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Two Controller Design Procedures Using Closed-Loop Pole Placement Technique

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Abstract: *In this paper, we present two procedures for controller design to an input-output control system using the closed-loop pole placement method, such that the open-loop system is either simple or double integrator. If all the poles of the closed-loop system are chosen real and negative to stabilize the system, then the controller is stable. In addition, the designed controller is also stable for any pole polynomial of the closed-loop system if the pole-zero excess of the plant is at most 4. The both controller design procedures provide a unique solution for the closed-loop pole placement to have a small response time of the closed-loop system, a given overshoot and an imposed value of the magnitude coefficient of the control variable. Notice that the first design procedure generalizes the controller tuning relations from the Kessler variant of the Modulus Criterion.*

Keywords: pole polynomial, semi-proper stable controller, steady-state error, simple/double integrator, magnitude coefficient.

1. INTRODUCTION

The synthesis of controllable and observable systems is frequently achieved in state space domain [2,5,7,8], either by classical design of state feedback pole placement and of state-estimator, or by H_2 optimal design of state feedback (LQ) and of state-estimator (Kalman). In frequency domain, the system synthesis with H_2 or H_∞ optimal control methods is predominantly parametric, since their target is to find the best tuning parameters of a given controller structure [1,6,9,10,11].

This paper treats the control linear system design starting from the idea that pole placement in the complex plane predominantly determines the control system dynamic performances, but the number of zeros and their placement in the complex plane also influence the system performances [3,4]. In our paper, the design of the controller by a suitable placement of the control system poles is based on three premises:

a) The plant is stable, minimum-phase, strictly proper and non-derivative (without null zeros);

b) The open-loop system must be either simple integrator (with one pole in origin) to have zero steady-state error to a step input, or double integrator (with two poles in origin) to have zero steady-state error to a ramp input;

c) The closed-loop system is either of order $n-k$ (in the first case) or $n-k+1$ (in the second case), where $n-k$ is the relative order of the plant transfer function (pole-zero excess).

2. POLE PLACEMENT FOR A SIMPLE INTEGRATOR OPEN-LOOP SYSTEM

Theorem 1. *Consider a stable, minimum-phase, non-derivative and strictly proper plant, described by a rational transfer function*

$$G_P(s)=\frac{r(s)}{p(s)},$$

where $r(s)$ *and* $p(s)$ *are coprime polynomials, and*

$$k=\deg r(s)<\deg p(s)=n.$$

For any Hurwitz polynomial $\mathcal{P}(s)$ *of degree* $n-k$, *there exists a semi-proper controller of at most order* n *and stable in the practical case when all the roots of* $\mathcal{P}(s)$ *are real and negative, such that the closed–loop system has* $\mathcal{P}(s)$ *as pole polynomial, and the open-loop system is of simple integrator type.*

Proof. Without loss of generality, assume that the free term of the Hurwitz polynomial $\mathcal{P}(s)$ is equal to 1; that is,

$$\mathcal{P}(s)=c_{n-k}s^{n-k}+c_{n-k-1}s^{n-k-1}+\cdots+c_1 s+1 \tag{1}$$

where all $c_i>0$.

We will show that the controller with the transfer function

$$G_C(s)=\frac{1}{G_P(s)[\mathcal{P}(s)-1]} \tag{2}$$

satisfies the imposed conditions in Theorem 1. We see that $G_C(s)$ is semi-proper, and has the numerator and denominator of degree n. The open-loop (direct) system is of simple integrator type, because its transfer function has the expression

$$G_d(s)=G_C(s)G_P(s)=\frac{1}{\mathcal{P}(s)-1}=\frac{1}{s\cdot p_1(s)},$$

where

$$p_1(s)=c_{n-k}s^{n-k-1}+c_{n-k-1}s^{n-k-2}+\cdots+c_2s+c_1,$$

with $p_1(0)=c_1>0$.

Moreover, the pole polynomial of the closed-loop system is even $\mathcal{P}(s)$, since

$$G_0(s)=\frac{G_d}{1+G_d}=\frac{1}{\mathcal{P}(s)}. \qquad (3)$$

We can write the controller transfer function (2) as

$$G_C(s)=\frac{p(s)}{s\cdot r(s)\cdot p_1(s)}, \qquad (4)$$

where $r(0)\neq 0$ (since the plant is non-derivative).

If the plant is of proportional type, that is $p(0)\neq 0$ and $r(0)\neq 0$, then the controller is simple integrator, with both the numerator and denominator of degree n.

If the plant is of simple integrator type, then $p(s)$ has a single root equal to 0, and the controller is of proportional type, with both the numerator and denominator of degree $n-1$.

If all the roots of $\mathcal{P}(s)$ are chosen real and negative, we can write it as

$$\mathcal{P}(s)=(T_1s+1)(T_2s+1)\cdots(T_{n-k}s+1),$$

where all time constants T_i are positive. From $sp_1(s)=\mathcal{P}(s)-1$, it follows that all the roots s_i of the polynomial $p_1(s)$ are roots of the equation $\mathcal{P}(s)=1$. Since the plant transfer function $G_P(s)$ is minimum-phase and non-derivative, $r(s)$ is a Hurwitz polynomial. Thus, in order to show that the controller (4) is stable we only need to prove that the real parts of the roots of the equation $\mathcal{P}(s)=1$, except the zero root, are negative. Let $s=x+\mathrm{j}y\neq 0$ be a root of the equation $\mathcal{P}(s)=1$. From $\mathcal{P}(x+\mathrm{j}y)=1$, we get

$$\prod_{i=1}^{n-k}[(T_ix+1)^2+T_i^2y^2]=1.$$

Clearly, this equality doesn't hold for $x\geq 0$, since

$$(T_ix+1)^2+T_i^2y^2>1$$

for all indices i. Therefore, the real part x of the root $s=x+\mathrm{j}y\neq 0$ is negative.

Remark 1. For any Hurwitz polynomial $\mathcal{P}(s)$, the controller (2) is *stable* if $n-k\leq 4$. Since $r(s)$ is a Hurwitz polynomial, from (4) it follows that the controller is stable if $p_1(s)$ is also a Hurwitz polynomial. For $n-k\leq 3$, $p_1(s)$ is a Hurwitz polynomial of degree at most 2, since all its coefficients are positive. Also, for $n-k=4$, the polynomial

$$p_1(s)=c_4s^3+c_3s^2+c_2s+c_1$$

is Hurwitzian because all its coefficients are positive and, in addition, $D_2=c_2c_3-c_1c_4>0$. The condition $D_2>0$ is true since

$$\mathcal{P}(s)=c_4s^4+c_3s^3+c_2s^2+c_1s+1$$

is Hurwitzian by hypothesis.

Remark 2. If we choose the pole polynomial $\mathcal{P}(s)$ of the closed-loop system such that all its roots are real and negative, then the step response of the closed-loop system is finite and monotonic. Theoretically, by a suitable choice of the pole polynomial $\mathcal{P}(s)$, the step response of the closed-loop system can be as rapid as desired. However, this is not possible in practice due to the plant model uncertainty, which imposes restrictions on the magnitude of the control variable. For a control system with proportional plant, $\mathcal{P}(s)$ must be chosen to have a step response time at most 2...5 times less than the step response time of the plant. Such a practical approach of the controller design reduces also the magnitude of the controller output $c(t)$ to a step setpoint or disturbance.

For a control system with *proportional plant* and semi-proper controller, we define the *magnitude coefficient* of the control variable c as the ratio between the initial value and the final value of this variable to a step setpoint:

$$M=\frac{c(0_+)}{c(\infty)}. \qquad (5)$$

In practical applications, the magnitude coefficient M must be bounded to a value less than 20 to reduce the wear and tear of the plant, and also the fuel and energy consumption. For a control system with proportional plant, the magnitude coefficient is given by

$$M = \frac{G_P(0)}{\lim\limits_{s\to\infty} \mathcal{P}(s) G_P(s)} . \tag{6}$$

Indeed, from the expression of the transfer function between the setpoint and the control output,

$$G_1(s) = \frac{G_0(s)}{G_P(s)} = \frac{1}{\mathcal{P}(s) G_P(s)} ,$$

using the initial and final value theorems, we get

$$c(0+) = \lim_{s\to\infty} G_1(s) = \frac{1}{\lim\limits_{s\to\infty} \mathcal{P}(s) G_P(s)} ,$$

and

$$c(\infty) = \lim_{s\to 0} G_1(s) = \frac{1}{\mathcal{P}(0) G_P(0)} = \frac{1}{G_P(0)} ,$$

respectively. For

$$G_P(s) = \frac{r(s)}{p(s)} = \frac{r_k s^k + r_{k-1} s^{k-1} + \cdots + r_0}{p_n s^n + r_{n-1} s^{n-1} + \cdots + p_0} ,$$

from (6) we get

$$M = \frac{r_0 p_n}{p_0 r_k c_{n-k}} .$$

Writing the pole polynomial $\mathcal{P}(s)$ of the closed-loop system as

$$\mathcal{P}(s) = (T_1 s + 1)(T_2 s + 1) \cdots (T_{n-k} s + 1) ,$$

where all time constants T_i are positive, then $c_{n-k} = T_1 T_2 \cdots T_{n-k}$, and hence

$$T_1 T_2 \cdots T_{n-k} = \frac{r_0 p_n}{M p_0 r_k} . \tag{7}$$

For a given value of the magnitude coefficient M , from the arithmetic mean-geometric mean inequality

$$T_1 + T_2 + \cdots + T_{n-k} \ge (n-k) \sqrt[n-k]{T_1 T_2 \cdots T_{n-k}} ,$$

it follows that the sum $T = T_1 + T_2 + \cdots + T_{n-k}$ is minimal when

$$T_1 = T_2 = \cdots = T_{n-k} = \sqrt[n-k]{\frac{r_0 p_n}{M p_0 r_k}} . \tag{8}$$

Since $Ts+1$ is the best first order approximation of $\mathcal{P}(s)$ by Padé approximation, we propose to chose the time constants T_i according to (8), to provide the least value of the equivalent time constant T of the closed-loop system and, in this way, a small closed-loop response time to a step setpoint.

For a control system with *simple integrator plant* and semi-proper controller, the *magnitude coefficient* of the control variable c is defined as the ratio between the initial value of the control variable c to a step setpoint and the setpoint magnitude (under the assumption that the setpoint and the control variable are expressed in percent). Thus, for a unit step setpoint,

$$M = c(0+) = \frac{1}{\lim\limits_{s\to\infty} \mathcal{P}(s) G_P(s)} . \tag{9}$$

In practice, the magnitude coefficient M must be bounded to a value less than 10. For

$$G_P(s) = \frac{r(s)}{p(s)} = \frac{r_k s^k + r_{k-1} s^{k-1} + \cdots + r_0}{p_n s^n + r_{n-1} s^{n-1} + \cdots + p_1 s} ,$$

from (6) we get

$$M = \frac{p_n}{r_k c_{n-k}} ,$$

or

$$T_1 T_2 \cdots T_{n-k} = \frac{p_n}{M r_k} . \tag{10}$$

Therefore, for the same reasons as above, we propose to chose the time constants T_i such that

$$T_1 = T_2 = \cdots = T_{n-k} = \sqrt[n-k]{\frac{p_n}{M r_k}} , \tag{11}$$

to provide a small closed-loop response time to a step setpoint for. To illustrate this proposed design procedure we present two applications for a proportional and integrator plant, respectively.

Example 1. Let us consider the *proportional plant* transfer function

$$G_P(s) = \frac{2(2s+1)}{(3s+1)^2 (6s+1)(8s+1)} .$$

For $M = 1$, from (8) , (2) and (3) we get

$$T_1 = T_2 = T_3 = 6 ,$$

$$G_C(s) = \frac{(3s+1)^2 (6s+1)(8s+1)}{36 s (2s+1)(12 s^2 + 6s + 1)} ,$$

$$G_0(s) = \frac{1}{(6s+1)^3} ,$$

while for $M=8$, we get $T_1=T_2=T_3=3$ and

$$G_C(s)=\frac{(3s+1)^2(6s+1)(8s+1)}{18s(2s+1)(3s^2+3s+1)},$$

$$G_0(s)=\frac{1}{(3s+1)^3}.$$

In figures 1 and 2 are shown the unit step response of the proportional plant and the closed-loop responses $c(t)$ and $y(t)$ to unit step setpoint for $M=1$ and $M=8$ (c - control variable, y - controlled variable).

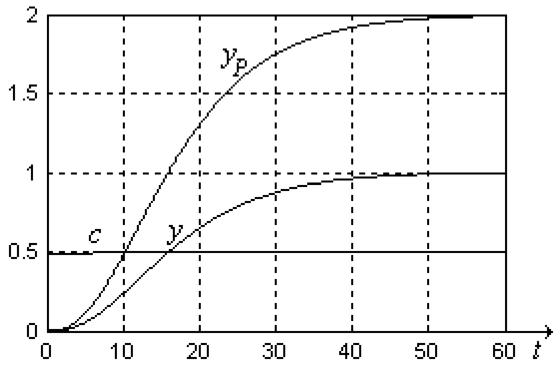

Fig. 1. Unit step responses of the plant (y_P) and closed-loop system (c and y), for $M=1$.

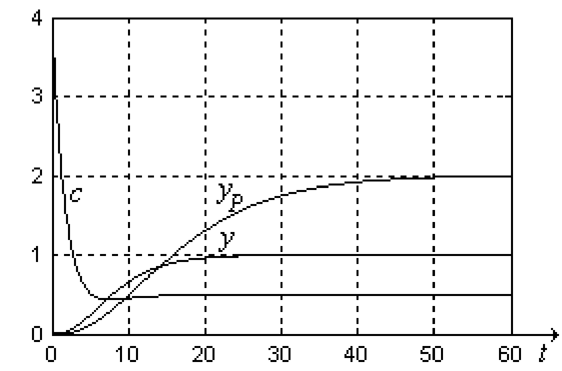

Fig. 2. Unit step responses of the plant (y_P) and closed-loop system (c and y), for $M=8$.

Example 2. Let us consider the *integrator plant* transfer function

$$G_P(s)=\frac{s+1}{s(2s+1)(4s+1)(8s+1)}.$$

For $M=1$, from (11), (2) and (3) we get

$T_1=T_2=T_3=4$,

$$G_C(s)=\frac{(2s+1)(4s+1)(8s+1)}{4(s+1)(16s^2+12s+3)}$$

$$G_0(s)=\frac{1}{(4s+1)^3},$$

while for $M=8$, we get

$T_1=T_2=T_3=2$,

$$G_C(s)=\frac{(2s+1)(4s+1)(8s+1)}{2(s+1)(4s^2+6s+3)},$$

$$G_0(s)=\frac{1}{(2s+1)^3}.$$

In figures 3 and 4 are shown the closed-loop responses $c(t)$ and $y(t)$ to unit step setpoint for $M=1$ and $M=8$, respectively (c - control variable, y - controlled variable).

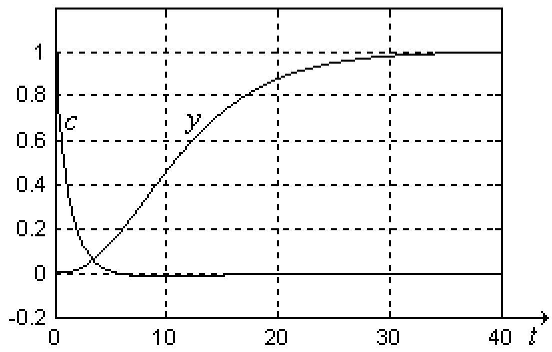

Fig. 3. Unit step responses of the closed-loop system with integrator plant, for $M=1$.

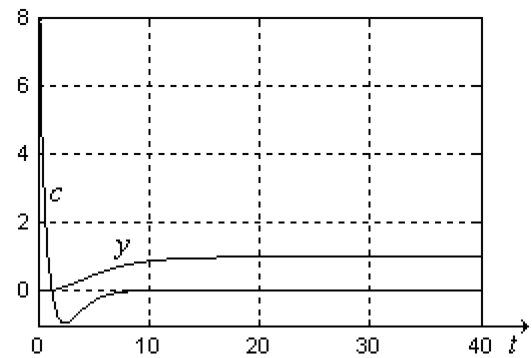

Fig. 4. Unit step responses of the closed-loop system with integrator plant, for $M=8$.

Remark 3. According to (4), if the polynomials $p(s)$ and $sp_1(s)$ have common roots, then the controller transfer function $G_C(s)$ is of order less than n. To illustrate this possibility, let us consider a plant with the transfer function

$$G_P(s)=\frac{K_P}{(T_1s+1)(T_2s+1)(T_\Sigma s+1)}, \tag{12}$$

where T_1 and T_2 are main time constants, and T_Σ is the sum of the parasitic time constants ($T_\Sigma << T_2 \le T_1$). Having in view that $sp_1(s) = P(s) - 1$, if the pole polynomial $\mathcal{P}(s)$ of the closed-loop system satisfies the condition

$$\mathcal{P}(\frac{-1}{T_\Sigma}) = 1, \tag{13}$$

then the controller order is less than $n = 3$, since both the numerator and denominator of the controller transfer function have the same factor $T_\Sigma s + 1$. So, if we choose the pole polynomial as

$$\mathcal{P}(s) = (\tau_1 s + 1)^2(\tau_2 s + 1), \tag{14}$$

with

$$\tau_1 = (x+1)T_\Sigma, \quad \tau_2 = (1 - \frac{1}{x^2})T_\Sigma, \quad x \ge 1,$$

in order to have $\mathcal{P}(\frac{-1}{T_\Sigma}) = 1$, then

$$\mathcal{P}(s) - 1 = \tau_1^2\tau_2 s^3 + \tau_1(\tau_1 + 2\tau_2)s^2 + (2\tau_1 + \tau_2)s =$$

$$= \frac{(x+1)^2}{x^2} T_\Sigma s(T_\Sigma s + 1)[(x^2 - 1)T_\Sigma s + 2x - 1]$$

and

$$G_C(s) = \frac{1}{G_P(s)[\mathcal{P}(s) - 1]} =$$

$$= \frac{K_C(T_1 s + 1)(T_2 s + 1)}{T_\Sigma s(aT_\Sigma s + 1)}, \tag{15}$$

where

$$K_C = \frac{x^2}{(x+1)^2(2x-1)K_P}, \quad a = \frac{x^2 - 1}{2x - 1}.$$

We see that the controller order is equal to 2. For $x = 1$, we get the improper form of the PID control algorithm,

$$G_C(s) = \frac{(T_1 s + 1)(T_2 s + 1)}{4K_P T_\Sigma s},$$

whereas, for $x = 2$, we get the proper form of the PID control algorithm,

$$G_C(s) = \frac{4(T_1 s + 1)(T_2 s + 1)}{27K_P T_\Sigma s(T_\Sigma s + 1)}.$$

The control system has the transfer function

$$G_0(s) = \frac{1}{\mathcal{P}(s)} = \frac{1}{(\tau_1 s + 1)^2(\tau_2 s + 1)},$$

and the magnitude coefficient of the control variable

$$M = \frac{G_P(0)}{\lim\limits_{s\to\infty} \mathcal{P}(s)G_P(s)} = \frac{x^2}{(x+1)^3(x-1)} \cdot \frac{T_1 T_2}{T_\Sigma^2}.$$

Note that for a given value of x, the system response does not depend on the dominant time constant T_1 and T_2. If the coefficient $x \in (1, \infty)$ is too small, then the magnitude coefficient M becomes too large.

In the case $T_\Sigma = 1$, the control system responses $y(t)$ to unit step setpoint are shown in figure 5 for different values of x.

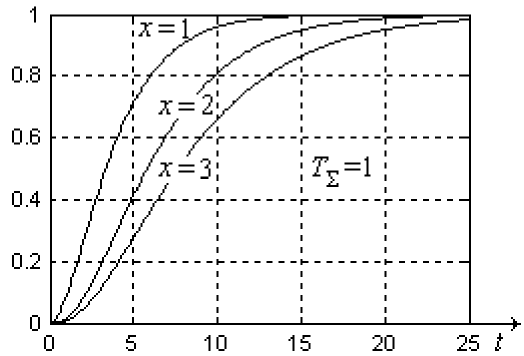

Fig. 5. Unit step responses of the closed-loop system with the plant (12) and controller (15).

For the same plant (12), let us choose now the closed-loop pole polynomial in the form

$$\mathcal{P}(s) = (2T_\Sigma^2 s^2 + 2T_\Sigma s + 1)(xT_\Sigma s + 1), \; x \ge 0. \tag{16}$$

Since

$$\mathcal{P}(s) - 1 = T_\Sigma s[2xT_\Sigma^2 s^2 + 2(1+x)T_\Sigma s + 2 + x],$$

we get

$$G_C(s) = \frac{1}{G_P(s)[\mathcal{P}(s) - 1]} =$$

$$= \frac{(T_1 s + 1)(T_2 s + 1)(T_\Sigma s + 1)}{K_P T_\Sigma s[2xT_\Sigma^2 s^2 + 2(1+x)T_\Sigma s + 2 + x]} \tag{17}$$

and

$$G_0(s) = \frac{1}{\mathcal{P}(s)} = \frac{1}{(2T_\Sigma^2 s^2 + 2T_\Sigma s + 1)(xT_\Sigma s + 1)}.$$

The magnitude coefficient of the control variable is

$$M = \frac{G_P(0)}{\lim\limits_{s\to\infty} \mathcal{P}(s)G_P(s)} = \frac{1}{2x} \cdot \frac{T_1 T_2}{T_\Sigma^2}.$$

For $x = 0$, the controller is of PID type, with the improper transfer function

$$G_C(s)=\frac{(T_1s+1)(T_2s+1)}{2K_PT_\Sigma s},$$

and the closed-loop system has the transfer function

$$G_0(s)=\frac{1}{2T_\Sigma^2s^2+2T_\Sigma s+1}.$$

Thus, we have obtained the controller tuning relations established by Kessler variant of the Modulus Criterion.

The control system responses $y(t)$ to unit step setpoint are presented in figure 6 for different values of x.

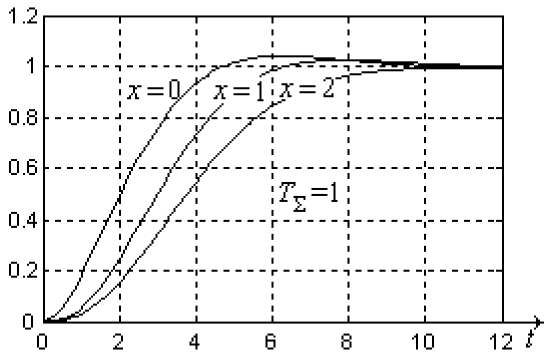

Fig. 6. Unit step responses of the closed-loop system with the plant (12) and controller (17).

3. POLE PLACEMENT FOR A DOUBLE INTEGRATOR OPEN-LOOP SYSTEM

Theorem 2. *Consider a stable, minimum-phase, non-derivative and strictly proper plant, described by a rational transfer function*

$$G_P(s)=\frac{r(s)}{p(s)},$$

where $r(s)$ and $p(s)$ are coprime polynomials, and

$k=\deg r(s)<\deg p(s)=n$.

For any Hurwitz polynomial $\mathcal{P}(s)$ of degree $n-k+1$, there exists a semi-proper controller of at most order $n+1$, such that the closed–loop system has $\mathcal{P}(s)$ as pole polynomial, and the open-loop system is of double integrator type.

Proof. As in the proof of Theorem 1, without loss of generality assume that the free term of the Hurwitz polynomial $\mathcal{P}(s)$ is equal to 1; that is,

$$\mathcal{P}(s)=c_{n-k+1}s^{n-k+1}+c_{n-k}s^{n-k}+\cdots+c_1s+1, \tag{18}$$

where all $c_i>0$.

We will show that the controller

$$G_C(s)=\frac{c_1s+1}{G_P(s)[\mathcal{P}(s)-c_1s-1]} \tag{19}$$

satisfies the imposed conditions in Theorem 2. We see that $G_C(s)$ is semi-proper, and has the numerator and denominator of degree $n+1$. The open-loop system is of double integrator type, because its transfer function has the expression

$$G_d(s)=G_C(s)G_P(s)=\frac{c_1s+1}{\mathcal{P}(s)-c_1s-1}=\frac{c_1s+1}{s^2p_2(s)},$$

where

$$p_2(s)=c_{n-k+1}s^{n-k-1}+c_{n-k}s^{n-k-2}+\cdots+c_3s+c_2,$$

with $p_2(0)=c_2>0$.

The pole polynomial of the closed-loop system is even $\mathcal{P}(s)$, since

$$G_0(s)=\frac{G_d}{1+G_d}=\frac{c_1s+1}{\mathcal{P}(s)}. \tag{20}$$

We can write the controller transfer function (19) as

$$G_C(s)=\frac{(c_1s+1)p(s)}{s^2\,r(s)\,p_2(s)}, \tag{21}$$

where $r(0)\neq 0$ (since the plant is non-derivative).

If the plant is of proportional type, that is $p(0)\neq 0$ and $r(0)\neq 0$, then the controller is double integrator, with both the numerator and denominator of degree $n+1$.

If the plant is of simple integrator type, then $p(s)$ has a single zero root, and the controller is simple integrator, with both the numerator and denominator of degree n.

We conjecture that the designed controller is *stable* in the practical case where all the roots of $P(s)$ are chosen real and negative.

Remark 4. For any Hurwitz polynomial $\mathcal{P}(s)$, the controller (19) is *stable* if $n-k\le 4$. Since the plant transfer function $G_P(s)$ is minimum-phase and non-derivative, $r(s)$ is a Hurwitz polynomial, and from (21) it follows that the controller is stable if $p_2(s)$ is also a Hurwitz polynomial. For $n-k\le 3$, $p_2(s)$ is a polynomial of degree at most 2, which is Hurwitzian because all its coefficients are positive. Also, for $n-k=4$, the polynomial

$$p_2(s)=c_5s^3+c_4s^2+c_3s+c_2$$

is Hurwitzian because all its coefficients are positive and $D_2=c_3c_4-c_2c_5>0$. The condition $D_2>0$ is true since

$$\mathcal{P}(s)=c_5s^5+c_4s^4+c_3s^3+c_2s^2+c_1s+1$$

is Hurwitzian by hypothesis.

Remark 5. In the practical case

$$\mathcal{P}(s)=(T_1s+1)(T_2s+1)\cdots(T_{n-k+1}s+1)\,,$$

with $T_1, T_2, \cdots, T_{n-k+1}>0$, from (17) we get the closed-loop system transfer function

$$G_0(s)=\frac{(T_1+T_2+\cdots T_{n-k+1})s+1}{(T_1s+1)(T_2s+1)\cdots(T_{n-k+1}s+1)}\,. \tag{22}$$

As shown in [3], the step response of the closed-loop system is non-monotonic, but the overshoot and the response time are small if we chose a dominant time constant significantly larger than the other time constants. Therefore, we propose to chose the time constants T_i such that

$$T_1>>T_2=T_3=\cdots=T_{n-k+1}\,. \tag{23}$$

Under this assumption, from (22) we get

$$G_0(s)=\frac{(j+n-k)T_2s+1}{(jT_2s+1)(T_2s+1)^{n-k}}\,, \tag{24}$$

where $j=T_1/T_2$. The overshoot σ of the closed-loop system with the transfer function (26) depends on j and $n-k$. For a given plant with the relative order $n-k$, using a MATLAB program, we can show that the overshoot $\sigma_{n-k}(j)$ of the closed-loop system is a strictly decreasing function; moreover,

$$\lim_{j\to\infty}\sigma_{n-k}(j)=0\,.$$

For a closed-loop system with *proportional plant* and semi-proper controller, the *magnitude coefficient* of the control variable is given by

$$M=\frac{c(0+)}{c(\infty)}=G_P(0)\lim_{s\to\infty}\frac{c_1s+1}{\mathcal{P}(s)G_P(s)}\,. \tag{25}$$

If

$$G_P(s)=\frac{r(s)}{p(s)}=\frac{r_ks^k+r_{k-1}s^{k-1}+\cdots+r_0}{p_ns^n+r_{n-1}s^{n-1}+\cdots+p_0}\,,$$

then

$$M=\frac{r_0p_nc_1}{p_0r_kc_{n-k+1}}\,,$$

or

$$M=\frac{r_0p_n}{p_0r_k}\cdot\frac{T_1+T_2+\cdots+T_{n-k+1}}{T_1T_2\cdots T_{n-k+1}}\,. \tag{26}$$

For the proposed choice (23), we get

$$M=\frac{(j+n-k)r_0p_n}{jp_0r_kT_2{}^{n-k}}\,, \tag{27}$$

where $j=T_1/T_2$.

Thus, the controller design procedure is as follows:

- determine $j=T_1/T_2$ to have an overshoot less than or equal to the imposed one;
- determine T_2 from (27) to have the imposed value of the magnitude coefficient M;
- determine $T_1=jT_2$, $\mathcal{P}(s)=(T_1s+1)(T_2s+1)^{n-k}$ and then $G_C(s)$ with (19), where $c_1=T_1+(n-k)T_2$.

For a control system with *simple integrator plant* and semi-proper controller, the *magnitude coefficient* of the control variable is

$$M=c(0+)=\frac{1}{\lim\limits_{s\to\infty}\mathcal{P}(s)G_P(s)}\,.$$

For

$$G_P(s)=\frac{r(s)}{p(s)}=\frac{r_ks^k+r_{k-1}s^{k-1}+\cdots+r_0}{p_ns^n+r_{n-1}s^{n-1}+\cdots+p_1s}\,,$$

we get

$$M=\frac{p_nc_1}{r_kc_{n-k+1}}\,,$$

or

$$M=\frac{p_n}{r_k}\cdot\frac{T_1+T_2+\cdots+T_{n-k+1}}{T_1T_2\cdots T_{n-k+1}}\,. \tag{28}$$

For the proposed choice (23), we get

$$M=\frac{(j+n-k)p_n}{jr_kT_2{}^{n-k}}\,, \tag{29}$$

where $j=T_1/T_2$.

Thus, the controller design procedure is as follows:

- determine $j=T_1/T_2$ to have an overshoot less than or equal to the imposed one;
- determine T_2 from (29) to have the imposed magnitude coefficient M;
- determine $T_1=jT_2$, $\mathcal{P}(s)=(T_1s+1)(T_2s+1)^{n-k}$ and then $G_C(s)$ with (19), where $c_1=T_1+(n-k)T_2$.

To illustrate this proposed design procedures we present two applications for a proportional and integrator plant, respectively.

Example 3. Let us consider the *proportional plant* transfer function with $n-k=2$:

$$G_P(s)=\frac{2(3s+1)}{(2s+1)(6s+1)(8s+1)}.$$

We propose to design a controller such that $\sigma \le 5.5$ % and $M=\frac{60}{7}$, where σ is the overshoot of the closed-loop system, and M is the magnitude coefficient of the control variable to a step setpoint.

By (24), we have

$$G_0(s)=\frac{(j+2)T_2s+1}{(jT_2s+1)(T_2s+1)^2}.$$

For $j=28$, using a MATLAB program we find $\sigma=5.46$ %. Using this value of j, from (27) we get $T_2=2$. Next,

$$G_0(s)=\frac{60s+1}{(56s+1)(2s+1)^2},$$

$$T_1=jT_2=56,$$

$$\mathcal{P}(s)=(56s+1)(2s+1)^2,$$

$$c_1=T_1+2T_2=60,$$

$$G_C(s)=\frac{(2s+1)(6s+1)(8s+1)(60s+1)}{8s^2(3s+1)(56s+57)}.$$

In figure 7 are shown the closed-loop responses $c(t)$ and $y(t)$ to unit step setpoint (c - control variable, y - controlled variable).

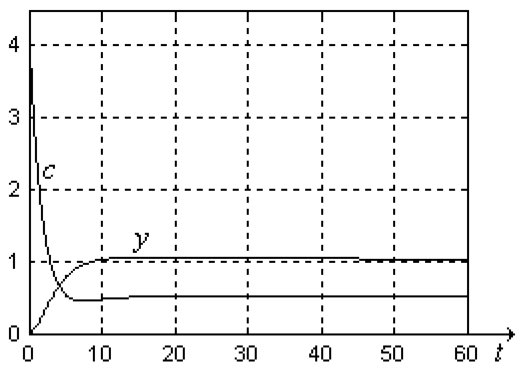

Fig. 7. Unit step responses of the closed-loop system with proportional plant ($\sigma=5.46$ %, $M=60/7$).

Example 4. Let us consider the *integrator plant* transfer function with $n-k=2$:

$$G_P(s)=\frac{3s+1}{s(6s+1)(8s+1)}.$$

We propose to design a controller such that $\sigma \le 5.5$ % and $M=\frac{30}{7}$, where σ is the overshoot of the closed-loop system, and M is the magnitude coefficient of the control variable to a step setpoint.

As shown at the example 3, we have

$$G_0(s)=\frac{(j+2)T_2s+1}{(jT_2s+1)(T_2s+1)^2},$$

with $\sigma=5.46$ % for $j=28$. Using this value of j, from (29) we get $T_2=2$. Next,

$$G_0(s)=\frac{60s+1}{(56s+1)(2s+1)^2},$$

$$T_1=jT_2=56,$$

$$\mathcal{P}(s)=(56s+1)(2s+1)^2,$$

$$c_1=T_1+2T_2=60,$$

$$G_C(s)=\frac{(6s+1)(8s+1)(60s+1)}{4s(3s+1)(56s+57)}.$$

In figure 8 are shown the closed-loop responses $c(t)$ and $y(t)$ to unit step setpoint (c - control variable, y - controlled variable).

Remark 6. If the polynomials $(c_1s+1)p(s)$ and $s^2r(s)p_2(s)$ have common roots, then the controller transfer function (21) has the order less than $n+1$.

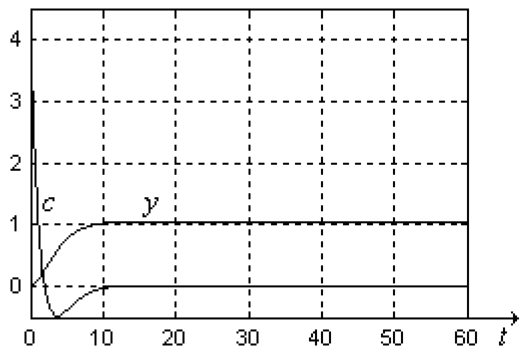

Fig. 8. Unit step responses of the closed-loop system with integrator plant ($\sigma=5.46$ %, $M=30/7$).

Remark 7. Using the proposed procedures we can design a two degree of freedom controller such that the control magnitude coefficient has a great value related to a step disturbance additive to the controlled variable and a smaller value related to a step setpoint change. We can do this by means of a suitable first-order filter for the setpoint signal.

4. CONCLUSIONS

Two theorems concerning the controller design using the closed-loop pole placement method have been proposed.

Using the pole placement approach in accordance to Theorem 1, the open-loop system is simple integrator, the controller is semi-proper and has the order less than or equal to the plant order, while the order of the closed-loop system is equal to the relative order (pole-zero excess) of the plant. In addition, the controller is stable in the practical case where all the poles of the closed-loop system are chosen real and negative to stabilize the system. For both the proportional and integrator plants, the controller design procedure provides a unique solution for the closed-loop pole placement (with equal poles) to have a small response time of the closed-loop system and a given value of the control magnitude coefficient.

Using the pole placement approach in accordance to Theorem 2, the open-loop system is of double integrator type, the controller is semi-proper and has the order less than or equal to the plant order plus 1, while the order of the closed-loop system is equal to the plant relative order plus 1. The designed controller is stable if the relative order of the plant is at most 4, but it seems that this property holds for any relative order of the plant if all the poles of the closed-loop system are chosen real and negative. For both the proportional and integrator plants, the controller design procedure provides a unique solution for the closed-loop pole placement (where all but one of the poles are equal) to have a given overshoot of the closed-loop system response and an imposed value of the magnitude coefficient of the control variable.

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