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Three Lectures on Neutral Functional Differential Equations ⋆

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Abstract: The main idea of this cycle is that mixed initial boundary value problems for partial differential equations of hyperbolic type in two dimensions modeling lossless propagation are a valuable source of functional differential equations, in particular of neutral type. Starting from the simplest examples there are discussed such topics as basic theory (including various explanations for "what could actually define a neutral equation"), stability and forced oscillations. Rather than giving strictly rigorous proofs, the good motivations and final results are given priority. It is author's strong belief that well formulated applied problems are able to supply interesting, appealing while not always easy to solve problems.

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1. LECTURE ONE. BASICS. WHAT IS A FUNCTIONAL DIFFERENTIAL EQUATION OF NEUTRAL TYPE

1.1 Some examples

There are several ways of introducing Functional Differential Equation of Neutral type both from the point of view of a mathematician and of an applied scientist. The following exposition is a personal point of view that starts from a quite well known pioneering reference (Pinney (1958)) where it is stated that the first reference about a differential equation with deviating argument goes back to the XVIIIth century and is due to Johann Bernoulli (Bernoulli (1728)). As the (Latin) title of the paper shows, he considered a weighted stretched vibrating cord with distributed masses on it. He finally was lead to the equation

$$\dot{y} = y(t-1)$$

This *is not* a Functional Differential Equation of Neutral type but it is said that there were several mistakes in deducing the equation. For our goal it is important to note that this equation arose from a problem of partial differential equations moreover, of hyperbolic type. The fact that functional equations may be associated to various boundary value problems for PDE became an elementary truth in the XXth century or even earlier.

A. If we go closer to our time, we shall find, first of all, the very interesting but almost forgotten papers of I. P. Kabakov and A. A. Sokolov, all three published in 1946 but worked and written probably several years earlier (Kabakov (1946); Kabakov and Sokolov (1946); Sokolov (1946)). They deal with propagation phenomena associated with combined heat electricity generation where lossless steam pipes are involved. One of the models considered by these authors was as follows (the model is reproduced after the more recent paper (Halanay and Răsvan (1997))

$$\begin{cases} T_c \dfrac{\partial \xi_p}{\partial t} + \dfrac{\partial \xi_w}{\partial \lambda} = 0 \\ \psi_c^2 T_c \dfrac{\partial \xi_w}{\partial t} + \dfrac{\partial \xi_p}{\partial \lambda} = 0, \quad t \geq 0, 0 \leq \lambda \leq 1, \\ \xi_w(0,t) + \alpha_p \xi_p(0,t) = -\delta\zeta(t) \\ \xi_w(1,t) - \psi_s \xi_p(1,t) = 0 \\ \psi_m^2 \zeta''(t) + \psi_D \zeta'(t) + \zeta(t) = \xi_p(0,t) \end{cases} \tag{1}$$

and if we follow their approach we may apply (in a formal way) the Laplace transform to find the characteristic equation of (1)

$$\begin{aligned} &(\psi_m^2 s^2 + \psi_D s + 1)\left(1 - \frac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c} \cdot \frac{1-\psi_s\psi_c}{1+\psi_s\psi_c} e^{-2\psi_c T_c s}\right) + \\ &+ \frac{\delta\psi_c}{1+\alpha_p\psi_c}\left(1 + \frac{1-\psi_s\psi_c}{1+\psi_s\psi_c} e^{-2\psi_c T_c s}\right) = 0. \end{aligned} \tag{2}$$

Is is obvious that the RHS of (3) is a *quasi-polynomial* with a principal term which corresponds to the neutral case (see, e.g. Bellman and Cooke (1963); Hale and Lunel (1993)). Even this formal approach is able to indicate some connection between the mixed problem (1) and, let us say, the neutral functional differential equation (NFDE):

$$\begin{aligned} &\psi_m^2 y''(t) + \psi_D y'(t) + \left(1 + \frac{\delta\psi_c}{1+\alpha_p\psi_c}\right) y(t) - \\ &- \frac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c} \cdot \frac{1-\psi_s\psi_c}{1+\psi_s\psi_c} \left[\psi_m^2 y''(t-2\psi_c T_c) + \right. \\ &\left. + \psi_D y'(t-2\psi_c T_c) + y(t-2\psi_c T_c)\right] + \\ &+ \frac{\delta\psi_c}{1+\alpha_p\psi_c} \cdot \frac{1-\psi_s\psi_c}{1+\psi_s\psi_c} y(t-2\psi_c T_c) = 0, \end{aligned} \tag{3}$$

which is the simplest and the most straight forward time-domain "realization" of (2).

Similar models may be observed in hydraulic engineering - for the case of hydraulic plants incorporating galleries (water pipes) and in electrical and electronic engineering, where lossless LC transmission lines are included. We mention here

⋆ Presented at the Department of Automatic Control of the Research Institute for Advanced Studies CINVESTAV Mexico City NOVEMBER 2006 the classical (now) application presented by Brayton and Miranker (Brayton and Miranker (1964)) - a tunnel diode circuit incorporating a lossless transmission line

$$\begin{cases} \dfrac{\partial u}{\partial \lambda} + L\dfrac{\partial i}{\partial t} = 0 \\ \dfrac{\partial i}{\partial \lambda} + C\dfrac{\partial u}{\partial t} = 0, \quad t \geq 0, \quad 0 \leq \lambda \leq 1 \\ E(t) = R_0 i(0,t) + u(0,t), \quad u(1,t) = v(t) \\ C_0\dfrac{dv(t)}{dt} + \psi(v) = i(1,t). \end{cases} \tag{4}$$

We may see that this model is of the same type as the one of (1): a mixed initial boundary value problem with the boundary conditions of Dirichlet type connected in feedback with a system of ODE.

B. Another source of such models is nuclear engineering where the model of the so-called *circulating fuel nuclear reactor* is described again by a mixed initial boundary value problem for a system of first order partial differential equations, the problem having this time periodic boundary conditions. We give below the equations for the sake of completeness

$$\begin{cases} \dfrac{d}{dt}n(t) = (\rho - \beta)n(t) + \displaystyle\sum_{i=1}^{m} \beta_i \bar{c}_i(t), \quad \beta = \sum_{i=1}^{m} \beta_i, \\ \bar{c}_i(t) = \displaystyle\int_0^h \phi(\eta) c_i(\eta, t) d\eta, \quad i = \overline{1,m} \\ \dfrac{\partial c_i}{\partial t} + \dfrac{\partial c_i}{\partial \eta} + \sigma_i c_i = \sigma_i \phi(\eta) n(t), \quad 0 \leq \eta \leq h, \quad t \geq t_0 \\ c_i(0,t) = c_i(h,t), \quad i = \overline{1,m} \\ c_i(\eta, t_0) = q_i^0(\eta), \quad n(t_0) = n_0, \quad 0 \leq \eta \leq h. \end{cases} \tag{5}$$

1.2 The analysis of the simplest example. Heuristics

The above models showed themselves interesting as a good motivation for introducing FDE. This is quite transparent from the papers of Cooke and Krumme (Cooke and Krumme (1968)) and Cooke (Cooke (1970)) where it has been shown that integration along the characteristics will associate some kind of FDE to the boundary value problems for hyperbolic PDE. The same idea appears earlier in a more general setting in the paper (Abolinia and Myshkis (1960)) and further in (Myshkis and Filimonov (1981)). It is worth mentioning that during the year when (Abolinia and Myshkis (1960)) was published, several other papers dealing with linear PDE of hyperbolic type with constant coefficients had been published also and we suggest that interesting FDE could be picked up from there.

A. In order to make motivation more clear for the application oriented reader, we shall start with the one of the simplest examples of the previous section, described by (4), in its linear version i.e. with $\psi(v) = G_D v$, the parameter G_D - the diode conductance being assumed constant

$$\begin{cases} \dfrac{\partial u}{\partial \lambda} + L\dfrac{\partial i}{\partial t} = 0 \\ \dfrac{\partial i}{\partial \lambda} + C\dfrac{\partial u}{\partial t} = 0, \quad t \geq 0, \quad 0 \leq \lambda \leq 1 \\ E(t) = R_0 i(0,t) + u(0,t), \quad u(1,t) = v(t) \\ C_0\dfrac{dv(t)}{dt} + G_D v = i(1,t). \end{cases} \tag{6}$$

Let us follow a rather classical textbook approach in Mathematical Physics (Smirnov (1964) vol. II, section 164). We focus on PDE and perform a change of variables

$$\tau = t - \lambda\sqrt{LC}\ ;\ \theta = t + \lambda\sqrt{LC} \tag{7}$$

thus obtaining the new functions

$$\begin{cases} \widehat{u}(\tau, \theta) = u\left(\dfrac{1}{2\sqrt{LC}}(\theta - \tau), \dfrac{1}{2}(\theta + \tau)\right) \\ \widehat{i}(\tau, \theta) = i\left(\dfrac{1}{2\sqrt{LC}}(\theta - \tau), \dfrac{1}{2}(\theta + \tau)\right) \end{cases} \tag{8}$$

with $u(\lambda, t)$, $i(\lambda, t)$ being some solution of the initial PDE. Simple manipulation shows that $\widehat{u}$ and $\widehat{i}$ are solutions of

$$\begin{cases} \dfrac{\partial}{\partial \theta}\left(\widehat{u}(\tau, \theta) + \dfrac{L}{C}\widehat{i}(\tau, \theta)\right) = 0 \\ \dfrac{\partial}{\partial \tau}\left(\widehat{u}(\tau, \theta) - \dfrac{L}{C}\widehat{i}(\tau, \theta)\right) = 0 \end{cases} \tag{9}$$

what gives

$$\begin{cases} \widehat{u}(\tau, \theta) + \dfrac{L}{C}\widehat{i}(\tau, \theta) \equiv \phi(\tau) \\ \widehat{u}(\tau, \theta) - \dfrac{L}{C}\widehat{i}(\tau, \theta) \equiv \psi(\theta) \end{cases} \tag{10}$$

where ϕ and ψ are (for a while) arbitrary functions with necessary smoothness properties. It follows that

$$\begin{cases} u(\lambda, t) = \dfrac{1}{2}[\phi(t - \lambda\sqrt{LC}) + \psi(t + \lambda\sqrt{LC})] \\ i(\lambda, t) = \dfrac{1}{2}\sqrt{C/L}[\phi(t - \lambda\sqrt{LC}) - \psi(t + \lambda\sqrt{LC})] \end{cases} \tag{11}$$

hence the general solution has been represented *via* two arbitrary functions ϕ and ψ, of a single argument; ϕ is called *progressive wave* since its argument $t - \sqrt{LC}$ is a solution of the increasing characteristic $dt/d\lambda = \sqrt{LC}$ while ψ is called *reflected wave* since its argument is a solution of the decreasing characteristic $dt/d\lambda = -\sqrt{LC}$. These physical notions being mentioned, we shall focus on determining the (by now) arbitrary functions ϕ and ψ.

Suppose (again for a while) that our problem is a standard Cauchy problem i.e. only some initial conditions $u_0(\lambda)$ and $i_0(\lambda)$ are given. We shall have

$$\begin{cases} \phi(-\lambda\sqrt{LC}) = u_0(\lambda) + \sqrt{L/C}\, i_0(\lambda) \\ \psi(\lambda\sqrt{LC}) = u_0(\lambda) - \sqrt{L/C}\, i_0(\lambda) \end{cases} \tag{12}$$

If no boundary value conditions are given, then we may assume two cases: a) the line is infinite hence $\lambda \in (-\infty, \infty)$. Therefore both ϕ and ψ are defined on $(-\infty, \infty)$ *by the initial conditions provided these ones are given for all* λ. From here we deduce that the solution of the Cauchy problem defined by the PDE in (6) and the initial conditions in the case of an infinite line is defined on the domain

$$\{(\lambda, t)|\ \lambda \in \mathbb{R},\ t \geq 0,\ t - \lambda\sqrt{LC} < 0,\ t + \lambda\sqrt{LC} > 0\}$$

called "the propagation (semi-infinite) angle"; b) the line is finite, as it should be in reality and we have $0 \leq \lambda \leq 1$ as in (6). Therefore ϕ is defined only on $(-\sqrt{LC}, 0)$ and ψ only on $(0, \sqrt{LC})$. We deduce that the solution of the Cauchy problem is defined on the domain

$$\{(\lambda, t)|\ 0 \leq \lambda \leq 1,\ t \geq 0,\ -\sqrt{LC} \leq t - \lambda\sqrt{LC} \leq 0,$$
$$0 \leq t + \lambda\sqrt{LC} \leq \sqrt{LC}\}$$

which is a "propagation triangle".

The solution may be further defined provided boundary conditions are given. To be as closer as possible to the classical textbook, assume for a while that "the line is disconnected at the right hand boundary" i.e; $i(1, t) = 0$. In this case we shall have the following boundary conditions

$$\begin{cases} u(0, t) + R_0 i(0, t) = E(t) \\ i(1, t) = 0 \end{cases} \tag{13}$$

which are expressed using ϕ and ψ as follows

$$\begin{cases} (1 + R_0\sqrt{C/L})\phi(t) + (1 - R_0\sqrt{C/L})\psi(t) = 2E(t) \\ \psi(t + \sqrt{LC}) = \phi(t - \sqrt{LC}) \end{cases} \tag{14}$$

Consider firstly the equations *as they are*; from the initial conditions ϕ is known on $(-\sqrt{LC}, 0)$ and ψ on $(0, \sqrt{LC})$; equations (14) allow a step-by-step construction of ϕ and ψ for increasing t. For instance, the first equation allows computation of ϕ on $(0, \sqrt{LC})$ since ψ is known on $(0, \sqrt{LC})$ and $E(t)$ is known for $t > 0$. Now let $t \in (0, \sqrt{LC})$ hence $t - \sqrt{LC} \in (-\sqrt{LC}, 0)$ and we may use the second equation to construct ψ on $(\sqrt{LC}, 2\sqrt{LC})$; from the first equation we obtain ϕ on $(\sqrt{LC}, 2\sqrt{LC})$. In this way we obtain *via* the boundary conditions, using the step-by-step construction, the functions ϕ and ψ - no longer arbitrary - which are used, as previously, to represent the solution of (this time) the initial-boundary value problem defined by the PDE, the initial and the boundary conditions.

B. We presented in some detail a textbook problem with a textbook approach not because of its intrinsic significance but because we want to look at this construction from a slightly different point of view. *Mainly we are interested in the step-by-step construction of the functions ϕ and ψ*. Note that the structure (11) of the solutions describes what is called "wave propagation": the arguments of ϕ and ψ originate from the equations of the two families of characteristics and it is said sometimes that "waves propagate along the characteristics"(Smirnov (1964)). But the construction by steps of ϕ and ψ is, roughly speaking, some kind of propagation from one time interval to the next. We want to look at *this* process in detail, *to understand within a more general mathematical framework that fortunately does already exist the mathematical object* (14). With this aim denote $\widetilde{\psi}(t) = \psi(t + \sqrt{LC})$ and since $\psi(\cdot)$ is known for $t > 0$, $\widetilde{\psi}(\cdot)$ will be known on $(-\sqrt{LC}, 0)$. We obtain the following system

$$\begin{pmatrix} \phi(t) \\ \widetilde{\psi}(t) \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{1 - R_0\sqrt{C/L}}{1 + R_0\sqrt{C/L}} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \phi(t - \sqrt{LC}) \\ \widetilde{\psi}(t - \sqrt{LC}) \end{pmatrix} + \tag{15}$$
$$+ \begin{pmatrix} \dfrac{2}{1 + R_0\sqrt{C/L}} \\ 0 \end{pmatrix} E(t)$$

This is a difference equation with continuous "time" (the independent variable). While obtained starting from a problem in PDE, this equation may be discussed independently of it. For the time being we shall just mention that (15) has the form

$$x(t) = D_1 x(t - \tau) = f(t) \tag{16}$$

or

$$(\mathcal{D}x)(t) = f(t) \tag{17}$$

where $\mathcal{D}$ is a *difference operator* defined by

$$\mathcal{D}\phi = \phi(0) - D_1\phi(-\tau) \tag{18}$$

Following (Hale and Lunel (1993), Chapter 9), we may see that a difference operator has to be defined on some n-dimensionally valued function space whose elements have the support some interval e.g. $(-\tau, 0)$; for instance we may take $\mathcal{D} : \mathcal{C}(-\tau, 0; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ where n is the dimension of the vectors x and f. The difference operator defined in (15) is *atomic at* 0 and also strongly stable in the sense of (*op. cit*) since according to Theorem 6.1 there (*op. cit*), a necessary and sufficient condition for the strong stability is that the spectral radius of D_1 be less than 1; here this radius is

$$\varrho = \left| \frac{1 - R_0\sqrt{C/L}}{1 + R_0\sqrt{C/L}} \right| < 1$$

for strictly positive R_0, C, L. We have looked ahead for the notion of *stability* of $\mathcal{D}$ - which means in fact asymptotic stability of the zero solution of (17) with $f \equiv 0$ - and *strong stability* which means validity of this property for all τ - *robustness in delay*).

The stability properties of the difference operator will be discussed later in detail and for a more general case; we mentioned them here for two reasons: firstly to show how known results apply to our mathematical object and secondly to have the opportunity to add something more about the application in discussion.

We have $\varrho < 1$ as long as $R_0 \neq 0$; physically this means a resistor at the left boundary i.e. some energy dissipation. The boundary conditions belong to the class of *dissipative boundary conditions*. If $R_0 = 0$ then $\varrho = 1$, no energy dissipation takes place and we deal with *conservative boundary conditions*; in this case $\mathcal{D}$ is no longer stable in the sense of the definition, self sustained oscillations of the free solutions of (16) (or (17)) are possible and their frequencies may depend on τ and may be sensible to variations of τ; we do not insist here on this matter.

We shall mention in brief another problem connected to (16): the discontinuities of the solution. It is quite obvious from (18) that the solution of (17) will be discontinuous at $t = 0$ even if the initial condition is a continuous function: let $x_0 : (-\tau, 0) \to \mathbb{R}^2$ be some initial condition and let $x(t)$ be the corresponding solution of (17). We shall have

$$\begin{aligned} &\lim_{t\to 0_+} x(t) := x(0_+) = D_1 x_0(-\tau) + f(0), \\ &\lim_{t\to 0_-} x(t) := x(0_-) = x_0(0) \end{aligned} \tag{19}$$

hence

$$x(0_+) - x(0_-) = D_1 x_0(-\tau) + f(0) - x_0(0) \neq 0$$

unless $x_0(\cdot)$ is such that the difference equation (16) holds at $t = 0$. Moreover this discontinuity at $t = 0$ "propagates": we shall have for $t = k\tau$

$$\begin{aligned} &x(k\tau + 0) - x(k\tau - 0) = \\ &= D_1(x((k-1)\tau + 0) - x((k-1)\tau - 0)) \end{aligned} \tag{20}$$

where we denoted by $x(k\tau \pm 0)$ the right hand and left hand limits at $k\tau$. It follows that the only way of avoiding discontinuities is to avoid discontinuities at 0 i.e. to have

$$D_1 x_0(-\tau) + f(0) = x_0(0) \tag{21}$$

Coming back to the basic problem we find, passing from (15) to (13) *via* (14) that

$$\begin{cases} u(0,0) + R_0 i(0,0) = u_0(0) + R_0 i_0(0) = E(0) \\ i(1,0) = i_0(1) = 0 \end{cases} \tag{22}$$

The above equalities put a restriction on the initial conditions; this restriction is in fact induced by the boundary conditions and is called *matching*. As known, *mismatching* of the initial conditions to the boundary ones propagates - the phenomenon is called *singularity propagation along the characteristics* and is considered typical for hyperbolic PDE. Here we *rediscover it* (on a simple example) *from the properties of the difference equation*.

C. To continue the parallelism of the IBVP for PDE and the system of functional equations, we consider now the line "connected" to the circuit at the right boundary to obtain exactly the boundary conditions in (6). Since now in (13) the second equality is replaced by the last two in (6) it is not difficult to find instead of (14) the system

$$\begin{aligned} &(1 + R_0\sqrt{C/L})\phi(t) + (1 - R_0\sqrt{C/L})\psi(t) = 2E(t) \\ &\phi(t - \sqrt{LC}) + \psi(t + \sqrt{LC}) = 2v(t) \\ &C_0\frac{dv}{dt} + G_D v = \frac{1}{2}\sqrt{C/L}\left[\phi(t - \sqrt{LC}) - \psi(t + \sqrt{LC})\right] \end{aligned} \tag{23}$$

where again ϕ is known on $(-\sqrt{LC}, 0)$ and ψ on $(0, \sqrt{LC})$. Denote as previously $\widetilde{\psi}(t) = \psi(t + \sqrt{LC})$ to obtain from (23) the system

$$\begin{aligned} &C_0\frac{dv}{dt} + (G_D + \sqrt{C/L})v = \phi(t - \sqrt{LC}) \\ &\begin{pmatrix} \phi(t) \\ \widetilde{\psi}(t) \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{1 - R_0\sqrt{C/L}}{1 + R_0\sqrt{C/L}} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi(t - \sqrt{LC}) \\ \widetilde{\psi}(t - \sqrt{LC}) \end{pmatrix} + \\ &+ \begin{pmatrix} 0 \\ 2 \end{pmatrix} v(t) + \begin{pmatrix} \dfrac{2}{1 + R_0\sqrt{C/L}} \\ 0 \end{pmatrix} E(t) \end{aligned} \tag{24}$$

This is a system of coupled differential and difference equations, associated to a specific IBVP for a system of PDE. We may proceed as in the case of (15) by a specific analysis. But such specific systems have been considered previously: the reader is sent to some recent surveys (Halanay and Răsvan (1997); Răsvan (1998)) as well as the book (Hale and Lunel (1993),Chapter 9). For these reasons an inference concerning the system of FDE will be possible, suggested by (24). Our main mathematical object will be in fact a system of functional differential equations consisting of delay-differential equations coupled with some difference equations with continuous independent variable ("time"); such systems have been called, more recently, *coupled differential and algebraic equations*.

1.3 The system of functional differential equations. Neutral equations

System (24) belongs to the following general class of functional differential equations

$$\begin{aligned} \dot{x}_1 &= A_0 x_1(t) + A_1 x_2(t-\tau) + f_1(t) \\ x_2(t) &= A_2 x_1(t) + A_3 x_2(t-\tau) + f_2(t) \end{aligned} \tag{25}$$

A. Our purpose is now to explain why this system is considered to be of neutral type. With respect to this it is interesting to see the various definitions given to neutral equations upon the time. We start from Bellman and Cooke (1963) where the simplest linear equation is considered

$$a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = f(t) \tag{26}$$

and the "natural" classification is introduced

(1) $a_0 \neq 0$, $a_1 = 0$ - equation of retarded type ;
(2) $a_0 = 0$, $a_1 \neq 0$, $b_0 \neq 0$ - equation of advanced type ;
(3) $a_0 \neq 0$, $a_1 \neq 0$ - equation of neutral type.

The "natural" character of this classification follows from the dependence of the simultaneous presence of the delayed and non-delayed derivative. If the neutral case is considered

$$y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1(t-\tau) = f(t) \tag{27}$$

(we took $a_0 = 1$ since $a_0 \neq 0$), let us denote

$$v(t) = y(t) + a_1 y(t-\tau) \tag{28}$$

to find

$$\begin{cases} v'(t) = -b_0 v(t) + (b_0 a_1 - b_1) y(t-\tau) + f(t) \\ y(t) = v(t) - a_1 y(t-\tau) \end{cases} \tag{29}$$

which belongs to the general class described by (25) introduced previously. Let us consider now the converse situation of (25) in the scalar case

$$\begin{cases} \dot{x}_1 = a_0 x_1(t) + a_1 x_2(t-\tau) + f_1(t) \\ x_2(t) = a_2 x_1(t) + a_3 x_2(t-\tau) + f_2(t), \end{cases} \tag{30}$$

multiply the first equation by a_2 and replace $a_2x_1(t)$ and its derivative by $x_2(t) - a_3x_2(t-\tau) - f_2(t)$ to obtain

$$\begin{aligned} &\dot{x}_2(t) - a_3\dot{x}_2(t-\tau) - a_0x_2(t) + \\ &+(a_0a_3 - a_2a_1)x_2(t-\tau) = \\ &= a_2f_1(t) + \dot{f}_2(t) - a_0f_2(t) \end{aligned} \tag{31}$$

B. Another characterization of the neutral type of some equation with deviated argument is valid also for linear (or linearized) equations since it is in connection with the roots of the characteristic equation

$$(a_0 + a_1e^{-\lambda\tau})\lambda + b_0 + b_1e^{-\lambda\tau} = 0 \tag{32}$$

In the retarded case the characteristic equation has a principal term (in the sense given by Pontryagin), in the advanced case it does not have such a term, but in the neutral case this term is multiplied by an entire function which is the characteristic equation of the "difference operator" which is differentiated in the neutral equation or defines a second equation in the more general case of (25) - see also (28).

The principal term is the one that gives the character of the *asymptotic root chains* of the characteristic equation

(1) If the principal term does not exist, there exist root chains that approach ∞ *via* $\mathbb{C}^+$.
(2) In the retarded case all asymptotic root chains approach ∞ *via* $\mathbb{C}^-$.
(3) In the neutral case there may exist root chains that approach ∞ along the imaginary axis $\imath\mathbb{R}$

C. The third characterization of the neutral type comes from smoothness analysis. We consider first the retarded case

$$y'(t) + b_0y(t) + b_1y(t-\tau) = f(t) \tag{33}$$

and construct the solution with the initial condition $y_0, \phi(\cdot)$ by steps

$$y'(t) = -b_0y(t) + f(t) - b_1\phi(t-\tau)\ ,\ 0 \le t < \tau$$

$$y(t) = e^{-b_0t}y_0 + \int_0^t e^{-b_0(t-\vartheta)}(f(\vartheta) - b_1\phi(\vartheta-\tau))d\vartheta$$

$$y'(t) = -b_0y(t) + f(t) - b_1\phi_1(t-\tau)\ ,\ \tau \le t < 2\tau$$

$$y(t) = e^{-b_0(t-\tau)}y_0 + \int_\tau^t e^{-b_0(t-\vartheta)}(f(\vartheta) - b_1\phi_1(\vartheta-\tau))d\vartheta$$

a.s.o. Even if it has a discontinuity at $t = 0$, the solution is continuous everywhere for $t > 0$ and has the smoothness imposed by the smoothness of f; if $f \equiv 0$, larger is p in considering $t \in [(p-1)\tau, p\tau)$, smoother is the solution, due to subsequent integration.

In the neutral case smoothness becomes crucial. The form of the equation (27) would require a differentiable initial condition on $[-\tau, 0)$ e.g. from $\mathcal{W}_2^{(1)}$ but we shall use the previously introduced functions and consider the initial condition $\phi(\cdot)$ at most continuous on $[-\tau, 0)$. We integrate by steps the first equation from (29) namely

$$v'(t) = -b_0v(t) + (b_0a_1 - b_1)y(t-\tau) + f(t)$$

with $v(t)$ defined above by (28) and the initial condition

$$v(0) = y_0 + a_1\phi(-\tau) \ne \phi(0)$$

thus obtaining

$$\begin{aligned} v(t) = &e^{-b_0t}v(0) + \\ &+\int_0^t e^{-b_0(t-\vartheta)}(f(\vartheta) - (b_1 - b_0a_1)\phi(\vartheta-\tau))d\vartheta \end{aligned}$$

for $0 \le t < \tau$ where it is at least absolutely continuous. Therefore

$$y(t) = v(t) - a_1\phi(t-\tau)$$

is as smooth as ϕ on $(0, \tau)$. Next step $(\tau, 2\tau)$ is made as follows. Since we may denote $\phi_1(t) := v(t) - a_1\phi(t-\tau)$ on $(0, \tau)$ we check that $\phi_1(0_+) = y_0 \ne \phi(0)$ and continue by

$$\begin{aligned} v(t) = &e^{-b_0(t-\tau)}v(\tau) + \\ &+\int_\tau^t e^{-b_0(t-\vartheta)}(f(\vartheta) - (b_1 - b_0a_1)\phi_1(\vartheta-\tau))d\vartheta \end{aligned}$$

for $\tau \le t < 2\tau$ to obtain again v at least absolutely continuous on that interval; again

$$y(t) = v(t) - a_1\phi_1(t-\tau)$$

and it is *as smooth as* ϕ_1 *which is as smooth as* ϕ. Therefore it is no smoothing of the solution. Moreover, v is continuous everywhere for $t > 0$. On the other hand

$$y(p\tau \pm 0) = v(p\tau) - a_1y((p-1)\tau \pm 0)$$

hence

$$y(p\tau+0) - y(p\tau-0) = -a_1(y((p-1)\tau+0) - y((p-1)\tau-0))$$

and the initial discontinuity propagates.

We may thus say that *neutral equations are those which preserve smoothness of the initial condition.*

1.4 A rather general model

In connection with considerations of the above subsection we would like to present here an application due to Cooke (1970) which was not (unfortunately) very circulated.

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + \tau_1(\lambda,t)\frac{\partial u_1}{\partial \lambda} = \Phi_1(\lambda,t) \\ &\frac{\partial u_2}{\partial t} + \tau_2(\lambda,t)\frac{\partial u_2}{\partial \lambda} = \Phi_2(\lambda,t), \quad 0 \le \lambda \le 1, \quad t \ge t_0, \end{aligned} \tag{34}$$

$$\begin{aligned} &\sum_{k=0}^m \left[a_{k1}(t)\frac{d^k}{dt^k}u_1(0,t) + a_{k2}(t)\frac{d^k}{dt^k}u_2(0,t)\right] = f_1(t) \\ &\sum_{k=0}^m \left[b_{k1}(t)\frac{d^k}{dt^k}u_1(1,t) + b_{k2}(t)\frac{d^k}{dt^k}u_2(1,t)\right] = f_2(t) \end{aligned} \tag{35}$$

with the initial conditions:

$$u_1(\lambda, t_0) = \omega_1(\lambda), \quad u_2(\lambda, t_0) = \omega_2(\lambda) \tag{36}$$

for $0 \le \lambda \le 1$ and under the assumption that $\tau_1(\lambda, t) > 0$, $\tau_2(\lambda, t) < 0$ in the considered domain (what gives the *hyperbolic character* to the system).

Remark that here the forward and backward waves (the Riemann invariants) are already separated, all the coefficients are time-varying, and the boundary conditions are derivative. This last fact will impose additional smoothness conditions on the solutions.

In order to associate a system of functional equations to (34)-(36), we shall consider the system of characteristics:

$$\frac{dt}{d\lambda} = \frac{1}{\tau_i(\lambda, t)}, \quad i = \overline{1,2}, \tag{37}$$

on the domain $0 \le \lambda \le 1$, $t \ge t_0$.

Let $t_1(\lambda; 0, t)$ be an increasing characteristic curve (i.e. verifying $\frac{dt}{d\lambda} = \frac{1}{\tau_1(\lambda, t)}$) starting from the point $(0, t)$ located on the boundary of the considered domain. Since $\tau_1(\lambda, t)$ is continuous and $(0, 1)$ is a bounded interval, the solution $t_1(\lambda; 0, t)$ is continuable on $(0, 1)$; we may thus define $T_1(t)$ as $T_1(t) = t_1(1; 0, t) - t$.

Consider now the forward wave $u_1(\lambda, t)$ along the characteristic $t_1(\lambda; 0, t)$ as follows:

$$\phi_1(\lambda) = u_1(\lambda, t_1(\lambda; 0, t)).$$

We shall have:

$$\begin{aligned} \frac{d\phi_1}{d\lambda} &= \frac{\partial u_1}{\partial \lambda}(\lambda, t_1(\lambda; 0, t)) + \frac{\partial u_1}{\partial t}(\lambda, t_1(\lambda; 0, t))\frac{dt_1}{d\lambda} \\ &= \frac{1}{\tau_1(\lambda, t_1(\lambda; 0, t))}\Phi_1(\lambda, t_1(\lambda; 0, t)). \end{aligned}$$

We may integrate with respect to λ:

$$\phi_1(\lambda) = \phi_1(0) + \int_0^\lambda \frac{\Phi_1(\sigma, t_1(\sigma; 0, t))}{\tau_1(\sigma, t_1(\sigma; 0, t))} d\sigma$$

and obtain

$$\begin{aligned} \phi_1(1) &= u_1(1, t_1(1; 0, t)) = u_1(1, t + T_1(t)) \\ &= u_1(0, t) + \int_0^1 \frac{\Phi_1(\sigma, t_1(\sigma; 0, t))}{\tau_1(\sigma, t_1(\sigma; 0, t))} d\sigma, \end{aligned} \tag{38}$$

in fact a functional relation between the boundary values of the forward wave.

Similarly, let $t_2(\lambda; 1, t)$ be a decreasing characteristic curve (i.e. verifying $\frac{dt}{d\lambda} = \frac{1}{\tau_2(\lambda, t)}$) starting from $(1, t)$; as previously, we may define $T_2(t)$ as $t_2(0; 1, t) - t$. Consider then the backward wave $u_2(\lambda, t)$ along the characteristic:

$$\phi_2(\lambda) = u_2(\lambda, t_2(\lambda; 1, t)).$$

We deduce:

$$\frac{d\phi_2}{d\lambda} = \frac{1}{\tau_2(\lambda, t_2(\lambda; 1, t))}\Phi_2(\lambda, t_2(\lambda; 1, t))$$

and this gives, after an integration:

$$u_2(0, t + T_2(t)) = u_2(1, t) + \int_0^1 \frac{\Phi_2(\sigma, t_2(\sigma; 1, t))}{\tau_2(\sigma, t_2(\sigma; 1, t))} d\sigma, \tag{39}$$

a functional relation between the boundary values of the backward wave.

Denoting as previously

$$y_1(t) = u_1(1, t), \quad y_2(t) = u_2(0, t), \tag{40}$$

the above relations become:

$$\begin{aligned} u_1(0, t) &= y_1(t + T_1(t)) - \psi_1(t) \\ u_2(1, t) &= y_2(t + T_2(t)) - \psi_2(t), \end{aligned} \tag{41}$$

where we denoted:

$$\psi_1(t) = \int_0^1 \frac{\Phi_1(\sigma, t_1(\sigma; 0, t))}{\tau_1(\sigma, t_1(\sigma; 0, t))} d\sigma$$

$$\psi_2(t) = \int_0^1 \frac{\Phi_2(\sigma, t_2(\sigma; 1, t))}{\tau_2(\sigma, t_2(\sigma; 1, t))} d\sigma.$$

We substitute the expressions of $u_i(0, t)$ and $u_i(1, t)$ in the boundary conditions (35) to obtain:

$$\begin{aligned} &\sum_{k=0}^{m} \left[a_{k1}(t)\frac{d^k}{dt^k}y_1(t + T_1(t)) + a_{k2}(t)\frac{d^k}{dt^k}y_2(t)\right] = \\ &= f_1(t) + \sum_{k=0}^{m} a_{k1}(t)\frac{d^k}{dt^k}\psi_1(t) \\ &\sum_{k=0}^{m} \left[b_{k1}(t)\frac{d^k}{dt^k}y_1(t) + b_{k2}(t)\frac{d^k}{dt^k}y_2(t + T_2(t))\right] = \\ &= f_2(t) + \sum_{k=0}^{m} b_{k1}(t)\frac{d^k}{dt^k}\psi_2(t) \end{aligned} \tag{42}$$

We obtained, as in the previous case that if $(u_1(\lambda, t), u_2(\lambda, t))$ is a solution of (34)-(36) then $(y_1(t), y_2(t))$ is a solution of the system (42) of FDE.

Formulae (42) suggest not only a classification but also a natural way of finding FDE of retarded, neutral or even advanced type. In fact, since all these coefficients and delays are time dependent, one could not exclude that the type of the equations might change with the time.

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