PID Controller Design Based on Laguerre Orthonormal Functions

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Abstract: In this paper, an analytical method for designing Proportional-Integral-Derivative (PID) controllers based on Laguerre orthonormal functions is presented. The PID controller parameters are calculated after matching the first three coefficients of Laguerre series of the open loop gain with the ideal one. To reach this goal, the plant, the PID controller and the ideal open loop gain transfer functions are represented in terms of their corresponding Laguerre series. The pole of the Laguerre basis function could be appropriately selected to assure the closed loop stability. Moreover, the optimum value of this parameter is determined to reach the best fitting to the desired step response considering the control signal constraint. The simulation results and the experimental tests on a DC servomotor system are given to show the effectiveness of the proposed method.

Keywords: PID controller, Orthogonal functions, Laguerre functions, PD controller, PI controller, Moment matching.

1. INTRODUCTION

The PID controllers are the most popular controllers in industry applications. It is due to their simplicity in design, auto tuning and compatibility with the real plants especially those described with the First Order Plus Dead Time (FOPDT) models (Åström and Hägglund, 1995; Åström and Hägglund, 2001; Bennett, 2001).

Several design approaches have been proposed to design PID controllers in the literature. The results of some of these methods have been summarized in the provided tuning tables such as well-known Ziegler and Nichols (Ziegler and Nichols, 1993). Many of the well-known PID controllers have been designed based on minimization of some performance indices using optimization algorithms (Kim et al., 2008; Zeng et al., 2014). These approaches couldn't give straightforward expressions for PID controller parameters. Therefore, proposing explicit relations to tune PID controllers could be considered as a challenge for the control engineers. Internal Model Controllers (IMC) have been utilized in this area (Jin and Liu, 2014; Skogestad, 2003). But, the IMC-PID controllers could be utilized only for special plants. (Ramasamy and Sundaramoorthy, 2008) proposed a moment matching based PID controller by matching the McLaurin expansion coefficients of the closed loop system with the desired one. The proposed method couldn't assure the closed loop system stability. (Xue, 2014) employed Taylor series to design a Proportional Differential (PD) controller for FOPDT plants. The application of the proposed method is confined to FOPDT plants.

The Laguerre polynomials are one of the well-known orthogonal functions which are widely used in signal processing and control applications (Akcay and Ninness, 1999; Dattoli et al., 2006). Employing these orthonormal functions in system identification and model reduction has been considered in the literature (Makila, 1990; Malti et al., 1998; Olivier, 1994; Van Den Hof et al., 1995; Wang and Jiang, 2011). These orthonormal functions could be utilized for controller design, too. Applying these functions in model predictive control has been studied in the literature (Elshafei et al., 1994; Kannan et al., 2013; Saha et al., 2004; Wang, 2004). The Laguerre functions have been employed to design adaptive controllers, too (Zervos and Dumont, 1988; Zhang, et al., 2009). (Yuan, 2007) has verified the effectiveness of the Laguerre filters in active noise control.

On the other hand, employing orthogonal functions to design PID controllers has been considered in the literature (Ayadi and Braiek, 2005; Bouafoura and Braiek, 2010; Horng and Chou, 1988; Olivier, 2009). (Ayadi and Braiek, 2005) proposed a MIMO PID controller based on Legendre functions. In this work, the differential state space equations have been converted to a set of algebraic equations by expanding the system input and output variables in terms of their Legendre orthogonal series. Olivier (2009) employed Laguerre series to design PID controllers. The proposed method is limited to some special plants. Moreover, the proposed method needs some numerical calculations to obtain PID parameters without giving any explicit formulas.

The aim of this paper is to employ Laguerre functions to present an analytical design method for PID controllers. The PID controller parameters are calculated through clear relations. The proposed Laguerre based PID controller could be employed to control a variety of plant models. The PID controller parameters are the result of matching the first three Laguerre coefficients of the open loop gain with the ideal one. The closed loop system stability analysis is performed with the consequent PID controller parameters. To assure the closed loop system stability, the pole of the Laguerre basis function should be considered in a definite bound. The optimal value of this parameter is selected based on the minimization of a loss function. This loss function is defined as the integral square error between the ideal step response and the closed loop system step response. The control signal limitations are considered in the optimization procedure, too. The simulation results on several plant models and experimental results in an actual servomechanism system demonstrate the performance of the proposed method as well.

Finally, this paper has three outstanding superiorities comparing with the other moment matching based PID controller design methods. Firstly, the proposed method could be utilized for a variety of plants (stable, unstable, non-minimum phase and integrating plants). In the already published work by (Xue, 2014), the application of the proposed moment matching based PD controller is only confined to FOPDT plants. Explicit relations obtained for the family of PID controllers (PI, PD and PID) in terms of the plant parameters is the second superiority. In other words, to derive the PID controller parameters, there is no need to numerical methods employed in the literature (Olivier, 2009). Thirdly, the stability analysis is considered in the design procedure. The Laguerre basis function pole could be appropriately selected to ensure the closed loop system stability unlike the McLaurin series based methods (Ramasamy and Sundaramoorthy, 2008; Xue, 2014). Moreover, this parameter could be determined to attain the optimal closed loop system step response by minimizing an integral square error performance index.

The rest of this paper is organized as follows. Section 2 describes Laguerre polynomials and their specifications. Section 3 presents the Laguerre based PD, PI and PID controllers. The way to select the Laguerre basis function pole to minimize the integral square error index is illustrated in section 4. Numerical examples and experimental tests are given in section 5. Section 6 concludes the paper.

2. LAGUERRE ORTHOGONAL FUNCTIONS

Consider the ordinary definition of the scalar product of two functions $\varphi_i(\mathrm{t})$ and $\varphi_j(\mathrm{t})$ as:

$$<\varphi_i(\mathrm{t}),\varphi_j(\mathrm{t})>=\int_0^\infty \varphi_i(\mathrm{t})\varphi_j(\mathrm{t})\,\mathrm{dt} \tag{1}$$

The root square of the scalar product of each function with itself is called 2-norm which is defined as:

$$\|\varphi_i(\mathrm{t})\|_2=\sqrt{<\varphi_i(\mathrm{t}),\varphi_i(\mathrm{t})>} \tag{2}$$

Two functions $\varphi_i(\mathrm{t})$ and $\varphi_j(\mathrm{t})$ are called orthonormal if they have normal length and their scalar product equals to zero. Or

$$<\varphi_i(\mathrm{t}),\varphi_j(\mathrm{t})>=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases} \tag{3}$$

The functions with finite norm form L_2 norm space. Any function $f(t)\in L_2$ could be represented as the weighted summation of orthonormal function as follows

$$f(t)=\sum_{k=1}^{\infty} f_k\varphi_k(\mathrm{t}) \tag{4}$$

The equation (4) could be written in the s-domain as

$$F(s)=\sum_{k=1}^{\infty} f_k\varphi_k(s) \tag{5}$$

where series coefficients (f_k) are calculated as

$$f_k=\langle F(s),\varphi_k(s)\rangle \tag{6}$$

where $\langle p,q\rangle$ means the inner product of two functions p and q. According to the Residue theorem, the series coefficients in (5), could be obtained as follows (Olivier, 2009)

$$f_k=\frac{1}{2\pi j}\oint_{RHP} F(-s)\varphi_k(\mathrm{s})\,\mathrm{ds}=-\sum_{\substack{residues\,@\\ RHPpoles}}[F(-s)\varphi_k(s)] \tag{7}$$

The Laguerre orthonormal basis functions could be defined as follows

$$\varphi_k(\mathrm{t})=\sqrt{2\lambda}\,\frac{e^{\lambda t}}{(k-1)!}\frac{d^{k-1}(\mathrm{t}^{k-1}e^{-2\lambda t})}{dt^{k-1}},\quad k=1,\ldots,\infty \tag{8}$$

where λ is a positive real constant. The Laplace transform of equation (8) yields the s-domain representation of Laguerre orthonormal functions as

$$\varphi_k(s)=\sqrt{2\lambda}\,\frac{(s-\lambda)^{k-1}}{(s+\lambda)^k},\quad k=1,\ldots,\infty \tag{9}$$

This means that the Laguerre function pole depends on λ. According to (9), the first three Laguerre basis functions are

$$\varphi_1(s)=\frac{\sqrt{2\lambda}}{(s+\lambda)},\ \varphi_2(s)=\frac{\sqrt{2\lambda}(s-\lambda)}{(s+\lambda)^2},\quad \varphi_3(s)=\frac{\sqrt{2\lambda}(s-\lambda)^2}{(s+\lambda)^3} \tag{10}$$

The product of two Laguerre basis functions could be calculated as (Olivier, 2009)

$$\varphi_i(s)\varphi_j(s)=\frac{\varphi_{i+j-1}(s)-\varphi_{i+j}(s)}{\sqrt{2\lambda}} \tag{11}$$

The proof is evident according to the equation (9).

3. USING LAGUERRE FUNCTIONS TO DESIGN PID CONTROLLERS

In this section, Laguerre orthonormal functions are employed to design the family of PID controllers. The main idea is that a strictly proper stable transfer function $F(s)$ could be approximately represented with the first three terms of its Laguerre series by neglecting higher terms. Or

$$F(s) \approx \sum_{k=1}^{3} g_k \varphi_k(s) \tag{12}$$

The system $F(s)$ is a low pass filter. Thus, the neglected terms representing the high frequency parts of $F(s)$ have little impact on its step response.

In many control design applications, the controller design algorithm is based on matching the product of the controller $C(s)$ and the plant model $G(s)$ with an ideal open loop gain transfer function $L(s)$

$$G(s)C(s) = L(s) \tag{13}$$

The plant model is considered as

$$G(s) = \frac{p(s)(1-\theta s)}{q_s(s)q_u(s)} \tag{14}$$

where $p(s)$ and $q_s(s)$ are Hurwitz polynomials and $q_u(s)$ is the unstable part of the denominator of $G(s)$ (containing unstable poles of the plant). The degree of $p(s)$, $q_u(s)$ and $q_s(s)$ are considered m, r and n, respectively. The nonnegative parameter θ determines the unstable zero of the plant. To have a strictly proper transfer function, the following relation should be satisfied

$$\begin{cases} m < n+r & if\ \theta = 0 \\ m < n+r-1 & if\ \theta \neq 0 \end{cases} \tag{15}$$

Thus, a strictly-proper rational transfer function which could be considered minimum-phase or non-minimum phase (with utmost one unstable zero), stable or unstable is considered for the plant model.

In many control design applications, the standard second-order transfer function is considered as the ideal closed loop transfer function (Åström and Hägglund, 1995). Moreover, the unstable zero couldn't be cancelled through controller pole and it will be appeared in the closed loop transfer function. Therefore, the following ideal closed loop transfer function is considered

$$M(s) = \frac{\omega_n^2(1-\theta s)}{s^2 + 2\eta\omega_n s + \omega_n^2}, \qquad 0.5 \le \eta \le 1 \tag{16}$$

where η and ω_n could be adjusted to reach the desired closed loop step response. The parameter η is considered smaller than 1 and greater than 0.5 to reach a fast step response with little overshoot. The parameter ω_n could be selected to adjust the transient response speed (considering the control signal constraints). Considering a unit negative feedback structure, the ideal open loop gain transfer function is calculated as

$$L(s) = \frac{M(s)}{1-M(s)} = \frac{\omega_n^2(1-\theta s)}{s(s+2\eta\omega_n + \omega_n^2\theta)} \tag{17}$$

The Laguerre series expansion could be written only for the strictly proper stable transfer functions. Therefore, the ideal open loop gain transfer function should be rewritten in the following form

$$L(s) = \frac{\dfrac{\omega_n^2(1-\theta s)}{(s+2\eta\omega_n+\omega_n^2\theta)(s+\lambda)^2}}{\dfrac{s}{(s+\lambda)^2}} = \frac{\sum_{i=1}^{\infty} l_i\varphi_i(s)}{\sum_{i=1}^{\infty} l'_i\varphi_i(s)} \tag{18}$$

The first three Laguerre series coefficients in (18) are

$$\begin{aligned} l_1 &= \frac{\omega_n^2(1-\theta\lambda)}{2\lambda\sqrt{2\lambda}(\lambda+2\eta\omega_n+\omega_n^2\theta)}, \\ l_2 &= \frac{\omega_n^2(\theta\lambda^2-(\theta^2\omega_n^2+2\eta\omega_n\theta+3)\lambda-(2\eta\omega_n+\omega_n^2\theta))}{2\lambda\sqrt{2\lambda}(\lambda+2\eta\omega_n+\omega_n^2\theta)^2}, \\ l_3 &= \frac{\sqrt{2\lambda}\omega_n^2(1+2\eta\omega_n\theta+\omega_n^2\theta^2)}{(\lambda+2\eta\omega_n+\omega_n^2\theta)^3}, l'_1 = l'_2 = \frac{1}{2\sqrt{2\lambda}}, l'_3 = 0 \end{aligned} \tag{19}$$

If the plant transfer function in (14) is stable ($q_u(s)=1$), it could be directly expanded in terms of its Laguerre basis functions as follows

$$G(s) = \frac{p(s)(1-\theta s)}{q_s(s)} = \sum_{i=1}^{\infty} g_i\varphi_i(s) \tag{20}$$

But, an unstable plant could be rewritten as the following Laguerre series expansion form

$$G(s) = \frac{p(s)(1-\theta s)}{q_s(s)q_u(s)} = \frac{\dfrac{p(s)(1-\theta s)}{q_s(s)(s+\lambda)^{r+1}}}{\dfrac{q_u(s)}{(s+\lambda)^{r+1}}} = \frac{\sum_{i=1}^{\infty} g_i\varphi_i(s)}{\sum_{i=1}^{\infty} g'_i\varphi_i(s)} \tag{21}$$

The controller is a PID controller with the following transfer function

$$C(s) = k_c(1+\frac{1}{T_i s}+T_d\, s) \tag{22}$$

The controller in (22) could be converted to conventional Proportional Integral (PI) ($T_d = 0$) or PD ($T_i = \infty$) controller. In this section, the family of PID controllers are designed based on the Laguerre series expansion method.

3.1 PI Controller design

In the first case, assuming $T_d = 0$ in equation (22) yields a PI controller with the following transfer function

$$C(s) = k_c(1+\frac{1}{T_i s}) \tag{23}$$

The Laguerre series representation of the PI controller (23) is

$$C(s)=\frac{\dfrac{1}{T_i}\dfrac{T_is+1}{(s+\lambda)^2}}{\dfrac{1}{k_c}\dfrac{s}{(s+\lambda)^2}}=\frac{\sum_{i=1}^{\infty}c_i\varphi_i(s)}{\sum_{i=1}^{\infty}c'_i\varphi_i(s)} \tag{24}$$

The Laguerre series coefficients in (24) are calculated as

$$c_1=\frac{1+\lambda T_i}{2\lambda\sqrt{2\lambda}T_i},\quad c_2=\frac{\lambda T_i-1}{2\lambda\sqrt{2\lambda T_i}},\quad c'_1=c'_2=\frac{1}{2k_c\sqrt{2\lambda}} \tag{25}$$

To design a PI controller for stable plants represented by the Laguerre series in (20), the following relation should be satisfied (according to relations (13), (18), (20) and (24))

$$\sum_{i=1}^{\infty}g_i\varphi_i(s)\sum_{i=1}^{\infty}c_i\varphi_i(s)\sum_{i=1}^{\infty}l'_i\varphi_i(s)=\sum_{i=1}^{\infty}l_i\varphi_i(s)\sum_{i=1}^{\infty}c'_i\varphi_i(s) \tag{26}$$

According to the multiplication property of the Laguerre functions in (11), both sides of equation (26) could be simplified as two Laguerre series. Matching the first two coefficients in these two Laguerre series yields

$$\frac{g_1c_1l'_1}{\sqrt{2\lambda}}=l_1c'_1,\qquad \frac{(g_1c_2+g_2c_1-g_1c_1)l'_1}{\sqrt{2\lambda}}=l_2c'_1 \tag{27}$$

Solving equation (27) gives the following expressions for the PI controller parameters

$$k_c=\frac{2\lambda((2g_1-g_2)l_1+g_1l_2)}{g_1^2},\quad T_i=\frac{(2g_1-g_2)l_1+g_1l_2}{\lambda(g_2l_1-g_1l_2)} \tag{28}$$

If the plant is considered as an unstable transfer function, then relation (21) should be employed. Therefore, equation (26) should be rewritten as

$$\begin{aligned}&\sum_{i=1}^{\infty}g_i\varphi_i(s)\sum_{i=1}^{\infty}c_i\varphi_i(s)\sum_{i=1}^{\infty}l'_i\varphi_i(s)=\\&\sum_{i=1}^{\infty}l_i\varphi_i(s)\sum_{i=1}^{\infty}c'_i\varphi_i(s)\sum_{i=1}^{\infty}g'_i\varphi_i(s)\end{aligned} \tag{29}$$

Simplifying (29) according to the multiplication property of the Laguerre basis functions and matching the first two terms, yields the following equations

$$\begin{aligned}&g_1c_1l'_1=g'_1l_1c'_1,\\&(g_1c_2+g_2c_1-g_1c_1)l'_1=((g'_2-g'_1)l_1+g'_1l_2)c'_1\end{aligned} \tag{30}$$

The PI controller parameters obtained from (30) are

$$\begin{aligned}k_c&=\frac{\sqrt{2\lambda}((g_1g'_1+g_1g'_2-g'_1g_2)l_1+g_1g'_1l_2)}{g_1^2},\\T_i&=\frac{(g'_1g_2-g_1g'_1-g_1g'_2)l_1-g_1g'_1l_2}{\lambda((g_1g'_2-g_1g'_1-g'_1g_2)l_1+g_1g'_1l_2)}\end{aligned} \tag{31}$$

3.2 PD Controller design

Considering $T_i=\infty$ in equation (22) yields the following PD controller

$$C(s)=k_c(1+T_d\,s) \tag{32}$$

Tracking constant reference signals, necessitates us to employ the PD controllers for type $N,(N\geq 1)$ systems. Therefore, the plant model should be an integrating transfer function which could be described with the Laguerre series expansion in (21). This means that relation (29) could be employed to design PD controllers, too.

The Laguerre series representation of PD controller needs the following different description of (32)

$$C(s)=\frac{\dfrac{T_ds+1}{(s+\lambda)^2}}{\dfrac{1}{k_c}\dfrac{1}{(s+\lambda)^2}}=\frac{\sum_{i=1}^{\infty}c_i\varphi_i(s)}{\sum_{i=1}^{\infty}c'_i\varphi_i(s)} \tag{33}$$

where the Laguerre series coefficients in (33) are given by

$$c_1=\frac{\lambda T_d+1}{2\lambda\sqrt{2\lambda}},\quad c_2=\frac{\lambda T_d-1}{2\lambda\sqrt{2\lambda}},\quad c'_1=-c'_2=\frac{1}{2\lambda k_c\sqrt{2\lambda}} \tag{34}$$

Simplifying equation (29) in accordance with relations (11), (19) and (34) and matching the first Laguerre series terms gives the following expression

$$\begin{aligned}&g_1c_1l'_1=g'_1l_1c'_1,\\&(g_1c_2+g_2c_1-g_1c_1)l'_1=((g'_2-3g'_1)l_1+g'_1l_2)c'_1\end{aligned} \tag{35}$$

Solving equation (35) results the following PD controller parameters

$$\begin{aligned}k_c&=\frac{\sqrt{2\lambda}((3g_1g'_1+g'_1g_2-g_1g'_2)l_1-g_1g'_1l_2)}{g_1^2},\\T_d&=\frac{(g_1g'_2-g'_1g_2-g_1g'_1)l_1+g_1g'_1l_2}{\lambda((3g_1g'_1+g'_1g_2-g_1g'_2)l_1-g_1g'_1l_2)}\end{aligned} \tag{36}$$

3.3 PID Controller design

In the general case, the transfer function of PID controller could be rewritten in the following Laguerre series form

$$C(s)=\frac{\dfrac{k_c(T_is+1+T_iT_ds^2)}{T_i(s+\lambda)^3}}{\dfrac{s}{(s+\lambda)^3}}==\frac{\sum_{i=1}^{\infty}c_i\varphi_i(s)}{\sum_{i=1}^{\infty}c'_i\varphi_i(s)} \tag{37}$$

where the Laguerre series coefficients in (37) are computed as follows

$$\begin{aligned}&c_1=\frac{k_c(T_i\lambda+T_iT_d\lambda^2+1)}{4\lambda^2\sqrt{2\lambda}T_i},\quad c_2=\frac{k_c(T_iT_d\lambda^2-1)}{2\lambda^2\sqrt{2\lambda}T_i},\\&c_3=\frac{k_c(T_iT_d\lambda^2-T_i\lambda+1)}{4\lambda^2\sqrt{2\lambda}T_i},\ c'_1=-c'_3=\frac{1}{4\lambda\sqrt{2\lambda}},\ c'_2=0\end{aligned} \tag{38}$$

If a stable plant with the Laguerre series representation in (20) is considered, then relation (26) could be utilized in the design procedure as well. Matching the first three Laguerre series yields the following three relations

$$\begin{aligned}&\sqrt{2\lambda}g_1c_1 = l_1, \qquad \sqrt{2\lambda}[(g_1-g_2)c_1-g_1c_2]=l_1-l_2,\\&\sqrt{2\lambda}[(g_3-g_1-g_2)c_1+(g_2-g_1)c_2+g_1c_3]=l_3-l_2-l_1\end{aligned} \tag{39}$$

After some simplifications, the PID controller parameters acquired from (39) are given by

$$\begin{aligned}k_c&=\frac{2\lambda(g_1l_1+g_2l_2-g_1\,l_3)}{g_1^2},\\T_i&=\frac{2(g_1l_1+g_2l_2-g_1\,l_3)}{\lambda[(g_1+g_2)(l_1-l_2)+g_1l_3]},\\T_d&=\frac{[(g_1-g_2)(l_1+l_2)+g_1l_3]}{2\lambda(g_1l_1+g_2l_2-g_1\,l_3)}\end{aligned} \tag{40}$$

For unstable plants, the equation (29) should be employed. In this case, the following relations will be derived

$$\begin{aligned}&2\lambda g_1c_1=g_1'l_1,\\&2\lambda(g_1c_2+g_2c_1-g_1c_1)=(g_2'-2g_1')l_1+g_1'l_2,\\&2\lambda(g_2c_2+g_1c_3+g_3c_1-g_1c_2-g_2c_1-g_1c_1)=\\&(g_3'-2g_2')l_1+(g_2'-2g_1')l_2+g_1'l_3\end{aligned} \tag{41}$$

The PID controller parameters determined from equation (41) are

$$\begin{aligned}&k_c=\frac{\sqrt{2\lambda}f}{g_1^3},\quad T_i=\frac{2f}{\lambda h},\quad T_d=\frac{e}{2\lambda f}\\&f=(g_1^2g_1'+g_1^2g_2'-g_1^2g_3'-g_1g_2g_1'+g_1g_2g_2'-g_1'g_2^2\\&+g_1g_3g_1')l_1+(g_1^2g_1'-g_1^2g_2'+g_1g_2g_1')l_2-g_1^2g_1'l_3,\\&h=(2g_1^2g_1'-2g_1^2g_2'+g_1^2g_3'+2g_1g_2g_1'-g_1g_2g_2'+\\&g_1'g_2^2-g_1g_3g_1')l_1+(g_1^2g_2'-2g_1^2g_1'-g_1g_2g_1')l_2+g_1^2g_1'l_3,\\&e=(g_2^2g_1'+g_1^2g_3'-g_1g_2g_2'-g_1g_3g_1')l_1+\\&(g_1^2g_2'-g_1g_2g_1')l_2+g_1^2g_1'l_3\end{aligned} \tag{42}$$

4. SELECTION OF THE LAGUERRE BASIS FUNCTION POLE

The proposed method is based on neglecting higher terms in the open loop gain Laguerre series expansion. Therefore, the closed loop system stability couldn't be guaranteed in the general case. To assure the closed loop stability, all of the closed loop poles should lie in the open left-half plane (LHP). This means that the closed loop characteristic equation should be a Hurwitz polynomial.

For the PI controller, the closed loop characteristic equation becomes

$$k_c(1+T_is)p(s)(1-\theta s)+T_isq_s(s)q_u(s)=0 \tag{43}$$

The closed loop characteristic polynomial for the PD controller is

$$k_c(1+T_ds)p(s)(1-\theta s)+q_s(s)q_u(s)=0 \tag{44}$$

The closed loop characteristic equation obtained for the PID controller is

$$k_c(1+T_iT_ds^2+T_is)p(s)(1-\theta s)+T_isq_s(s)q_u(s)=0 \tag{45}$$

The Laguerre basis function pole (λ) should be appropriately selected such that the polynomials (43), (44) and (45) have no roots in the open right-half plane (RHP). This means that the pole of the Laguerre function should be located in some appropriate ranges to assure the stability requirements. But, what is the optimal pole location? The Laguerre basis function pole location should be adjusted so that the difference between the desired closed loop unit step response ($y_d(t)$) and the closed loop unit step response ($y(t)$) is minimized. This means that the following Integral Square Error (ISE) criteria should be minimized

$$J=\frac{\int_0^T(y(t)-y_d(t))^2\,dt}{\int_0^T y_d^2(t)dt} \tag{46}$$

where T could be considered as the settling time of the closed loop system step response. Optimizing the cost function (46), improves the transient step response of the closed loop system but increases the control signal $u(t)$. The control signal must have permissible values considering practical limitations such as actuator saturation and so on. Therefore, the following constrained optimization problem should be solved

$$\begin{aligned}&\min:\ J(\lambda)\\&s.t:\ u^-<u(t)<u^+\end{aligned} \tag{47}$$

where u^+ and u^- are the upper and lower admissible bounds for the control signal which are determined according to the practical limitations. The ideal open loop gain parameter ω_n and the Laguerre pole location (λ) should be appropriately selected to minimize the loss function (47).

5. APPLYING THE PROPOSED METHOD ON DIFFERNET PLANT MODELS

In this section, the ability of the proposed Laguerre based PI, PD and PID controllers to control a variety of plants is verified. Moreover, the sufficient conditions in terms of the Laguerre pole location (λ) and other design parameters to assure the closed loop stability are derived.

5.1 PI and PID controller design for the First-Order (FO) plants

Consider a FO plant with the following transfer function.

$$G(s)=\frac{k}{\tau s+1}, \qquad \tau>0 \tag{48}$$

where k and τ are the steady state gain and the time constant of the plant, respectively.

According to (7) and (10), the first three Laguerre series coefficients of (48) are obtained as

$$g_1 = \frac{k\sqrt{2\lambda}}{\tau\lambda+1}, g_2 = \frac{k\sqrt{2\lambda}(1-\tau\lambda)}{(\tau\lambda+1)^2}, g_3 = \frac{k\sqrt{2\lambda}(1-\tau\lambda)^2}{(\tau\lambda+1)^3} \quad (49)$$

According to (28), the following PI controller parameters are obtained

$$k_c = \frac{\omega_n^2(2\eta\omega_n\tau-1)}{k(\lambda+2\eta\omega_n)^2}, \qquad T_i = \frac{2\eta\omega_n\tau-1}{\tau\lambda^2+2\lambda+2\eta\omega_n} \quad (50)$$

Regarding the plant transfer function in (48), the closed loop characteristic polynomial in (43) could be written as

$$\tau s^2 + (1+kk_c)s + \frac{kk_c}{T_i} = 0 \quad (51)$$

According to (51), the closed loop system stability could be guaranteed if the following conditions will be satisfied

$$\frac{kk_c}{T_i} > 0, \qquad 1+kk_c > 0 \quad (52)$$

Incorporating equations (50) and (52) yields the following relations

$$\frac{kk_c}{T_i} = \frac{\omega_n^2(\tau\lambda^2+2\lambda+2\eta\omega_n)}{(\lambda+2\eta\omega_n)^2} \quad (53)$$

$$1+kk_c = \frac{\lambda^2+4\eta\omega_n\lambda+(4\eta^2-1)\omega_n^2+2\eta\omega_n^3\tau}{(\lambda+2\eta\omega_n)^2} \quad (54)$$

Equation (53) is always positive for positive design parameters. Equation (54) is also positive considering $0.5 \le \eta \le 1$ (according to 16). Therefore, the proposed Laguerre based PI controller could assure the closed loop stability for all values of parameter λ for FO plants.

The PID controller parameters obtained from (40) are

$$k_c = \frac{\omega_n^2(2\eta\omega_n\tau-1)(3\lambda+2\eta\omega_n)}{k(\lambda+2\eta\omega_n)^3},$$
$$T_i = \frac{(2\eta\omega_n\tau-1)(3\lambda+2\eta\omega_n)}{\tau\lambda^3+3\lambda^2+6\eta\omega_n\lambda+4\eta^2\omega_n^2}, \ T_d = \frac{-1}{3\lambda+2\eta\omega_n} \quad (55)$$

The characteristic polynomial of the closed loop system including the PID controller in (22) and the plant model in (48) could be computed as

$$(kk_cT_d+\tau)s^2 + (1+kk_c)s + \frac{kk_c}{T_i} = 0 \quad (56)$$

The polynomial in (56) is Hurwitz if the following conditions will be meet

$$kk_cT_d+\tau > 0, \quad 1+kk_c > 0, \quad \frac{kk_c}{T_i} > 0 \quad (57)$$

According to (55), the relations in (57) are calculated as

$$\frac{kk_c}{T_i} = \frac{\omega_n^2(\tau\lambda^3+3\lambda^2+4\eta^2\omega_n^2+6\lambda\eta\omega_n)}{(\lambda+2\eta\omega_n)^3} \quad (58)$$

$$1+kk_c = \frac{f(\lambda)}{(\lambda+2\eta\omega_n)^3}, \quad f(\lambda) = \lambda^3+6\eta\omega_n\lambda^2+$$
$$(3\omega_n^2(4\eta^2-1)+6\eta\omega_n^3\tau)\lambda+4\eta^2\omega_n^4\tau+2\eta\omega_n^3(4\eta^2-1) \quad (59)$$

$$kk_cT_d+\tau =$$
$$\frac{\tau\lambda^3+6\eta\omega_n\tau\lambda^2+12\eta^2\omega_n^2\tau\lambda+2\eta\omega_n^3\tau(4\eta^2-1)+\omega_n^2}{(\lambda+2\eta\omega_n)^3} \quad (60)$$

According to (16), it is obvious that the right-side of equations (58), (59) and (60) are always positive. Therefore, a Laguerre based PID controller for the FO plants leads to a stable closed loop system for all values of λ. The following example shows the performance of the proposed method in angular velocity control of a DC servomechanism.

Example 1: to verify the robust performance of the proposed design method in an actual situation, a DC servomechanism system is considered. Fig. 1 shows the DC servomotor control system components. A permanent magnet DC motor coupled with a tachometer to measure its angular velocity is considered. Advantech A/D and D/A interfaces are utilized to implement the designed controllers in MTLAB real time environment. To verify the effect of the load torque, a magnetic brake causing angular velocity decrement is applied.

The transfer function of the plant (obtained from a non-parametric identification approach) is calculated as

$$G(s) = \frac{\omega(s)}{V(s)} = \frac{3.45}{0.1s+1} \quad (61)$$

where $\omega(t)$ is the motor angular velocity and $v(t)$ is the voltage applied to the servo amplifier block. The reference angular velocity is selected as $1000\,RPM$. To have a well damped transient response four times faster than the plant step response, the following design parameters are considered

$$\eta = 0.7, \ \omega_n = \frac{4}{0.7\times0.1} = 57.1429\frac{rad}{\sec} \quad (62)$$

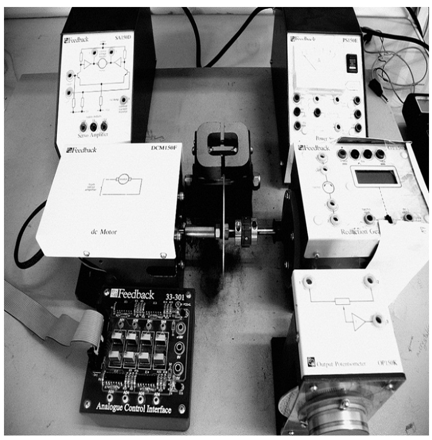

Fig. 1. The dc servomotor plant in Example 1.

The upper and lower bounds for the voltage applied to the servo amplifier are considered as -2.5 and 2.5 volts. The MATLAB FMINCON function is employed to solve the optimization problem in (47). The optimal choices for the Laguerre pole location and the resultant PI and PID controllers with their corresponding loss functions are given in Table. 1.

Table. 1. Controller properties corresponding to examples 1-5.

	$C(s)$	λ	J
Example 1, PI	$0.85(1+\frac{1}{0.068s})$	7.98	0.0021
Example 1, PID	$1+\frac{1}{0.084s}-\frac{0.009s}{1+0.01s}$	10	0.000442
Example 2, PD	$0.019(1+\frac{35.1s}{1+0.35s})$	0.26	0.0014
Example 3, PD	$0.106(1+\frac{3.53s}{1+0.1768s})$	1.078	0.000562
Example 4, PID	$6.01(1+\frac{1}{6.8s}-\frac{0.011s}{1+0.01s})$	11.95	0.00044
Example 5, PI	$0.1(1+\frac{1}{0.44s})$	0.23	0.000545

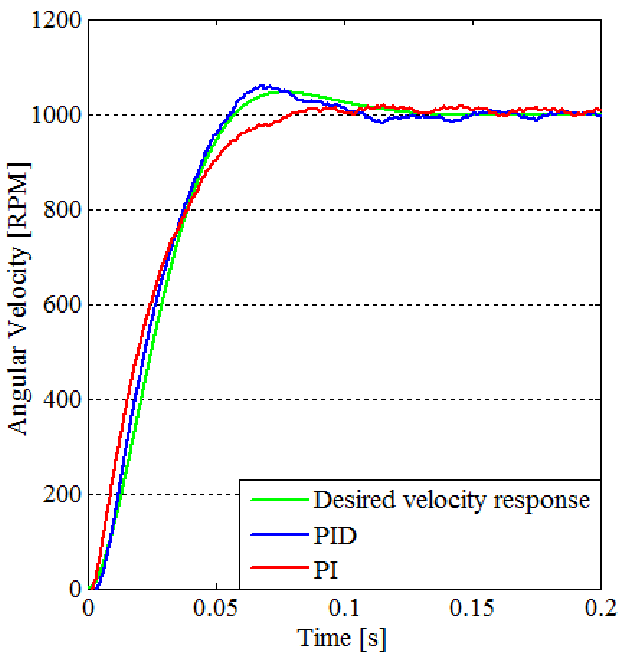

Fig. 2. The closed loop step responses in Example 1.

To implement controllers (22) and (32), the term $T_d s$ is replaced with $\frac{T_d s}{1+\beta s}$ in which β is a small positive constant. This extra pole is far enough from the imaginary axis. Thus, it has no significant effect on the dominant closed loop poles and on the closed loop system step response. Fig. 2 compares the motor angular velocity with the desired one in PI and PID cases. The PID controller leads to better performance comparing with PI controller as shown in Fig. 2. The minimum and maximum control signal admissible values are fulfilled as could be seen in Fig. 3. The load torque ($T_L = 0.16\,N.m$) applied in $t = 0.4$ sec is immediately eliminated as could be seen in Fig. 4.

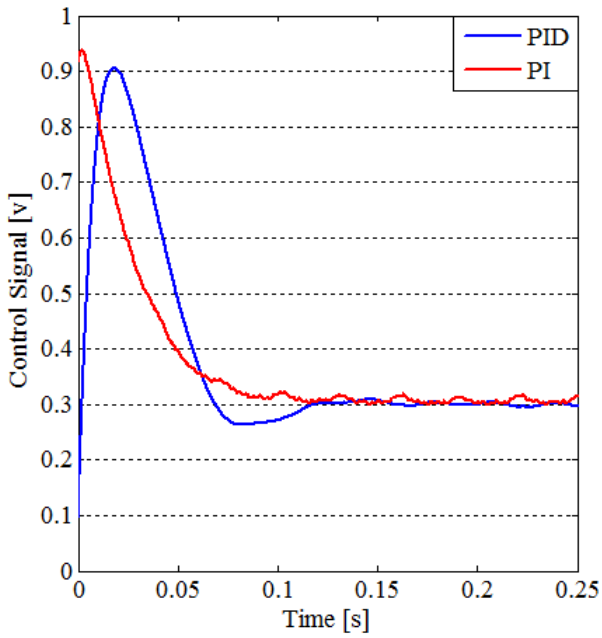

Fig. 3. The control signals in Example 1.

5.2 PD controller design for double-integrator plants

Consider a double-integrator plant with the following transfer function

$$G(s) = \frac{k}{s^2} \tag{63}$$

The plant is unstable and should be represented in the form of (21). Or:

$$G(s) = \frac{\frac{k}{(s+\lambda)^3}}{\frac{s^2}{(s+\lambda)^3}} = \frac{\sum_{i=1}^{\infty} g_i \varphi_i(s)}{\sum_{i=1}^{\infty} g'_i \varphi_i(s)} \tag{64}$$

The first two Laguerre series coefficients in (64) are obtained as

$$g_1 = \frac{k}{4\lambda^2\sqrt{2\lambda}},\ g_2 = \frac{-k}{2\lambda^2\sqrt{2\lambda}},\ g_1{}' = \frac{1}{4\sqrt{2\lambda}},\ g_2{}' = \frac{1}{2\sqrt{2\lambda}} \tag{65}$$

According to (36), the PD controller parameters are determined as

$$k_c = \frac{\lambda^2\omega_n^2}{k(\lambda+2\eta\omega_n)^2},\quad T_d = \frac{2\eta\omega_n}{\lambda^2} \tag{66}$$

The closed loop characteristic equation is given by

$$s^2 + kk_c T_d s + kk_c = 0 \quad (67)$$

Therefore, the closed loop system stability conditions are

$$kk_c T_d > 0, \qquad kk_c > 0 \quad (68)$$

According to relations (66) and (68), it is obvious that the closed loop system is always stable regardless of the value of λ. The next example illustrates the ability of the proposed PD controller in stabilization of a double-integrator plant.

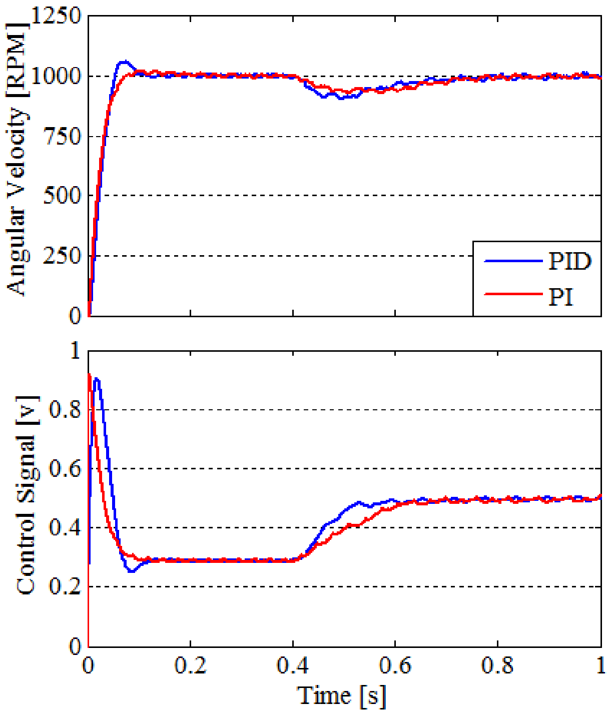

Fig. 4. The closed loop step responses with $0.16\,N.m$ load torque applying in $t = 0.4$ sec in Example 1.

Example 2: consider the plant (63) with $k = 2$. The design parameters are selected as

$$\omega_n = 2, \eta = 0.6, u^+ = 5, u^- = -5 \quad (69)$$

The optimal value of the parameter λ and the PD controller parameters are given in Table 1. Fig. 5 shows that the closed loop system unit step response is closely similar to the desired step response. Moreover, Fig. 5 shows that the control signal is in the admissible range determined by the designer.

5.3 PD controller design for third-order integrating plants

Consider the following third-order integrating plant with two real poles.

$$G(s) = \frac{k}{s(1+\tau_1 s)(1+\tau_2 s)}, \qquad \tau_1, \tau_2 > 0 \quad (70)$$

The equation (70) should be rewritten in the following Laguerre series form

$$G(s) = \frac{\dfrac{k}{(1+\tau_1 s)(1+\tau_2 s)(s+\lambda)^2}}{\dfrac{s}{(s+\lambda)^2}} = \frac{\sum_{i=1}^{\infty} g_i \varphi_i(s)}{\sum_{i=1}^{\infty} g'_i \varphi_i(s)} \quad (71)$$

where

$$g_1 = \frac{k}{2\lambda\sqrt{2\lambda}(1+\lambda\tau_1)(1+\lambda\tau_2)},$$
$$g_2 = \frac{-k(5\tau_1\tau_2\lambda^2 + 3(\tau_1+\tau_2)\lambda + 1)}{2\lambda\sqrt{2\lambda}(1+\lambda\tau_1)^2(1+\lambda\tau_2)^2}, g'_1 = g'_2 = \frac{1}{2\sqrt{2\lambda}} \quad (72)$$

The PD controller parameters calculated according to (36) are

$$k_c = \frac{\omega_n^2((\tau_1+\tau_2-2\eta\omega_n\tau_1\tau_2)\lambda^2 + 2\lambda + 2\eta\omega_n)}{k(\lambda+2\eta\omega_n)^2},$$
$$T_d = \frac{\tau_1\tau_2\lambda^2 + 4\eta\omega_n\tau_1\tau_2\lambda + 2\eta\omega_n(\tau_1+\tau_2) - 1}{(\tau_1+\tau_2-2\eta\omega_n\tau_1\tau_2)\lambda^2 + 2\lambda + 2\eta\omega_n} \quad (73)$$

The closed loop system characteristic equation is obtained as

$$\tau_1\tau_2 s^3 + (\tau_1+\tau_2)s^2 + (1+kk_c T_d)s + kk_c = 0 \quad (74)$$

The closed loop system is stable if and only if the following inequalities are fulfilled

$$1 + kk_c T_d > 0, \ \ kk_c > 0, \ \ (\tau_1+\tau_2)(1+kk_c T_d) - \tau_1\tau_2 kk_c > 0 \quad (75)$$

According to (73), the left side of relations in (75) are calculated as

$$1 + kk_c T_d = \frac{h(\lambda)}{(\lambda+2\eta\omega_n)^2}, \quad h(\lambda) = (1+\omega_n^2\tau_1\tau_2)\lambda^2$$
$$+4\eta\omega_n(1+\omega_n^2\tau_1\tau_2)\lambda + \omega_n^2(4\eta^2 - 1 + 2\eta\omega_n(\tau_1+\tau_2)) \quad (76)$$

$$kk_c = \frac{\omega_n^2((\tau_1+\tau_2-2\eta\omega_n\tau_1\tau_2)\lambda^2 + 2\lambda + 2\eta\omega_n)}{(\lambda+2\eta\omega_n)^2} \quad (77)$$

$$(\tau_1+\tau_2)(1+kk_c T_d) - \tau_1\tau_2 kk_c = \frac{\psi(\lambda)}{(\lambda+2\eta\omega_n)^2}$$
$$\psi(\lambda) = (\tau_1+\tau_2+2\eta\omega_n^3\tau_1^2\tau_2^2)\lambda^2 +$$
$$\frac{\omega_n}{4\eta(\tau_1+\tau_2)}(\tau_1\tau_2(4\eta\omega_n(\tau_1+\tau_2)-1)^2 + \quad (78)$$
$$16\eta^2(\tau_1^2+\tau_2^2) + (32\eta^2-1)\tau_1\tau_2)\lambda +$$
$$\omega_n^2((4\eta^2-1)(\tau_1+\tau_2) + 2\eta\omega_n(\tau_1^2+\tau_1\tau_2+\tau_2^2))$$

It is obvious that right sides of equations (76) and (78) are always positive. Therefore, the closed loop system is stable if the right side of equation (77) is positive. This results in the following stability conditions

$$(\nabla > 0) \quad or \; ((\nabla < 0) \quad and \quad (\lambda < \frac{-(1+\sqrt{1-2\eta\omega_n\nabla})}{\nabla})) \qquad (79)$$

$$\nabla = \tau_1 + \tau_2 - 2\eta\omega_n\tau_1\tau_2$$

The closed loop system performance with the so-called Laguerre based PD controller is verified in example 3.

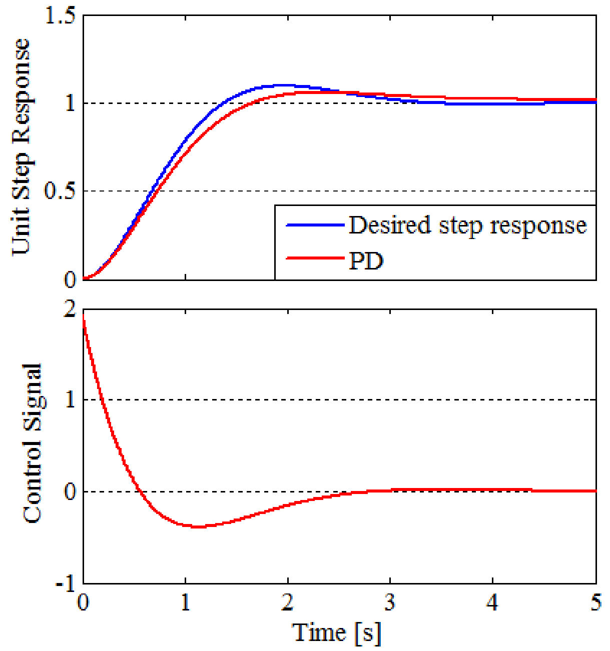

Fig. 5. The closed loop unit step responses and the control signal in Example 2.

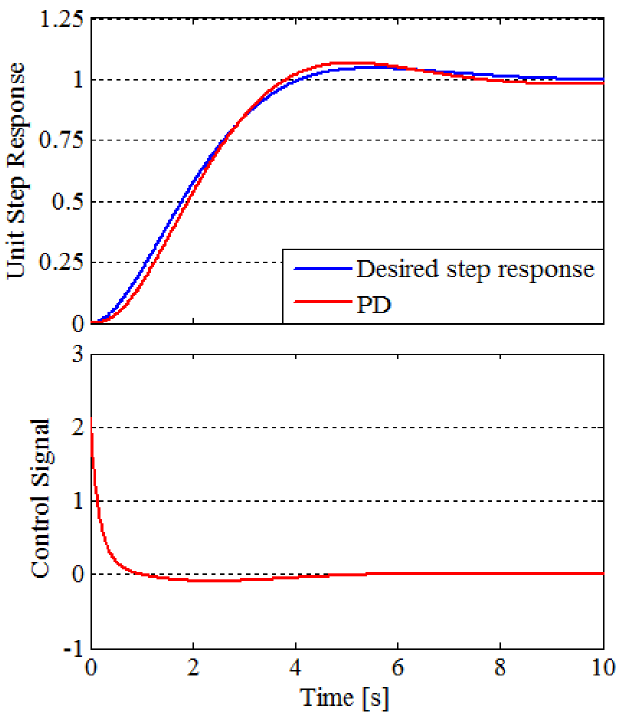

Fig. 6. The closed loop unit step responses and the control signal in Example 3.

Example 3: in this example, the plant parameters in (70) are considered as $k = 5$, $\tau_1 = 1$ and $\tau_2 = 3$. The design parameters are chosen as

$$\omega_n = 0.8, \eta = 0.7, u^+ = 5, u^- = -5 \qquad (80)$$

Table 1 shows the optimal value of the Laguerre basis function and the corresponding parameters for the PD controller. Fig.6 shows that the closed loop system step response is similar to the desired step response, as well. Also, the control signal is in the permissible range.

5.4 PID controller design for an unstable first-order plant

In this section, the ability of the Laguerre base PID controller to control an unstable plant is verified. The following unstable first-order plant is considered

$$G(s) = \frac{k}{(\tau s - 1)}, \qquad \tau > 0 \qquad (81)$$

The transfer function (81) should be modified as

$$G(s) = \frac{\dfrac{k}{(s+\lambda)^2}}{\dfrac{\tau s - 1}{(s+\lambda)^2}} = \frac{\sum_{i=1}^{\infty} g_i \varphi_i(s)}{\sum_{i=1}^{\infty} g'_i \varphi_i(s)} \qquad (82)$$

The first three Laguerre series coefficients in (82) are computed as

$$g_1 = -g_2 = \frac{k}{2\lambda\sqrt{2\lambda}}, \quad g_3 = g'_3 = 0,$$
$$g'_1 = \frac{\tau\lambda - 1}{2\lambda\sqrt{2\lambda}}, \quad g'_2 = \frac{\tau\lambda + 1}{2\lambda\sqrt{2\lambda}} \qquad (83)$$

The PID controller parameters obtained from (42) are

$$k_c = \frac{\omega_n^2(2\eta\omega_n\tau + 1)(2\eta\omega_n + 3\lambda)}{k(\lambda + 2\eta\omega_n)^3},$$
$$T_i = \frac{(2\eta\omega_n\tau + 1)(2\eta\omega_n + 3\lambda)}{\tau\lambda^3 - 3\lambda^2 - 6\eta\omega_n\lambda - 4\eta^2\omega_n^2}, \quad T_d = \frac{-1}{2\eta\omega_n + 3\lambda} \qquad (84)$$

The closed loop system characteristic equation is obtained as

$$(kk_cT_d + \tau)s^2 + (kk_c - 1)s + \frac{kk_c}{T_i} = 0 \qquad (85)$$

The closed loop system stability conditions are

$$kk_cT_d + \tau > 0, \qquad kk_c - 1 > 0, \qquad \frac{kk_c}{T_i} > 0 \qquad (86)$$

The left sides of inequalities (86) are obtained according to (84) as follows

$$kk_cT_d + \tau = \frac{\tau(\lambda + 2\eta\omega_n)^3 - \omega_n^2(2\eta\omega_n\tau + 1)}{(\lambda + 2\eta\omega_n)^3} \qquad (87)$$

$$kk_c - 1 = \frac{(\omega_n^2(2\eta\omega_n\tau+1)-(\lambda+2\eta\omega_n)^2)}{(\lambda+2\eta\omega_n)^2} + \frac{2\lambda\omega_n^2(2\eta\omega_n\tau+1)}{(\lambda+2\eta\omega_n)^3} \quad (88)$$

$$\frac{kk_c}{T_i} = \frac{\omega_n^2(\tau\lambda^3-3\lambda^2-6\eta\omega_n\lambda-4\eta^2\omega_n^2)}{(\lambda+2\eta\omega_n)^3} \quad (89)$$

According to relations (86), (87), (88) and (89), the following closed loop system stability conditions are derived

$$(\lambda < \omega_n(\sqrt{2\eta\omega_n\tau+1}-2\eta) \ \ and \ \ (\omega_n > \frac{4\eta^2-1}{2\eta\tau})$$
$$and \ \ (\lambda > \sqrt[3]{\frac{\omega_n^2(2\eta\omega_n\tau+1)}{\tau}} - 2\eta\omega_n) \quad (90)$$
$$and \ \ (\tau\lambda^3-3\lambda^2-6\eta\omega_n\lambda-4\eta^2\omega_n^2>0)$$

To investigate the usefulness of the mentioned PID controller, the next example is given.

Example 4: consider the unstable plant in (81) with $k=10$ and $\tau=5$. The following design parameters are considered

$$\omega_n = 26.35, \eta=1, u^+=5, u^-=-5 \quad (91)$$

The optimal value of λ and PID controller parameters are shown in Table 1. The desired step response, the closed loop system step response and the control signal shown in Fig.7 demonstrate that the closed loop system is stable and control signal constraints are satisfied.

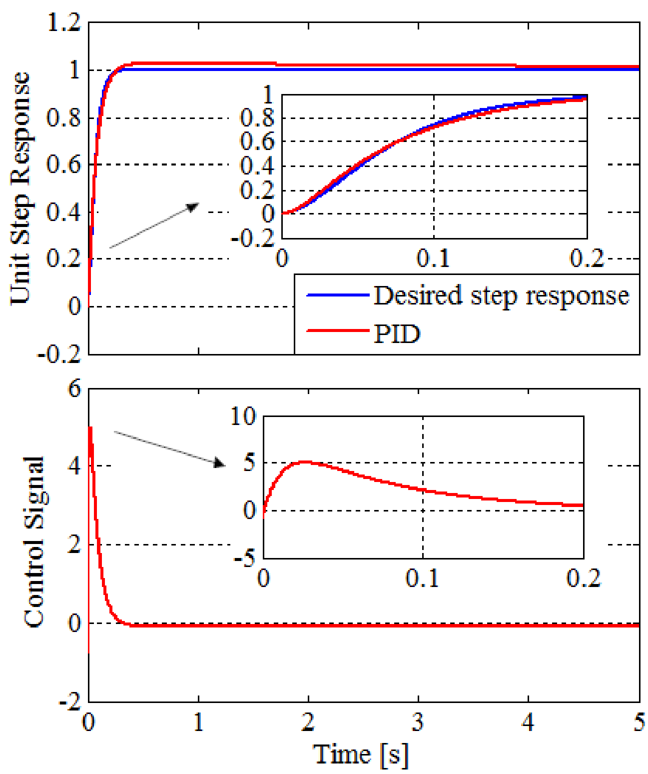

Fig. 7. The closed loop unit step responses and the control signal in Example 4.

5.5 PI controller design for FOPDT plants

In this section, the efficiency of the proposed Laguerre based controller to control time-delay systems is investigated. Consider the following FOPDT plant

$$G(s) = \frac{ke^{-\theta s}}{(1+\tau s)}, \qquad \tau,\theta>0 \quad (92)$$

The proposed method could be utilized only for rational transfer functions. Thus, the delay term in (92) should be approximated as

$$e^{-\theta s} \approx 1-\theta s \quad (93)$$

According to (93), the following approximated model for the plant (92) is obtained

$$\tilde{G}(s) = \frac{k(1-\theta s)}{(1+\tau s)}, \qquad \tau,\theta>0 \quad (94)$$

The approximated model in (94) isn't a strictly proper transfer function and should be converted to the following form

$$\tilde{G}(s) = \frac{\frac{k(1-\theta s)}{(s+\lambda)^2}}{\frac{1+\tau s}{(s+\lambda)^2}} = \frac{\sum_{i=1}^{\infty} g_i\varphi_i(s)}{\sum_{i=1}^{\infty} g'_i\varphi_i(s)} \quad (95)$$

where

$$g_1 = \frac{k(1-\theta\lambda)}{2\lambda\sqrt{2\lambda}}, \quad g_2 = \frac{-k(1+\theta\lambda)}{2\lambda\sqrt{2\lambda}},$$
$$g_3 = g'_3 = 0, \quad g'_1 = \frac{1+\tau\lambda}{2\lambda\sqrt{2\lambda}}, \quad g'_2 = \frac{\tau\lambda-1}{2\lambda\sqrt{2\lambda}} \quad (96)$$

The Laguerre based PI controller parameters calculated according to (31) are

$$k_c = \frac{\omega_n^2(2\eta\omega_n\tau+\omega_n^2\theta\tau-1)}{k(\lambda+2\eta\omega_n+\omega_n^2\theta)^2}, T_i = \frac{2\eta\omega_n\tau+\omega_n^2\theta\tau-1}{\tau\lambda^2+2\lambda+2\eta\omega_n+\omega_n^2\theta} \quad (97)$$

The closed loop system characteristic equation considering the approximated model is

$$(\tau-kk_c\theta)s^2+(1+kk_c-\frac{\theta kk_c}{T_i})s+\frac{kk_c}{T_i}=0 \quad (98)$$

The approximated closed loop system is stable if the following inequalities are fulfilled

$$\frac{kk_c}{T_i}>0, \qquad \tau-kk_c\theta>0, \qquad 1+kk_c-\frac{\theta kk_c}{T_i}>0 \quad (99)$$

Considering the controller parameters in (97), the following relations for the left sides of inequalities (99) will be obtained

$$\frac{kk_c}{T_i} = \frac{\omega_n^2(\tau\lambda^2+2\lambda+2\eta\omega_n+\omega_n^2\theta)}{(\lambda+2\eta\omega_n+\omega_n^2\theta)^2} \quad (100)$$

$$\tau - kk_c\theta = \frac{\tau\lambda^2 + 2\tau(2\eta\omega_n + \omega_n^2\theta)\lambda + 2\eta\omega_n\tau(2\eta\omega_n + \omega_n^2\theta) + \omega_n^2\theta}{(\lambda + 2\eta\omega_n + \omega_n^2\theta)^2} \tag{101}$$

$$1 + kk_c - \frac{\theta kk_c}{T_i} = \frac{l(\lambda)}{(\lambda + 2\eta\omega_n + \omega_n^2\theta)^2},$$
$$l(\lambda) = \lambda^2 + (4\eta - \theta\tau\omega_n\lambda)\omega_n\lambda + \omega_n^2(4\eta^2 - 1 + 2\eta\omega_n\theta + 2\eta\omega_n\tau + \theta\tau\omega_n^2) \tag{102}$$

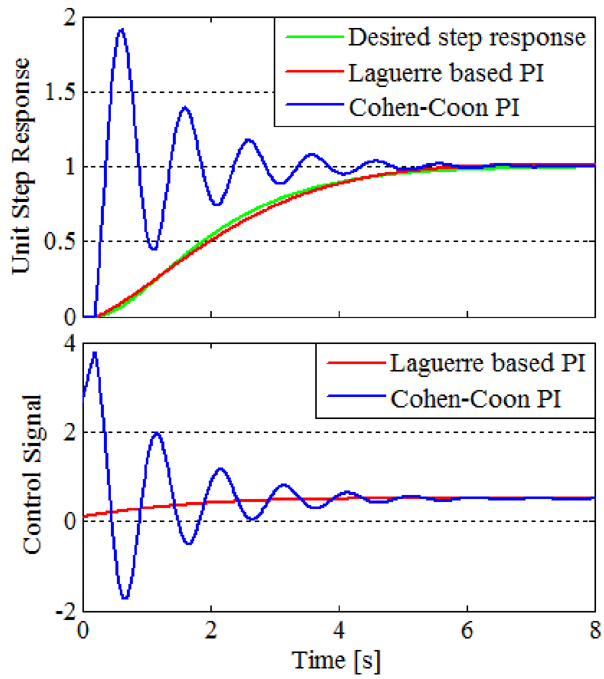

Fig. 8. The closed loop unit step responses and the control signal in Example 5.

It is obvious that relations (100) and (101) are always positive. Thus, the positivity of (102) results in the closed loop system stability. Finally, the sufficient condition for the closed loop system stability is

$$\lambda \le \frac{4\eta}{\theta\tau\omega_n} \tag{103}$$

The next example investigates the performance of the proposed PI controller to control FOPDT plants.

Example 5: consider the FOPDT plant (92) with $k = 2$, $\tau = 1$ and $\theta = 0.2$. The design parameters are selected as

$$\omega_n = 1, \eta = 1, u^+ = 1, u^- = 0 \tag{104}$$

The optimal value of λ and PI controller parameters are shown in Table 1. The PI controller is designed based on model (94) but is applied to plant (92). To compare the proposed PI performance with the common PI controllers for FOPDT plants, the Cohen-Coon PI method is selected for comparison (Cohen and Coon, 1953). In this method, the PI controller parameters are calculated according to the following relations

$$k_c = \frac{0.9}{a}\left(1 + \frac{0.92\chi}{1-\chi}\right), \quad T_i = \frac{(3.3 - 3\chi)\theta}{1 + 1.2\chi} \tag{105}$$

where

$$\chi = \frac{\theta}{\theta + \tau}, \quad a = \frac{k\theta}{\tau} \tag{106}$$

The following values are obtained for the Cohen-Coon PI controller parameters

$$k_c = 2.66, \; T_i = 0.466 \tag{107}$$

Fig. 8 compares the closed loop step response obtained from Laguerre based PI and the Cohen-Coon PI with the desired step response. The transient response of the Laguerre based PI method is much superior comparing with the Cohen-Coon PI. The control signals shown in Fig. 8 confirm the superiority of the proposed method in satisfying control signal limitations, too.

To verify the robust performance of the proposed PI controller, the plant parameters are increased 20% than their nominal values (one parameter is changed and the others are kept unchanged). Fig. 9 shows that 20% change in the plant parameters causes little change in the transient response which demonstrates the robust performance of the Laguerre based PI controller.

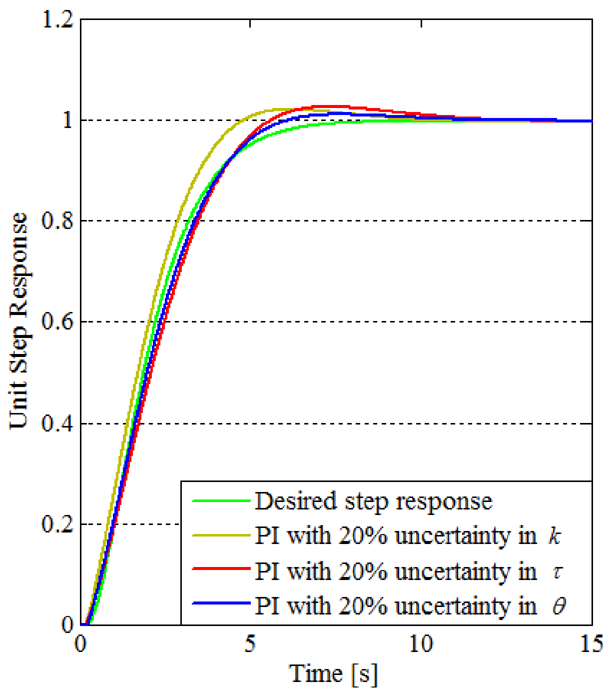

Fig. 9. The closed loop unit step responses with 20% increase in plant parameters in Example 5.

6. CONCLUSIONS

The proposed Laguerre series based approach could be utilized as an efficient analytical tool to design the family of PID controllers as well. The explicit relations obtained for the

PID controller parameters makes the proposed design method distinguished from other usual PID design methods.

The pole of the Laguerre basis function could be appropriately selected to ensure the closed loop system stability. Moreover, the optimal value of this parameter is calculated based on an optimization method. The performance of the proposed method is verified for a variety of plants with different transfer functions. The proposed method could be employed to control unstable, integrating and non-minimum phase plants. Employing other orthogonal functions to design PID controllers such as Legendre functions could be considered as a future research topic.

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