 & \dots & j \end{pmatrix} > 0 \quad \forall j \in 1, \dots, n \quad (5)$$

- For any positive real numbers $\eta = (\eta_1, \dots, \eta_n)^T$ the algebraic equations $Ax = \eta$ have a positive solution $w = (w_1, \dots, w_n)^T$.

Remark 1. $A = (a_{ij})_{1 \leq i, j \leq n}$ is the opposite of an M -matrix if $(-A)$ is an M -matrix.

Remark 2. A continuous-time system characterized by $A(\cdot)$ is stable if $A(\cdot)$ is the opposite of an M -matrix. In this case, the main minors of $A(\cdot)$ are alternating sign with the first is negative and the Kotelyanski lemma allows us to conclude about stability of the system characterized by $A(\cdot)$. Below, we present the definition of an overvaluing system.

Definition 2. (Borne et al., 2008) The matrix $M_c(\cdot)$ is said a pseudo-overvaluing matrix of the system $\dot{x}(t) = A(\cdot)x(t)$ with respect to the vector norm $p(x) = [|x_1| \dots |x_n|]^T$, if the following inequality is met:

$$D^+ p(x) \leq M_c(\cdot) p(x) \quad (6)$$

where D^+ denotes the right hand derivative.

Consequently, the stability of the comparison system: $\dot{z}(t) = M_c(\cdot)z(t)$ with the initial conditions such as $z_0 = p(x_0)$, implies the same property for the initial system.

2.2. Continuous-time switched systems: main results

In this section, we give the main result of the closed-loop system (1).

Theorem 1. System (1) with polytopic uncertainties (3) is globally robust asymptotically stabilizable with the state feedback controller (4) under arbitrary switching rule (1) if $M_c(\cdot)$ is the opposite of an M -matrix,

with:

$$M_c(\cdot) = \max_{1 \leq i \leq N} \left((A_i^c(\cdot))^* \right) \quad (7)$$

where:

$$A_i^c(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(t) (A_{il}(\cdot) - B_{il}(\cdot)K_{il}(\cdot))$$

$$= \begin{bmatrix} \sum_{l=1}^{N_l} \mu_{il}(t) (a_i^{c(11)}(\cdot)) & \dots & \sum_{l=1}^{N_l} \mu_{il}(t) (a_i^{c(1n)}(\cdot)) \\ \vdots & & \vdots \\ \sum_{l=1}^{N_l} \mu_{il}(t) (a_i^{c(n1)}(\cdot)) & \dots & \sum_{l=1}^{N_l} \mu_{il}(t) (a_i^{c(nn)}(\cdot)) \end{bmatrix} \quad (8)$$

Proof. Let $w \in \mathbb{R}^n$ with components $(w_m > 0, \forall m = 1, \dots, n)$ and $x(t) \in \mathbb{R}^n$ is the state vector.

Define the radially unbound common Lyapunov function below for the closed-loop system (1) with polytopic uncertainties (3):

$$V(x(t), t) = \langle |x(t)|, w \rangle \quad (9)$$

it is clear that $V(x(t_0), t_0) < \infty$.

The right hand derivative of $V(x(t), t)$ along the trajectory of the closed-loop system (1) under the switching signal $\sigma(t)$ is given as follows:

$$D^+ V(x(t), t) = \left\langle \frac{d^+ |x(t)|}{d^+ t}, w \right\rangle = \left\langle \text{sgn}(x(t)) \frac{d^+ x(t)}{d^+ t}, w \right\rangle \quad (10)$$

where:

$$\text{sgn}(x(t)) = \begin{bmatrix} \text{sgn}(x_1(t)) & & \\ & \ddots & \\ & & \text{sgn}(x_n(t)) \end{bmatrix} \quad (11)$$

In this way, we obtain:

$$D^+ V(x(t), t) = \langle \text{sgn}(x(t)) (A_{\sigma(t)}^c(\cdot)) x(t), w \rangle$$

$$\leq \left\langle (A_{\sigma(t)}^c(\cdot))^* |x(t)|, w \right\rangle < \langle M_c(\cdot) |x(t)|, w \rangle \quad (12)$$

To complete this proof, we assume that $M_c(\cdot)$ is the opposite of an M -matrix. Therefore, we can find a vector $\rho \in \mathbb{R}_+^n$ ($\rho_m \in \mathbb{R}_+ \quad m = 1, \dots, n$) satisfying that $(M_c(\cdot))^T w = -\rho, \forall w \in \mathbb{R}_+^n$.

Hence, we obtain:

$$\langle (M_c(\cdot)) |x(t)|, w \rangle = \langle (M_c(\cdot))^T w, |x(t)| \rangle = \langle -\rho, |x(t)| \rangle \quad (13)$$

Taking into account (13), relation (12) becomes:

$$D^+ V(x(t), t) < \langle -\rho, |x(t)| \rangle = -\sum_{m=1}^n \rho_m |x(t)| < 0 \quad (14)$$

This completes the proof of Theorem 2. $\blacksquare$

2.3. Application to uncertain switched systems defined by differential equations

In this section, a new state feedback stabilization design is presented for a class of continuous-time uncertain switched nonlinear systems described by N subsystems. All the subsystems are modeled by a family of differential equations given as below:

$$y^n(t) + \left(\sum_{j=0}^{n-1} \sum_{l=1}^{N_l} \mu_{il}(t) a_{il}^{n-j}(\cdot) y^{(j)}(t) \right) = u(t) \quad (15)$$

where $y(t) \in \mathbb{R}^n$, $a_{il}^j(\cdot)$ are nonlinear coefficients for each $i \in \underline{N}$, ($l = 1, \dots, N_l$) and ($j = 0, \dots, n-1$). $\mu_{il}(t)$ are time-varying polytopic uncertain parameters given in (3) and $u(t) \in \mathbb{R}$ is the control input.

Consider the following state variables for system (15):

$$x_{j+1}(t) = \frac{dy^{(j)}(t)}{dt^{(j)}}, \quad j = 0, \dots, n-1 \quad (16)$$

By substituting (16) into (15), we obtain:

$$\begin{cases} \dot{x}_j(t) = x_{j+1}(t), & j = 0, \dots, n-1 \\ \dot{x}_n(t) = \left(-\sum_{p=0}^{n-1} \sum_{l=1}^{N_l} \mu_{il}(t) a_{il}^{n-j}(\cdot) x_{j+1}(t) \right) + u(t) \end{cases} \quad (17)$$

or under matrix representation, we obtain:

$$\dot{x}(t) = \sum_{l=1}^{N_l} \mu_{il}(t) (A_i(\cdot) x(t) + B_i(\cdot) u(t)), \quad i \in \underline{N} \quad (18)$$

Due to (17), it is easy to see that system (18) is given in the companion form such as:

$$A_{il}(\cdot) = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ -a_{il}^n(\cdot) & -a_{il}^{n-1}(\cdot) & \dots & -a_{il}^1(\cdot) \end{bmatrix}, \quad B_{il} = B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (19)$$

Led to the following a state feedback controller:

$$u(t) = -\sum_{l=1}^{N_l} \mu_{il}(t) K_{il}(\cdot) x(t) \quad (20)$$

where $K_{il}(\cdot)$ are the vectors gains of the controller for each $i \in \underline{N}$, and $(l = 1, \dots, N_l)$.

So, all the closed loop subsystems are characterized by the following state space description:

$$\begin{aligned} \dot{x}(t) &= \sum_{l=1}^{N_l} \mu_{il}(t) (A_{il}(\cdot) - B_{il}(\cdot) K_{il}(\cdot)) x(t) \\ &= \sum_{l=1}^{N_l} \mu_{il}(t) A_{il}^c(\cdot) x(t), \quad i \in \underline{N} \end{aligned} \quad (21)$$

with:

$$A_{il}^c(\cdot) = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ -a_{il}^{c(n)}(\cdot) & -a_{il}^{c(n-1)}(\cdot) & \dots & -a_{il}^{c(1)}(\cdot) \end{bmatrix} \quad (22)$$

where:

$$a_{il}^{c(j)}(\cdot) = a_{il}^j(\cdot) + k_{il}^{n+1-j}(\cdot), \quad (j = 1, \dots, n) \quad (23)$$

and $a_{il}^{c(j)}(\cdot)$ are the coefficients of the instantaneous characteristic polynomial $P_{A_{il}(\cdot)}(\lambda)$ of the vertex matrix $A_{il}^c(\cdot)$. It is given by:

$$P_{A_{il}(\cdot)}(\lambda) = \lambda^n + \sum_{p=0}^{n-1} a_{il}^{c(n-p)}(\cdot) \lambda^p \quad (24)$$

Now, by considering the switching signal (1), the closed loop switched nonlinear system is deduced as below:

$$\dot{x}(t) = A_{\sigma(t)}^c(\cdot) x(t) \quad (25)$$

Next, the following basic change will be adopted in order to simplify the application of the Kotelyanski lemma:

$$\dot{z}(t) = M_i(\cdot) z(t), \quad i \in \underline{N} \quad (26)$$

where $z(t) = Px(t)$ is the new state vector, P is the

corresponding passage matrix and $M_i(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(t) M_{il}(\cdot)$

where $M_{il}(\cdot)$ are vertex matrices in the arrow form given as following:

$$M_{il}(\cdot) = P^{-1} A_{il}^c(\cdot) P = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 & \beta_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \alpha_{n-1} & \beta_{n-1} \\ \gamma_{il}^1(\cdot) & \dots & \dots & \gamma_{il}^{n-1}(\cdot) & \gamma_{il}^n(\cdot) \end{bmatrix} \quad (27)$$

$$P = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & 0 \\ (\alpha_1)^2 & (\alpha_2)^2 & \dots & (\alpha_{n-1})^2 & \vdots \\ \vdots & \vdots & \dots & \vdots & 0 \\ (\alpha_1)^{n-1} & (\alpha_2)^{n-1} & \dots & (\alpha_{n-1})^{n-1} & 1 \end{bmatrix} \quad (28)$$

$$\begin{cases} \beta_j = \prod_{\substack{q=1 \\ q \neq j}}^{n-1} (\alpha_j - \alpha_q)^{-1} \quad \forall j = 1, \dots, n-1 \\ \gamma_{il}^j(\cdot) = -P_{A_{il}^c(\cdot)}(\alpha_j) \quad \forall j = 1, \dots, n-1 \\ \gamma_{il}^n(\cdot) = -a_{il}^{c(1)}(\cdot) - \sum_{j=1}^{n-1} \alpha_j \end{cases} \quad (29)$$

and $\alpha_j, j = 1 \dots n-1$ are distinct arbitrary constant parameters.

Next, the comparison system associated to the vector norm

$p(z) = [|z_1| \dots |z_n|]^T$ is defined by:

$$\dot{z}(t) = M_c(\cdot) z(t) \quad (30)$$

where $M_c(\cdot)$ is the comparison matrix relative to system (25), it is deduced as below:

$$M_c(\cdot) = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \alpha_{n-1} & |\beta_{n-1}| \\ \bar{t}(\cdot)^1 & \dots & \dots & \bar{t}(\cdot)^{n-1} & \bar{t}(\cdot)^n \end{bmatrix} \quad (31)$$

with:

$$\begin{cases} \bar{t}^n(\cdot) = \max_{1 \leq i \leq N} \left(\sum_{l=1}^{N_l} \mu_{il}(t) \gamma_{il}^n(\cdot) \right) \\ \bar{t}^j(\cdot) = \max_{1 \leq i \leq N} \left(\sum_{l=1}^{N_l} \mu_{il}(t) \gamma_{il}^j(\cdot) \right), \quad j = 1, \dots, n-1 \end{cases} \quad (32)$$

The aim of this work consist to design a state feedback controller (20) by using the poles placement control which guarantee asymptotic stability of closed loop system (25).

Next, the following theorem presents a new pole placement control stabilization for system (25).

Theorem 2. If all the n poles $\{p_1, \dots, p_n\}$ are chosen to be real, distinct and negative. Then, the closed loop system (25) with polytopic uncertainties (3) is robust stabilizing by the controller (20) under arbitrary switching (1) and the following conditions are satisfied:

$$\begin{cases} \bar{\tau}^j(\cdot) = \max_{1 \leq i \leq N} \left(\sum_{l=1}^{N_l} \mu_{il}(t) \gamma_{il}^j(\cdot) \right) = 0, j = 1, \dots, n-1 \\ \bar{\tau}^n(\cdot) = \max_{1 \leq i \leq N} \left(\sum_{l=1}^{N_l} \mu_{il}(t) \gamma_{il}^n(\cdot) \right) = p_n < 0 \end{cases} \quad (33)$$

where:

$$\begin{cases} \beta_j = \prod_{\substack{q=1 \\ q \neq j}}^{n-1} (p_j - p_q)^{-1} \quad \forall j = 1, \dots, n-1 \\ \gamma_{il}^j(\cdot) = -P_{A_{il}}(p_j) \quad \forall j = 1, \dots, n-1 \\ \gamma_{il}^n(\cdot) = -a_{il}^{c(1)}(\cdot) - \sum_{j=1}^{n-1} p_j \end{cases} \quad (34)$$

Proof of this theorem is given in appendix A.

3. DISCRETE-TIME SWITCHED SYSTEMS

3.1 Problem formulation and Preliminaries

Consider the following discrete-time switched nonlinear systems:

$$x(k+1) = A_{\sigma(k)}(\cdot) x(k) + B_{\sigma(k)}(\cdot) u(k) \quad (35)$$

where $x(k) \in \mathbb{R}^n$ is the state vector of the system at time k , $u(k) \in \mathbb{R}$ is the control input, $A_{\sigma(k)}(\cdot)$ and $B_{\sigma(k)}(\cdot)$ are matrices have nonlinear elements with appropriate dimensions and $\sigma(k): \mathbb{R}^+ \rightarrow \underline{N} = \{1, 2, \dots, N\}$ is the switching signal.

The switched system is composed of N discrete-time subsystems which are given by:

$$\dot{x}(t) = A_i(\cdot) x(k) + B_i(\cdot) u(k), \quad i \in \underline{N} \quad (36)$$

when the uncertain model is presented, the matrices $A_i(\cdot)$

and $B_i(\cdot)$ are given by $A_i(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(k) A_{il}(\cdot)$ and

$B_i(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(k) B_{il}(\cdot)$, where $\mu_{il}(k)$ ($l = 1, \dots, N_l$) are

time-variant uncertainties given in (3).

The state feedback controller is given as below:

$$u(k) = -K_{\sigma(k)} x(k) \quad (37)$$

Now, we present some definition and remarks which will play important roles in deriving our main results for discrete-time switched systems subsequently.

Remark 3. (Borne, et al., 2007) A discrete-time system given by a matrix $A(\cdot)$ is stable if matrix $(I_n - A(\cdot))$ verified the

Kotelyanski conditions. In this case $(I_n - A(\cdot))$ is an M -matrix and all the principal minors of $(I_n - A(\cdot))$ are positive.

Definition 3. (Benrejeb, et al., 2006) The matrix $M_D(\cdot)$ is the comparison matrix of the system given by a matrix $A(\cdot)$ with respect to the vector norm p if the inequality below is satisfied:

$$p(x(k+1)) \leq M_D(\cdot) p(x(k)) \quad (38)$$

Then, the stability of the comparison system: $z(k+1) = M_D(\cdot) z(k)$ with the initial conditions such as $z_0 = p(x_0)$ implies the same property for the initial system.

3.2 Discrete-time switched systems-main results

This subsection presents a new state feedback stabilization of the closed-loop system (35).

Theorem 3. The closed-loop system (35) is robustly asymptotically stabilizable with the state feedback controller (37) under arbitrary switching (35) and all admissible uncertainties (3) if $(I_n - M_D)$ is an M -matrix, with:

$$M_D(\cdot) = \left(\max_{1 \leq i \leq N} \left(\|A_i^c(\cdot)\| \right) \right) \quad (39)$$

and:

$$\begin{aligned} A_i^c(\cdot) &= \sum_{l=1}^{N_l} \mu_{il}(k) (A_{il}(\cdot) - B_{il}(\cdot) K_{il}(\cdot)) \\ &= \begin{bmatrix} \sum_{l=1}^{N_l} \mu_{il}(k) (a_{i1}^{c(11)}(\cdot)) & \dots & \sum_{l=1}^{N_l} \mu_{il}(k) (a_{i1}^{c(1n)}(\cdot)) \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^{N_l} \mu_{il}(k) (a_{in}^{c(11)}(\cdot)) & \dots & \sum_{l=1}^{N_l} \mu_{il}(k) (a_{in}^{c(1n)}(\cdot)) \end{bmatrix} \end{aligned} \quad (40)$$

Proof. Consider the following common Lyapunov function for the closed-loop system (35):

$$V(x(k), k) = \langle |x(k)|, w \rangle \quad (41)$$

where $w \in \mathbb{R}^n$ with components $(w_m > 0, \forall m = 1, \dots, n)$ and $x(k) \in \mathbb{R}^n$ is the state vector.

Along the trajectories of system (35), the difference of the Lyapunov function is given as below:

$$\begin{aligned} \Delta v(x(k), k) &= v(x(k+1), k+1) - v(x(k), k) \\ &= \langle |x(k+1)|, w \rangle - \langle |x(k)|, w \rangle \end{aligned} \quad (42)$$

Subsequently, under arbitrary switching $\sigma(k)$, it is easy to verify that:

$$\langle |A_{\sigma(k)}^c(\cdot) x(k)|, w \rangle - \langle |x(k)|, w \rangle$$

$$\begin{aligned}
&\leq \left\langle \max_{1 \leq i \leq N} \left(\|A_i^c(\cdot)\| \right) |x(k)|, w \right\rangle - \left\langle |x(k)|, w \right\rangle \\
&\leq \left\langle \left(\max_{1 \leq i \leq N} \left(\|A_i^c(\cdot)\| \right) - I_n \right) |x(k)|, w \right\rangle \\
&= \left\langle -(I_n - M_D(\cdot)) |x(k)|, w \right\rangle \quad (43)
\end{aligned}$$

where $M_D(\cdot)$ is introduced in (39).

We assume now that $(I_n - M_D(\cdot))$ is an M -matrix. In such condition, we can find a vector $\rho \in \mathbb{R}_+^{*n}$ ($\rho_p \in \mathbb{R}_+^*$ $p=1, \dots, n$) satisfying the following relation $(I_n - M_D(\cdot))^T w = \rho, \forall w \in \mathbb{R}_+^{*n}$, then we can write:

$$\begin{aligned}
&\left\langle (I_n - M_D(\cdot)) |x(k)|, w \right\rangle \\
&= \left\langle (I_n - M_D(\cdot))^T w, |x(k)| \right\rangle = \langle \rho, |x(k)| \rangle \quad (44)
\end{aligned}$$

According to (44) it holds that:

$$\left\langle -(I_n - M_D(\cdot)) |x(k)|, w \right\rangle = \langle -\rho, |x(k)| \rangle \quad (45)$$

Finally, we have:

$$\begin{aligned}
\Delta V(k) &\leq \left\langle -(I_n - M_D(\cdot)) |x(k)|, w \right\rangle \\
&\leq -\sum_{p=1}^n \rho_p |x_p(k)| < 0 \quad (46)
\end{aligned}$$

This completes the proof of Theorem 3. $\blacksquare$

3.3. Application to switched systems defined by difference equations

In this section, we consider a class of discrete-time uncertain switched nonlinear systems composed of N subsystems, all of them are modeled by the following difference equation:

$$y(k+n) + \left(\sum_{l=1}^{N_l} \mu_{il}(k) \sum_{j=0}^{n-1} a_{il}^{n-j}(\cdot) y(k+j) \right) = u(k) \quad (47)$$

where $y(k) \in \mathbb{R}^n$, $a_{il}^j(\cdot)$ are nonlinear coefficients for each $i \in \underline{N}$, ($l=1, \dots, N_l$) and ($j=1, \dots, n-1$), $\mu_{il}(k)$ are time-varying polytopic uncertain parameters given in (3) and $u(k) \in \mathbb{R}$ is the control input.

To solve this problem, we introduce the following state variables:

$$x_{j+1}(k) = y(k+j), \quad j=0, \dots, n-1 \quad (48)$$

Combining (47) and (48) yields for each $i \in \underline{N}$:

$$\begin{cases} x_j(k+1) = x_{j+1}(k), \quad j=1, \dots, n-1 \\ x_n(k+1) = - \left(\sum_{l=1}^{N_l} \mu_{il}(k) \sum_{j=0}^{n-1} a_{il}^{n-j} x_{j+1}(k) \right) + u(k) \end{cases} \quad (49)$$

or under matrix representation:

$$x(k+1) = \sum_{l=1}^{N_l} \mu_{il}(k) (A_{il}(\cdot) x(k) + B_{il}(\cdot) u(k)), \quad i \in \underline{N} \quad (50)$$

where $x(k)$ is the state, $A_{il}(\cdot)$ and $B_{il}(\cdot)$ are vertex matrices that have nonlinear elements of appropriate dimension.

It is clear to see that all the models are given in the controllable form given in (19).

The feedback controller is given as follows:

$$u(k) = - \sum_{l=1}^{N_l} \mu_{il}(k) K_{il}(\cdot) x(k), \quad i \in \underline{N} \quad (51)$$

So, all the closed-loop subsystems can be written as follows:

$$\begin{aligned}
x(k+1) &= \sum_{l=1}^{N_l} \mu_{il}(k) (A_{il}(\cdot) - B K_{il}(\cdot)) x(k) \\
&= \sum_{l=1}^{N_l} \mu_{il}(k) A_{il}^c(\cdot) x(k), \quad i \in \underline{N} \quad (52)
\end{aligned}$$

where the vertex matrix $A_{il}^c(\cdot)$ is defined in (22).

Therefore, by considering the switching law (35), the closed-loop switched nonlinear is given as follows:

$$x(k+1) = A_{\sigma(k)}^c(\cdot) x(k) \quad (53)$$

As done for the continuous time case a change of base for all the subsystems defined in (52) under the arrow form yields to:

$$z(k+1) = \sum_{l=1}^{N_l} \mu_{il}(k) M_{il}(\cdot) z(k), \quad i \in \underline{N} \quad (54)$$

where $M_{il}(\cdot)$ is given in (28).

and finally, the comparison discrete-time system $z(k) \in \mathbb{R}^n$ is given by:

$$z(k+1) = M_D(\cdot) z(k) \quad (55)$$

where the comparison matrix $M_D(\cdot)$ for discrete-time is given as below, it is deduced from the matrices

$M_i(\cdot) = \sum_{l=1}^{N_l} \mu_{il}(k) M_{il}(\cdot)$ by substituting all its elements by their absolute values:

$$M_D(\cdot) = \begin{bmatrix} |\alpha_1| & 0 & \dots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & |\alpha_{n-1}| & |\beta_{n-1}| \\ \bar{t}_D^{-1}(\cdot) & \dots & \dots & \bar{t}_D^{-n-1}(\cdot) & \bar{t}_D^{-n}(\cdot) \end{bmatrix} \quad (56)$$

with:

$$\bar{t}_D^{-j}(\cdot) = \max_{1 \leq i \leq N} \left(\sum_{l=1}^{N_l} \mu_{il}(t) \bar{t}_{il}^{-j}(\cdot) \right), \quad j=1, \dots, n \quad (57)$$

After this formulation, now we are in position to present a new robust stabilization of the closed-loop system (53) by using pole assignment control.

Theorem 4. If all the n poles $\{z_1, \dots, z_n\}$ imposed by the control law (51) of the closed-loop system (53) assumed to be real and have modules inferior to the unit. Then, system (53) is robust stabilizing by the control law (51) under arbitrary switching rule (35) and all admissible uncertainties (3) and the conditions below are met:

$$\left\{ \begin{array}{l} \bar{t}_D^j(.) = \max_{1 \leq i \leq N} \left(\left| \sum_{l=1}^{N_i} \mu_{il}(t) \gamma_{il}^j(.) \right| \right) = 0, j=1, \dots, n-1 \\ \bar{t}_D^n(.) = \max_{1 \leq i \leq N} \left(\left| \sum_{l=1}^{N_i} \mu_{il}(t) \gamma_{il}^n(.) \right| \right) = |z_n| < 1 \end{array} \right. \quad (58)$$

with:

$$\left\{ \begin{array}{l} \beta_j = \prod_{\substack{q=1 \\ q \neq j}}^{n-1} (z_j - z_q)^{-1} \quad \forall j=1, \dots, n-1 \\ \gamma_{il}^j(.) = -P_{A_{il}}(z_j) \quad \forall j=1, \dots, n-1 \\ \gamma_{il}^n(.) = -a_{il}^{c(1)}(.) - \sum_{j=1}^{n-1} z_j \end{array} \right. \quad (59)$$

Proof of this theorem is given in Appendix B. ■

4. ILLUSTRATIVE EXAMPLES

In the following section, an application to stabilize a real system is provided to demonstrate the effectiveness of the proposed methods.

Example 1. (Benrejeb et al. 2008) Consider a switched system which modeling a DC motor with shunt excitation under variable mechanical load. All the subsystems are characterized by a transfer function which is preceded by a nonlinear element $\varphi(.)$ corresponding to the nonlinear characteristic of the magnetic flux (Lur'e Postnikov problem). The system is given in Figure 1 with the following parameters:

$$A_1(.) = \mu_{11}(t)A_{11} + \mu_{12}(t)A_{12};$$

$$A_2(.) = \mu_{21}(t)A_{21} + \mu_{22}(t)A_{22}$$

$$\text{and: } A_3(.) = \mu_{31}(t)A_{31} + \mu_{32}(t)A_{32}$$

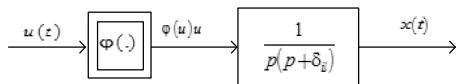


Fig. 1. Modelling of DC motor with shunt excitation

According to Figure 1, the state equation for all the subsystems $i \in \{1, 2, 3\}$ can be written such as:

$$\dot{x}(t) = \sum_{l=1}^2 \mu_{il}(t) (A_{il}(.)x(t) + Bv), \quad v(.) = \varphi(.)u$$

and: $u(t) = -K_{il}(.)x(t)$, where the vertex matrices are defined as follows:

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0 & 1 \\ 0 & -2.5 \end{bmatrix}, & A_{12} &= \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, & A_{21} &= \begin{bmatrix} 0 & 1 \\ 0 & -1.66 \end{bmatrix}, \\ A_{22} &= \begin{bmatrix} 0 & 1 \\ 0 & -1.42 \end{bmatrix}, & A_{31} &= \begin{bmatrix} 0 & 1 \\ 0 & -1.11 \end{bmatrix}, & A_{32} &= \begin{bmatrix} 0 & 1 \\ 0 & -0.76 \end{bmatrix} \quad \text{and} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Consider the following time-varying uncertainties parameters:

$$\mu_{11}(t) = \rho(.), \quad \mu_{12}(t) = 1 - \rho(.), \quad \mu_{21}(t) = \rho(.),$$

$$\mu_{22}(t) = 1 - \rho(.), \quad \mu_{31}(t) = \rho(.), \quad \text{and} \quad \mu_{32}(t) = 1 - \rho(.)$$

with $\rho(.)$ being a general nonlinearity such as $0 \leq \rho(.) \leq 1$.

In this application, we aim to design a state feedback controller which guarantee that the resulting closed-loop system is asymptotically stable under any selected mechanical load. Indeed, the state feedback controller is characterized with the following parameters:

$$K_{11}(.) = [k_{11}^1(.) \quad k_{11}^2(.)], \quad K_{12}(.) = [k_{12}^1(.) \quad k_{12}^2(.)],$$

$$K_{21}(.) = [k_{21}^1(.) \quad k_{21}^2(.)], \quad K_{22}(.) = [k_{22}^1(.) \quad k_{22}^2(.)],$$

$$K_{31}(.) = [k_{31}^1(.) \quad k_{31}^2(.)] \quad \text{and} \quad K_{32}(.) = [k_{32}^1(.) \quad k_{32}^2(.)]$$

So, the closed-loop system is given by:

$$A_{11}^c(.) = \begin{bmatrix} 0 & 1 \\ -K_{11}^1(.)\varphi(.) & -2.5 - \varphi(.)K_{11}^2(.) \end{bmatrix},$$

$$A_{12}^c(.) = \begin{bmatrix} 0 & 1 \\ -K_{12}^1(.)\varphi(.) & -2 - \varphi(.)K_{12}^2(.) \end{bmatrix},$$

$$A_{21}^c(.) = \begin{bmatrix} 0 & 1 \\ -K_{21}^1(.)\varphi(.) & -1.66 - \varphi(.)K_{21}^2(.) \end{bmatrix},$$

$$A_{22}^c(.) = \begin{bmatrix} 0 & 1 \\ -K_{22}^1(.)\varphi(.) & -1.42 - \varphi(.)K_{22}^2(.) \end{bmatrix}$$

$$A_{31}^c(.) = \begin{bmatrix} 0 & 1 \\ -K_{31}^1(.)\varphi(.) & -1.11 - \varphi(.)K_{31}^2(.) \end{bmatrix}$$

$$\text{and: } A_{32}^c(.) = \begin{bmatrix} 0 & 1 \\ -K_{32}^1(.)\varphi(.) & -0.76 - \varphi(.)K_{32}^2(.) \end{bmatrix}$$

Next, the vertex matrices in the arrow form are the following:

$$M_{11}(.) = P^{-1}A_{11}^c(.)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{11}^1(.) & \gamma_{11}^2(.) \end{bmatrix},$$

$$M_{12}(.) = P^{-1}A_{12}^c(.)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{12}^1(.) & \gamma_{12}^2(.) \end{bmatrix},$$

$$M_{21}(.) = P^{-1}A_{21}^c(.)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{21}^1(.) & \gamma_{21}^2(.) \end{bmatrix},$$

$$M_{22}(.) = P^{-1}A_{22}^c(.)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{22}^1(.) & \gamma_{22}^2(.) \end{bmatrix},$$

$$M_{31}(.) = P^{-1}A_{31}^c(.)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{31}^1(.) & \gamma_{31}^2(.) \end{bmatrix}$$

$$\text{and: } M_{32}(.) = P^{-1}A_{32}^c(.)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{32}^1(.) & \gamma_{32}^2(.) \end{bmatrix}$$

with:

$$\begin{cases} \gamma_{11}^1(.) = -P_{11}(\alpha) = -[\alpha^2 + (2.5 + \varphi(.)k_{11}^2(.))\alpha + \varphi(.)k_{11}^1(.)] \\ \gamma_{11}^2(.) = -(2.5 + \varphi(.)k_{11}^2(.) + \alpha) \end{cases}$$

$$\begin{cases} \gamma_{12}^1(\cdot) = -P_{12}(\alpha) = -\left[\alpha^2 + (2 + \varphi(\cdot)k_{12}^2(\cdot))\alpha + \varphi(\cdot)k_{12}^1(\cdot)\right], \\ \gamma_{12}^2(\cdot) = -(2 + \varphi(\cdot)k_{12}^2(\cdot) + \alpha) \\ \gamma_{21}^1(\cdot) = -P_{21}(\alpha) = -\left[\alpha^2 + (1.66 + \varphi(\cdot)k_{21}^2(\cdot))\alpha + \varphi(\cdot)k_{21}^1(\cdot)\right] \\ \gamma_{21}^2(\cdot) = -(1.66 + \varphi(\cdot)k_{21}^2(\cdot) + \alpha) \end{cases}$$

$$\begin{cases} \gamma_{22}^1(\cdot) = -P_{22}(\alpha) = -\left[\alpha^2 + (1.42 + \varphi(\cdot)k_{22}^2(\cdot))\alpha + \varphi(\cdot)k_{22}^1(\cdot)\right] \\ \gamma_{22}^2(\cdot) = -(1.42 + \varphi(\cdot)k_{22}^2(\cdot) + \alpha) \\ \gamma_{31}^1(\cdot) = -P_{31}(\alpha) = -\left[\alpha^2 + (1.11 + \varphi(\cdot)k_{31}^2(\cdot))\alpha + \varphi(\cdot)k_{31}^1(\cdot)\right] \\ \gamma_{31}^2(\cdot) = -(1.11 + \varphi(\cdot)k_{31}^2(\cdot) + \alpha) \end{cases}$$

and:

$$\begin{cases} \gamma_{32}^1(\cdot) = -P_{32}(\alpha) = -\left[\alpha^2 + (0.76 + \varphi(\cdot)k_{32}^2(\cdot))\alpha + \varphi(\cdot)k_{32}^1(\cdot)\right] \\ \gamma_{32}^2(\cdot) = -(0.76 + \varphi(\cdot)k_{32}^2(\cdot) + \alpha) \end{cases}$$

Next, by choosing the two poles $p_1 = -1$ and $p_2 = -3$ are real negative, this implies that $\alpha = -1$ and $\beta = 1$.

According to Theorem 2, for all admissible uncertainties and under any switching law (1), we can deduce the following stabilization conditions:

$$\begin{aligned} \text{i)} \quad & \alpha = p_1 = -1 < 0 \\ \text{ii)} \quad & \left| \mu_{11}(t)\gamma_{11}^1(\cdot) + \mu_{12}(t)\gamma_{12}^1(\cdot) \right| \\ & = \left| \mu_{21}(t)\gamma_{21}^1(\cdot) + \mu_{22}(t)\gamma_{22}^1(\cdot) \right| \\ & = \left| \mu_{31}(t)\gamma_{31}^1(\cdot) + \mu_{32}(t)\gamma_{32}^1(\cdot) \right| = 0 \\ \text{iii)} \quad & p_2 = -3 \\ & = \max \begin{pmatrix} \mu_{11}(t)\gamma_{11}^1(\cdot) + \mu_{12}(t)\gamma_{12}^1(\cdot), \\ \mu_{21}(t)\gamma_{21}^1(\cdot) + \mu_{22}(t)\gamma_{22}^1(\cdot), \\ \mu_{31}(t)\gamma_{31}^1(\cdot) + \mu_{32}(t)\gamma_{32}^1(\cdot) \end{pmatrix} < 0 \end{aligned}$$

for a particular choice $k_{11}^1(\cdot) = k_{12}^1(\cdot)$, $k_{11}^2(\cdot) = k_{12}^2(\cdot)$, $k_{21}^1(\cdot) = k_{22}^1(\cdot)$, $k_{21}^2(\cdot) = k_{22}^2(\cdot)$; $k_{31}^1(\cdot) = k_{32}^1(\cdot)$ and $k_{31}^2(\cdot) = k_{32}^2(\cdot)$ relations (ii) and (iii) allow us to deduce the following stabilization conditions:

$$\begin{cases} k_{11}^1(\cdot)\varphi(\cdot) = k_{12}^1(\cdot)\varphi(\cdot) = 3 \\ k_{11}^2(\cdot)\varphi(\cdot) = k_{12}^2(\cdot)\varphi(\cdot) = 2 - 0.5\rho(\cdot) \end{cases}$$

$$\begin{cases} k_{21}^1(\cdot)\varphi(\cdot) = k_{22}^1(\cdot)\varphi(\cdot) = 3 + 0.03\rho(\cdot) \\ k_{21}^2(\cdot)\varphi(\cdot) = k_{22}^2(\cdot)\varphi(\cdot) = 2.58 - 0.24\rho(\cdot) \end{cases}$$

and:

$$\begin{cases} k_{31}^1(\cdot)\varphi(\cdot) = k_{32}^1(\cdot)\varphi(\cdot) = 2.99 \\ k_{31}^2(\cdot)\varphi(\cdot) = k_{32}^2(\cdot)\varphi(\cdot) = 3.24 - 0.35\rho(\cdot) \end{cases}$$

Now, we assume that the nonlinear gain $\varphi(\cdot)$ is given by the following relationship: $\varphi(\cdot) = \varphi(u) = \frac{0.5u}{1 + 0.4u^2}$; where the nonlinear gain $\varphi(\cdot)$ is represented in the shape of Figure 2.

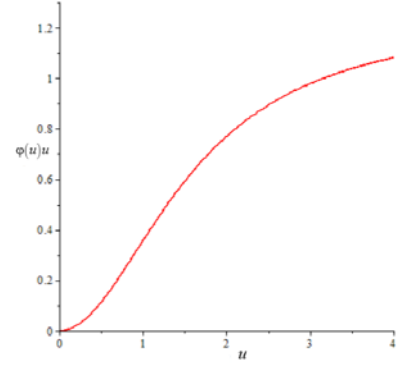


Fig 2. Characteristic of the nonlinear gain $\varphi(\cdot)$

If we choose the uncertain parameters $\rho(\cdot) = 0.4$, the initial state vector $x(t_0=0) = [2 \ 1]^T$ and the switching sequence given in Figure 3 The simulation result of the close-loop system are shown in Figures 4 and 5, respectively which correspond to the evolution of states with respect to time and the control input evolution.

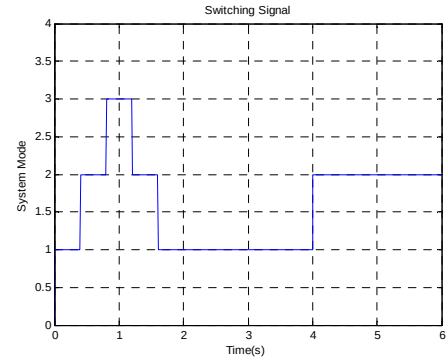


Fig 3. Switching function $\sigma(t)$ between subsystems

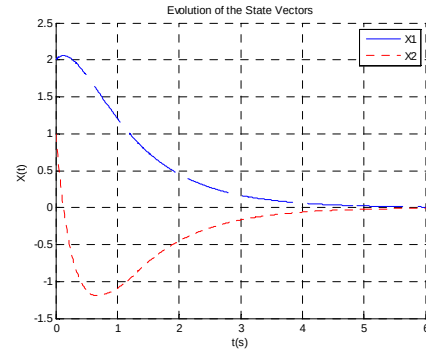


Fig 4. State responses of the close loop system

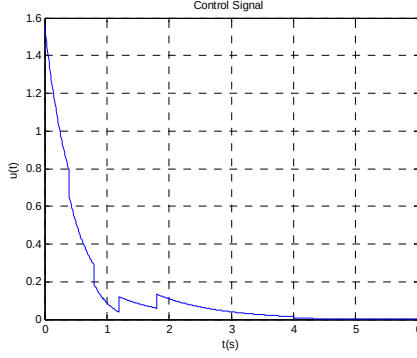


Fig 5. Control signal

Example 2. (Benrejeb *et al.* 2006) This example is deduced by the discretization of the linear part of the continuous-time switched system of example 1 by a zero-order holder with a sampling time $T_e = 0.2s$.

So, the vertex matrices are defined as:

$$A_{d11} = \begin{bmatrix} 0 & 1 \\ -0.6 & 1.6 \end{bmatrix}, A_{d12} = \begin{bmatrix} 0 & 1 \\ -0.67 & 1.67 \end{bmatrix},$$

$$A_{d21} = \begin{bmatrix} 0 & 1 \\ -0.71 & 1.71 \end{bmatrix}, A_{d22} = \begin{bmatrix} 0 & 1 \\ -0.75 & 1.75 \end{bmatrix},$$

$$A_{d31} = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.8 \end{bmatrix} \text{ and } A_{d32} = \begin{bmatrix} 0 & 1 \\ -0.85 & 1.85 \end{bmatrix}.$$

The varying-time uncertainties parameters are:

$$\mu_{11}(k) = \rho(\cdot), \mu_{12}(k) = 1 - \rho(\cdot), \mu_{21}(k) = \rho(\cdot),$$

$$\mu_{22}(k) = 1 - \rho(\cdot), \mu_{31}(k) = \rho(\cdot) \text{ and } \mu_{32}(k) = 1 - \rho(\cdot).$$

So, by considering the state feedback controller given in (51) characterized by the following gains vectors:

$$K_{11}(\cdot) = [k_{11}^1(\cdot) \ k_{11}^2(\cdot)], K_{12}(\cdot) = [k_{12}^1(\cdot) \ k_{12}^2(\cdot)],$$

$$K_{21}(\cdot) = [k_{21}^1(\cdot) \ k_{21}^2(\cdot)], K_{22}(\cdot) = [k_{22}^1(\cdot) \ k_{22}^2(\cdot)],$$

$$K_{31}(\cdot) = [k_{31}^1(\cdot) \ k_{31}^2(\cdot)] \text{ and } K_{32}(\cdot) = [k_{32}^1(\cdot) \ k_{32}^2(\cdot)]$$

Then, the closed-loop system is given by:

$$A_{11d}^c(\cdot) = \begin{bmatrix} 0 & 1 \\ -0.6 - K_{11}^1(\cdot)\varphi(\cdot) & 1.6 - \varphi(\cdot)K_{11}^2(\cdot) \end{bmatrix},$$

$$A_{12d}^c(\cdot) = \begin{bmatrix} 0 & 1 \\ -0.67 - K_{12}^1(\cdot)\varphi(\cdot) & 1.6 - \varphi(\cdot)K_{12}^2(\cdot) \end{bmatrix},$$

$$A_{21d}^c(\cdot) = \begin{bmatrix} 0 & 1 \\ -0.71 - K_{21}^1(\cdot)\varphi(\cdot) & 1.71 - \varphi(\cdot)K_{21}^2(\cdot) \end{bmatrix},$$

$$A_{22d}^c(\cdot) = \begin{bmatrix} 0 & 1 \\ -0.75 - K_{22}^1(\cdot)\varphi(\cdot) & 1.75 - \varphi(\cdot)K_{22}^2(\cdot) \end{bmatrix},$$

$$A_{31d}^c(\cdot) = \begin{bmatrix} 0 & 1 \\ -0.8 - K_{31}^1(\cdot)\varphi(\cdot) & 1.8 - \varphi(\cdot)K_{31}^2(\cdot) \end{bmatrix}$$

and: $A_{32d}^c(\cdot) = \begin{bmatrix} 0 & 1 \\ 0.85 - K_{32}^1(\cdot)\varphi(\cdot) & 1.85 - \varphi(\cdot)K_{32}^2(\cdot) \end{bmatrix}$

The vertex matrices in arrow form are the following:

$$M_{11}(\cdot) = P^{-1}A_{11d}^c(\cdot)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{11}^1(\cdot) & \gamma_{11}^2(\cdot) \end{bmatrix},$$

$$M_{12}(\cdot) = P^{-1}A_{12d}^c(\cdot)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{12}^1(\cdot) & \gamma_{12}^2(\cdot) \end{bmatrix},$$

$$M_{21}(\cdot) = P^{-1}A_{21d}^c(\cdot)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{21}^1(\cdot) & \gamma_{21}^2(\cdot) \end{bmatrix},$$

$$M_{22}(\cdot) = P^{-1}A_{22d}^c(\cdot)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{22}^1(\cdot) & \gamma_{22}^2(\cdot) \end{bmatrix},$$

$$M_{31}(\cdot) = P^{-1}A_{31d}^c(\cdot)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{31}^1(\cdot) & \gamma_{31}^2(\cdot) \end{bmatrix}$$

$$\text{and: } M_{32}(\cdot) = P^{-1}A_{32d}^c(\cdot)P = \begin{bmatrix} \alpha & 1 \\ \gamma_{32}^1(\cdot) & \gamma_{32}^2(\cdot) \end{bmatrix}$$

with:

$$\begin{cases} \gamma_{11}^1(\cdot) = -[\alpha^2 + (-1.6 + \varphi(\cdot)k_{11}^2(\cdot))\alpha + 0.6 + \varphi(\cdot)k_{11}^1(\cdot)] \\ \gamma_{11}^2(\cdot) = -(1.6 + \varphi(\cdot)k_{11}^2(\cdot) + \alpha) \\ \gamma_{12}^1(\cdot) = -[\alpha^2 + (-1.67 + \varphi(\cdot)k_{12}^2(\cdot))\alpha + 0.67 + \varphi(\cdot)k_{12}^1(\cdot)] \\ \gamma_{12}^2(\cdot) = -(-1.67 + \varphi(\cdot)k_{12}^2(\cdot) + \alpha) \\ \gamma_{21}^1(\cdot) = -[\alpha^2 + (-1.77 + \varphi(\cdot)k_{21}^2(\cdot))\alpha + 0.71 + \varphi(\cdot)k_{21}^1(\cdot)] \\ \gamma_{21}^2(\cdot) = -(-1.71 + \varphi(\cdot)k_{21}^2(\cdot) + \alpha) \\ \gamma_{22}^1(\cdot) = -[\alpha^2 + (-1.75 + \varphi(\cdot)k_{22}^2(\cdot))\alpha + 0.75 + \varphi(\cdot)k_{22}^1(\cdot)] \\ \gamma_{22}^2(\cdot) = -(-1.75 + \varphi(\cdot)k_{22}^2(\cdot) + \alpha) \\ \gamma_{31}^1(\cdot) = -[\alpha^2 + (-1.8 + \varphi(\cdot)k_{31}^2(\cdot))\alpha + 0.8 + \varphi(\cdot)k_{31}^1(\cdot)] \\ \gamma_{31}^2(\cdot) = -(-1.8 + \varphi(\cdot)k_{31}^2(\cdot) + \alpha) \end{cases}$$

and:

$$\begin{cases} \gamma_{32}^1(\cdot) = -[\alpha^2 + (-1.85 + \varphi(\cdot)k_{32}^2(\cdot))\alpha + 0.85 + \varphi(\cdot)k_{32}^1(\cdot)] \\ \gamma_{32}^2(\cdot) = -(-1.85 + \varphi(\cdot)k_{32}^2(\cdot) + \alpha) \end{cases}$$

Now, according to theorem 4, by choosing the two poles $z_1 = e^{RT_e} = 0.81$ and $z_2 = e^{P_2T_e} = 0.54$, this implies the following stabilization conditions for all admissible uncertainties and under any switching law (35):

$$\begin{aligned} \text{i)} \quad & |z_1| = 0.81 < 1 \\ \text{ii)} \quad & |\mu_{11}(t)\gamma_{11}^1(\cdot) + \mu_{12}(t)\gamma_{12}^1(\cdot)| \\ & = |\mu_{21}(t)\gamma_{21}^1(\cdot) + \mu_{22}(t)\gamma_{22}^1(\cdot)| \\ & = |\mu_{31}(t)\gamma_{31}^1(\cdot) + \mu_{32}(t)\gamma_{32}^1(\cdot)| = 0 \end{aligned}$$

$$\text{iii)} \quad |z_2| = 0.54$$

$$\max \left\{ \begin{aligned} & \left| \mu_{11}(t) \gamma_{11}^1(\cdot) + \mu_{12}(t) \gamma_{12}^1(\cdot) \right|, \\ & \left| \mu_{21}(t) \gamma_{21}^1(\cdot) + \mu_{22}(t) \gamma_{22}^1(\cdot) \right|, \\ & \left| \mu_{31}(t) \gamma_{31}^1(\cdot) + \mu_{32}(t) \gamma_{32}^1(\cdot) \right| \end{aligned} \right\} < 1$$

For a particular choice $k_{11}^1(\cdot) = k_{12}^1(\cdot)$, $k_{11}^2(\cdot) = k_{12}^2(\cdot)$, $k_{21}^1(\cdot) = k_{22}^1(\cdot)$, $k_{21}^2(\cdot) = k_{22}^2(\cdot)$; $k_{31}^1(\cdot) = k_{32}^1(\cdot)$ and $k_{31}^2(\cdot) = k_{32}^2(\cdot)$ conditions (ii) and (iii) allow us to deduce the following stabilization conditions:

$$\begin{cases} k_{11}^1(\cdot) \varphi(\cdot) = k_{12}^1(\cdot) \varphi(\cdot) = -1.102 + 0.068\rho(\cdot) \\ k_{21}^1(\cdot) \varphi(\cdot) = k_{22}^1(\cdot) \varphi(\cdot) = 1.4 - 0.07\rho(\cdot) \\ k_{31}^1(\cdot) \varphi(\cdot) = k_{32}^1(\cdot) \varphi(\cdot) = -1.1813 + 0.0425\rho(\cdot) \\ k_{21}^2(\cdot) \varphi(\cdot) = k_{22}^2(\cdot) \varphi(\cdot) = 1.48 - 0.04\rho(\cdot) \end{cases},$$

and:

$$\begin{cases} k_{31}^1(\cdot) \varphi(\cdot) = k_{32}^1(\cdot) \varphi(\cdot) = -1.2874 + 0.04\rho(\cdot) \\ k_{31}^2(\cdot) \varphi(\cdot) = k_{32}^2(\cdot) \varphi(\cdot) = 1.58 - 0.05\rho(\cdot) \end{cases}$$

As done for the continuous-time case, the nonlinear gain $\varphi(\cdot)$ is given by the following relationship:

$$\varphi(\cdot) = \varphi(u) = \frac{0.5u}{1 + 0.4u^2} \text{ given in figure 2.}$$

For this example, in case when the uncertain parameters $\rho(\cdot)$ is fixed at 0.4, the initial state vector $x(0) = [2 \ 1]^T$ and by considering the same switching sequence given in Figure 3, the evolution of the states and the control signal are given in Figures 6 and 7, respectively.

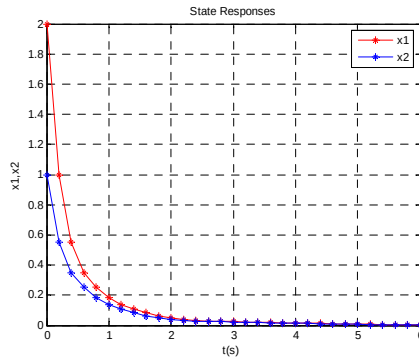


Fig. 6. State evolution of the closed-loop system

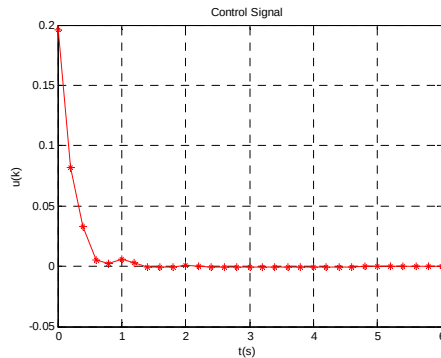


Fig. 7. Control signal of the closed-loop system

These two examples were treated to show the effectiveness of the proposed method. In fact, the controllers designed in example 1 and 2 can guarantee the robust stabilisation of the closed-loop systems, as well to see that the proposed stabilisation conditions are sufficient and closely necessary. By another side, according to the number of the subsystems and the complexity of those subsystems, it is very difficult to find a common Lyapunov function for all the subsystems given in the examples, in order to guarantee stability of the corresponding closed-loop systems under arbitrary switching.

5. CONCLUSIONS

This paper has investigated robust pole placement stabilization with state feedback controller for a class both continuous and discrete-time switched nonlinear systems with time-varying polytopic uncertainties. These conditions were deduced by constructing a common Lyapunov function and applying the Kotelyanski lemma combined with the M -matrix properties.

Compared with the existing results of switched systems, the main advantages of this approach consist in that these obtained conditions are formulated in terms of the time-varying polytopic uncertain parameters and they allow us to avoid searching a common Lyapunov function.

As application, the effectiveness of the theoretical results is illustrated for a DC motor with shunt excitation and polytopic uncertainties models under variable mechanical loads for the both continuous and discrete-time cases.

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Appendix A

Proof of theorem 2. (Borne *et al.*, 2008) Consider the close loop system (26) given in the arrow form, by choosing the poles $p_j = \alpha_j$ for $j=1, \dots, n-1$ are real and negative. Then, the closed loop system is stable if $M_c(\cdot)$ is the opposite of an M -matrix. In such conditions, the principal minors of $M_c(\cdot)$ are alternating sign with the first is negative, it becomes $\bar{t}^j(\cdot) = 0$, $j=1, \dots, n-1$ where: $\bar{t}^j(\cdot)$ are given in (33) and the last condition $j=n$ is:

$$(-1)^n \begin{vmatrix} p_1 & 0 & \cdots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & p_{n-1} & |\beta_{n-1}| \\ 0 & \cdots & \cdots & 0 & \bar{t}^n(\cdot) \end{vmatrix} > 0;$$

That is $p_n = \bar{t}^n(\cdot) < 0$. So, in such condition, the new dynamic is characterized by the distinct poles imposed on the system and the switched system is stable under arbitrary switching and the poles $p_j < 0$ $j=1, \dots, n$.

Appendix B

Proof of theorem 4. (Borne *et al.*, 2008) For system (53) with the control law (51), the dynamic of this system is characterized by the distinct poles imposed on the system by choosing the poles $z_j = \alpha_j$ for $j=1, \dots, n-1$, it permits us to conclude that the system is stable if all the principal minors of $(I_n - M_D)$ are positive, it becomes that $\bar{t}_D^j(\cdot) = 0$ for $j=1, \dots, n-1$ and $\bar{t}_D^n(\cdot) = |z_n|$, under these conditions, we have that switched system (54) is stable under arbitrary switching and $|z_j| < 1$ $j=1, \dots, n$.