Stability Analysis of Fractional Order Multiple Linear Uncertain Systems: Exposed Edge Sampling Approach

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Abstract: This study presents a robust stability analysis method for fractional order systems with multiple linear uncertainty structures. A numerical method, which is based on transforming fractional orders to expanded integer orders, has been used for the robust stability check of fractional order multiple linear uncertain control systems. Scan sampling method is used to investigate exact root region of fractional order multiple linear uncertain systems that will be a criterion for robust stability. Types of linear uncertainties are represented and multiple versions of these uncertainties are studied in these analyses. Exposed edge sampling is used to decrease computational complexity of the robust stability analysis procedure. The proposed method is demonstrated on illustrative examples.

Keywords: Fractional-order systems, multiple linear uncertainties, robust stability, roots region, edge sampling

1. INTRODUCTION

Nowadays, fractional order calculus (FOC) drew great interest since its superior consistency to represent real world processes comparing to classical mathematics was discovered (Podlubny, 1999a; Podlubny, 1999b; Petras, 2009). Increasing number of studies related to FOC have been come up since the first idea about the topic in late 17th century. Since then, particularly in the last decades, FOC has found applications in numerous fields such as physics (Parada et al., 2007; Rudolf, 2000), mathematics (Miller and Ross, 1993; Gutierraz et al., 2010), chemistry (Oldham and Spanier, 1974), electrics (Arena et al., 2000), bioengineering (Magin, 2012), signal processing (Vinagre et al., 2003) etc. Last decades also brought so many applications of FOC on control theory (Caponetto et al., 2010; Petras et al., 2002; Chen et al., 2009; Caponetto and Dongola., 2013).

One of the challenging issues in control practice is the uncertainties in control systems. Uncertainties in parameters or models mostly yield to unpredictable results in that, responses of simulations and real processes may differ from each other casually (Senol et al., 2014). In order to prevent the system from this unwanted behavior, one has to assure the system's response robustly within possible ranges of system parameters. Robust control of classical systems has been widely studied in the literature (Bhattacharyya et al., 1995; Matusu and Prokop, 2011; Bhattacharyya et al, 2010) as well as systems of fractional orders (Tan et al., 2009; Yeroglu et al., 2010; Senol and Yeroglu, 2013; Senol and Yeroglu, 2012a; Senol and Yeroglu, 2012b). Various types of uncertainties have been taken account in many works. A study related to the stability of fractional order polynomials (FOPs) with different types of uncertainties using the value set concept is presented in (Senol and Yeroglu, 2012b). A simplification method related to frequency response computation with uncertain parameters (Senol and Yeroglu, 2012a; Senol and Yeroglu, 2012b; Shaw and Jayasuriya, 1996; Tan and Atherton, 2000) and another frequency response study of uncertain systems with interval plants is given in (Yeroglu et al., 2010). Robust stability analysis of linear fractional order systems (FOS) with interval uncertainties can be found in (Senol et al., 2014). Magnitude and phase envelopes of control systems with affine linear uncertainty are shown in (Tan and Atherton, 1998). There are also some studies on robust stability and stabilization of uncertain FOS (Lu and Chen, 2010a; Lu and Chen, 2010b; Lu and Chen, 2009). Convex polytopic uncertainties are dealt with in (Lu and Chen, 2010a). FOS of the order $0<\alpha<1$ with interval uncertainties is studied in (Lu and Chen, 2010b) and a Linear Matrix Inequalities (LMI) approach on robust stability and stabilization of fractional order interval systems is presented in (Lu and Chen, 2009). A frequency response study on the stability of FOS with nonlinear uncertainties can also be found (Senol and Yeroglu, 2013). Stability analysis of FOS with interval coefficients and interval orders is given in[(Senol et al. 2014).

This paper deals with the roots region based stability analysis of FOS with multiple linear uncertainty structures which is composed of the multiplication of two or more FOS with linear uncertainties. There can be found some papers in the literature which studied different types of uncertainties (Tan, 2002; Hamelin and Boukhobza, 2005; Tan and Atherton, 2002; Yeroglu and Senol, 2013). Robust stability of multilinear affine polynomials is shown in (Tan and Atherton, 2002) and computation of the frequency response of multilinear affine systems is given in (Tan, 2002; Yeroglu and Senol, 2013). A simplified robust stability analysis of FOS with multilinear affine uncertainties can be found in

(Yeroglu and Senol, 2013) and a fault detection approach for multilinear affine systems is presented in (Hamelin and Boukhobza, 2005).

Stability analysis in this paper is based on the roots regions of the characteristic equation of the multiple linear FOS. Fractional orders of the characteristic equation are extended to minimal integer orders with a Least Common Multiplier (LCM). Then, roots of this new polynomial are computed and their absolute phase values are found. Stability is decided considering the locations of these roots. There can be found some studies based on the related method in the literature (Caponetto et al., 2010; Senol et al., 2014; Radwan et al., 2009). Roots region based stability analysis of FOS with interval coefficients and interval orders is presented in (Senol et al., 2014).

Computational complexity of a multiple linear system exponentially grows as the number of uncertain parameters increase; therefore, this is one of the most challenging problems of control systems with uncertainties. As a FOS with multiple linear uncertainties is built up of two or more uncertain FOS, it is possible to find great number of roots to form the roots region of such systems. This study utilizes the Edge theorem to decrease this complexity. Using the Edge Theorem, only the roots which are on the exposed edges of the roots region are computed, thus, the computational complexity is reduced substantially. There can be found wide application areas of the Edge Theorem in the literature (Senol et al., 2014; Bhattacharyya et al., 1995; Matusu and Prokop, 2011; Bhattacharyya et al., 2010; Senol and Yeroglu, 2012a; Senol and Yeroglu, 2012b). Usage of the Edge Theorem in roots region computation (Senol et al., 2014), value set computation (Bhattacharyya et al., 1995), Kharitonov's Theorem (Matusu and Prokop, 2011), etc. are some of the instances.

In this paper, Section 2 summarizes fractional order uncertain systems, linear uncertainty types, multiple linear uncertainty structure and roots region based stability analysis for fractional order interval polynomials. Stability investigation of fractional order multiple linear uncertain polynomials is studied in Section 3 and roots region of the characteristic equation of fractional order multiple linear uncertain systems is given in Section 4. Section 5 has illustrative examples and Section 6 has the concluding remarks.

2. PROBLEM STATEMENT AND PRELIMINARIES

2.1. Fractional-order uncertain systems with multiple linear uncertainty structures

This section gives brief information about fractional-order uncertain systems and multiple linear uncertainty structure. Consider the fractional-order polytopic polynomial family of the following form (Yeroglu and Senol, 2013).

$$P(s,\mathbf{q}) = I_0(\mathbf{q})s^{\alpha_0} + I_1(\mathbf{q})s^{\alpha_1} + \cdots + I_n(\mathbf{q})s^{\alpha_n} \tag{1}$$

where, the coefficients $I_i(\mathbf{q})$ linearly depend on $\mathbf{q} = [q_0, q_1, \cdots, q_q]^T$ and the uncertainty box is

$$Q = \{\mathbf{q} : q_i \in [\underline{q_i} \ \ \overline{q_i}], \quad i = 1, 2, \cdots, q\}. \tag{2}$$

$\underline{q_i}$ and $\overline{q_i}$ specify the lower and upper bounds of the i-th perturbation q_i respectively. $\alpha_0 < \alpha_1 < ... < \alpha_n$ are non-integer orders of the polynomial. Let $I_k(\mathbf{q})$ has linear uncertainty of general form. Then, uncertain parameters of Eq. 1 can be written as,

$$I_k(\mathbf{q}) = a_{k,0} + a_{k,1}q_1 + a_{k,2}q_2 + ... + a_{k,q}q_q \tag{3}$$

where, $a_{k,i}$ are constants and q_i are uncertain parameters ($k = 0,1,2,...,n$ and $i = 1,2,\cdots,q$).

Referring to Eq. 3, different types of $I_k(\mathbf{q})$ form different types of linear uncertainties. Three types of linear uncertainties have been represented in this paper. For instance, following form of $P(s,\mathbf{q})$ builds the single parameter uncertainty.

$$\begin{aligned} P(s,\mathbf{q}) = a_0 + \left(a_1 + b_1 q\right)s^{\alpha_1} + \left(a_2 + b_2 q\right)s^{\alpha_2} + \\ ... + \left(a_n + b_n q\right)s^{\alpha_n} \end{aligned} \tag{4}$$

where, a_i, b_i, $i = 1,2,...,n$ are constants and q is the uncertain parameter. Similarly, interval uncertainty can be defined as follows.

$$P(s,\mathbf{q}) = q_1 s^{\alpha_1} + q_2 s^{\alpha_2} + ... + q_n s^{\alpha_n} \tag{5}$$

where, q_i, $i = 1,2,...,n$ are uncertain parameters that lay in the uncertainty interval $\left[\underline{q}_i, \bar{q}_i\right]$. $\underline{q}_i$ and $\bar{q}_i$ are the lower and the upper limits of the uncertain parameters respectively. Similarly, following form of $P(s,\mathbf{q})$ can be named as affine linear uncertainty structure.

$$\begin{aligned} P(s,\mathbf{q}) = (a_{1,0} + a_{1,1}q_1 + a_{1,2}q_2 + ... + a_{1,k}q_k)s^{\alpha_1} + \\ (a_{2,0} + a_{2,1}q_1 + a_{2,2}q_2 + ... + a_{2,k}q_k)s^{\alpha_2} + \\ ... + (a_{n,0} + a_{n,1}q_1 + a_{n,2}q_2 + ... + a_{n,k}q_k)s^{\alpha_n} \end{aligned} \tag{6}$$

where, $a_{i,j}$ are constants, α_i are arbitrary real orders and q_j are uncertain parameters, $i = 1,2,...,n$, $j = 1,2,...,k$. As a fractional order uncertain plant can have fractional order uncertain polynomials in its numerator and denominator, one can say that a fractional order plant with any linear uncertainty structure can be written as follows.

$$\begin{aligned} G(s,\mathbf{a}_i,\mathbf{b}_k) = \frac{N_k(s,\mathbf{b}_k)}{D_i(s,\mathbf{a}_i)} = \\ \frac{N_0(\mathbf{b})s^{\alpha_{N0}} + N_1(\mathbf{b})s^{\alpha_{N1}} + \cdots + N_m(\mathbf{b})s^{\alpha_{Nm}}}{D_0(\mathbf{a})s^{\alpha_{D0}} + D_1(\mathbf{a})s^{\alpha_{D1}} + \cdots + D_n(\mathbf{a})s^{\alpha_{Dn}}} \end{aligned} \tag{7}$$

where, $N_k(s,\mathbf{b}_k)$ and $D_i(s,\mathbf{a}_i)$ are fractional order uncertain polynomials of the numerator and the denominator respectively ($i = 0,1,2,\cdots,n$ and $k = 0,1,2,\cdots,m$).

Now, consider a fractional order polytopic family of polynomials with the multiple linear uncertainty structure, which is multiplication of two or more linear uncertain family of polynomials as follows.

$$F(s,\mathbf{h}) = A(s)P_1(s,\mathbf{h}_1)P_2(s,\mathbf{h}_2)\cdots P_n(s,\mathbf{h}_n) \quad (8)$$

where, $\mathbf{h}_i = [q_{i1}, q_{i2},\cdots,q_{ik}]$, $A(s)$ is a fixed polynomial and $P_i(s,\mathbf{h}_i)$, $i = 1,2,\cdots,n$ are polynomials with any linear uncertainty structure. Thus, considering Eq. 7 and Eq. 8, a fractional order system with multiple linear uncertainty structure can be given in the following form.

$$L(s,\mathbf{c}) = \frac{L_N(s,\mathbf{b})}{L_D(s,\mathbf{a})} = \frac{C_N(s)}{C_D(s)} \cdot \frac{P_{N1}(s,\mathbf{b_1})P_{N2}(s,\mathbf{b}_2)...P_{Nm}(s,\mathbf{b_m})}{P_{D1}(s,\mathbf{a_1})P_{D2}(s,\mathbf{a_2})...P_{Dn}(s,\mathbf{a_n})} \quad (9)$$

where, $C_N(s)$ and $C_D(s)$ are numerator and denominator polynomials of the controller, $P_{Ni}(s,\mathbf{b}_i)$ and $P_{Di}(s,\mathbf{a}_i)$, $i = 1,2,...,t$ are numerator and denominator polynomials of the plant with any linear uncertainty and $\mathbf{c} = [\mathbf{b}_1,\mathbf{b}_2,...\mathbf{b}_m,\mathbf{a}_1,\mathbf{a}_2,...\mathbf{a}_n]$. Closed loop characteristic equation of the system in Eq. 9 can be given in the following form.

$$\begin{aligned}\Delta(s,\mathbf{c}) = C_N(s)P_{N1}(s,\mathbf{b_1})P_{N2}(s,\mathbf{b}_2)...P_{Nm}(s,\mathbf{b}_m) + \\ C_D(s)P_{D1}(s,\mathbf{a_1})P_{D2}(s,\mathbf{a_2})...P_{Dn}(s,\mathbf{a}_n)\end{aligned} \quad (10)$$

Fig. 1 shows the typical block diagram of a fractional order system with multiple linear uncertainty [31, 34].

2.2 Exposed Edge Based Stability Analysis for Fractional Order Interval Polynomials

This section introduces the exposed edge stability analysis to fractional order interval systems (Senol et al., 2014). Consider the fractional order interval polynomial in Eq. 5. In order to obtain the roots region, one can apply $s = v^m$ mapping to the fractional order polynomial. This converts the fractional orders to expanded integer orders, thus, the fractional order stability problem turns into an integer order stability problem considering the Riemann sheets. Assuming $s = v^m$, m sheets of Riemann surfaces are defined in the complex root space (Chen et al., 2006; Ahn et al., 2007; Monje et al., 2010). Then, the stability analysis can be performed by root locus analysis of the expanded integer order polynomials in their root spaces. For $s = v^m$ transformation, expanded integer order form of Eq. 5 can be written as follows.

$$P^m(v,\mathbf{q}) = q_1 v^{\beta_1} + q_2 v^{\beta_2} + ... + q_n v^{\beta_n} \quad (11)$$

where, $P^m(v,\mathbf{q})$ is the expanded order polynomial and $\beta_i = m\alpha_i$ are expanded integer orders of the polynomial, $i = 1,2,\cdots,n$. $\mathbf{q} = [q_1, q_2,\cdots,q_n]$ is the set of uncertain parameters. Then, lower and upper limits of the uncertain parameter vector $\mathbf{q}$ define a hypercube Q in the coefficient space as follows.

$$Q = \{q : \underline{q}_i \le q_i \le \overline{q}_i, i = 1,2,\cdots,n\} \quad (12)$$

Let $\mathbf{q}$ be the set of uncertain parameters of the polynomial in Eq. 1 and consider the hypercube given in Eq. 12. Hypercube Q represents a polynomial family defined by perturbations of the uncertain parameter q in lower and upper bounds of elements. Let us denote this polynomial family set by $\Omega \in R^n$ .Then, the following theorem holds.

Theorem 1: The complex roots of the polynomial family Ω forms a set of root points which compose a roots region as,

$$R(\Omega) = \{v_r : P_u^m(q,v_r) = 0,\ \forall q \in Q, r \in 1,2,\cdots,\xi\} \quad (13)$$

where, $R(\Omega)$ is defined as the roots region of the fractional order polynomial in Eq. 1 (Senol et al., 2014). This root region determines the stability of fractional order uncertain polynomial.

Proof: This proof is summarized from (Senol et al., 2014). Expansion coefficient is the least common multiplier (LCM) which expands the fractional orders to minimal integer orders. Degree of the new polynomial family in this case can be written as thus, every member of the polynomial family has roots on the Riemann surface (Monje et al., 2010). Let us denote as this roots as . Roots located in the first Riemann sheet, which coincides with the phase range , are physically meaningful and this region is used for the stability analysis of fractional order uncertain polynomial (Senol et al., 2014; Radwan et al., 2009; Monje et al., 2010). Thus, the roots region of the expanded integer order polynomial is obtained.

Based on the stability analysis given in Appendix 1, followings are integrated in proof. Corner points of the hypercube Q are formed by considering all lower and upper limits of the uncertain coefficients, $q_i \in \left[\underline{q}_i, \overline{q}_i\right]$ thus, the hypercube has 2^n corners and $n2^{n-1}$ exposed edges. n is the number of uncertain coefficients. Exposed edges are the outer line segments connecting the corner points. Fig. 2 shows an instance hypercube of a fractional order interval polynomial with 3 uncertain coefficients.

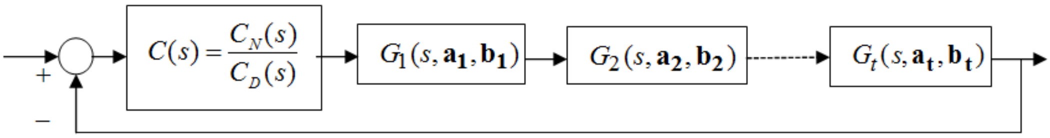


Figure 1. Closed loop fractional order system with multiple linear uncertainty.

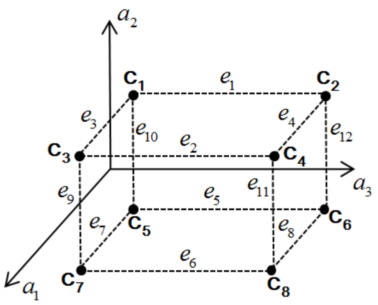


Figure 2. Corner points and exposed edges of hypercube Q for a fractional order interval polynomial with 3 uncertain coefficients.

Set of corner points and exposed edges are named as $c=\left[c_1,c_2,\cdots,c_{2^n}\right]$ and $e=\left[e_1,e_2,\cdots,e_{n2^{n-1}}\right]$ respectively. The edge between two corners can be obtained using the following equation.

$$e_{i-j}=\left(\lambda c_i+(\lambda-1)c_j\right),\lambda\in\left[0,1\right] \tag{14}$$

where, c_i and c_j are the corner polynomials and e_{i-j} is the edge between these corners. Thus, the roots region which is constituted by exposed edges, compose the exact roots region of fractional order interval polynomials. Then, the stability of the interval polynomial can be investigated with the aid of Appendix 1. This completes the proof. ¤

3. STABILITY ANALYSIS FOR FRACTIONAL ORDER MULTIPLE LINEAR UNCERTAIN POLYNOMIALS

This section presents exposed edge stability analysis for fractional order polynomials of multiple linear uncertainties. Consider the fractional order multiple linear uncertain polynomial given in Eq. 8. Extended integer order form of the multiple linear uncertain polynomial family can be written as follows.

$$F^m(v,\mathbf{h})=A^m(s)P_1^m(v,\mathbf{h}_1)P_2^m(v,\mathbf{h}_2)\cdots P_n^m(v,\mathbf{h}_n) \tag{15}$$

where, $P_i^m(v,\mathbf{h}_1)$ are extended integer order polynomials of any linear uncertainty. Considering the lower and upper limits of the uncertain parameters vector $\mathbf{h}$, the hypercube $Q_\mathbf{h}$ is defined in the following form.

$$Q_\mathbf{h}=\{\mathbf{h}:\underline{q}_{ij}\le q_{ij}\le\overline{q}_{ij},\, i=1,2,\cdots,n \quad j=1,2,\cdots,k\} \tag{16}$$

Hypercube $Q_\mathbf{h}$ stands for a polynomial family defined by perturbations of lower and upper limits of uncertain parameters q_{ij}. Let us denote this polynomial family by $\Omega_\mathbf{h}\in R^n$. Then, the following corollary is valid.

Corollary 1: The complex roots of the multiple linear polynomial family $\Omega_\mathbf{h}$ forms a set of root points defined as,

$$R(\Omega_\mathbf{h})=\{v_r:F_u^m(\mathbf{h},v_r)=0,\ \forall\mathbf{h}\in Q_\mathbf{h},\, r\in 1,2,\cdots,\xi\} \tag{17}$$

where, $R(\Omega_\mathbf{h})$ is defined as the roots region of the fractional order polynomial in Eq. 8, which is used to investigate the stability of fractional order multiple linear polynomial.

Proof: $\Omega_\mathbf{h}$ is a subset of interval polynomial family Ω. In this case, the root region of $\Omega_\mathbf{h}$, denoted by $R(\Omega_h)$, is also subset of $R(\Omega)$. So, if all elements of $R(\Omega)$ are stable roots, all elements of $R(\Omega_h)$ are also stable roots. ¤

Then, stability investigation of fractional order polynomials with multiple linear uncertainties can be done as given in Appendix 1. Similar to fractional order polynomials with interval uncertainty, corner points of the hypercube $Q_\mathbf{h}$ are formed by considering all lower and upper limits of the uncertain coefficients, $q_{ij}\in\left[\underline{q}_{ij},\overline{q}_{ij}\right]$ thus, the hypercube has 2^n corners and $n2^{n-1}$ exposed edges. n is the number of uncertain coefficients.

4. STABILITY ANALYSIS FOR FRACTIONAL ORDER MULTIPLE LINEAR UNCERTAIN SYSTEMS

This section presents exposed edge stability analysis for fractional order systems of multiple linear uncertainties. Consider the fractional order multiple linear uncertain plant given in Eq. 9. Extended integer order form of the multiple linear uncertain plant family can be written as follows.

$$\begin{aligned}L^m(v,\mathbf{c})&=\frac{L_N^m(v,\mathbf{b})}{L_D^m(v,\mathbf{a})}=\\&\frac{C_N^m(s)}{C_D^m(s)}\cdot\frac{P_{N1}^m(s,\mathbf{b_1})P_{N2}^m(s,\mathbf{b}_2)\ldots P_{Nt}^m(s,\mathbf{b_t})}{P_{D1}^m(s,\mathbf{a_1})P_{D2}^m(s,\mathbf{a_2})\ldots P_{Dt}^m(s,\mathbf{a_t})}\end{aligned} \tag{18}$$

where, $P_{Ni}^m(v,\mathbf{b}_i)$ and $P_{Di}^m(v,\mathbf{a}_i)$ are extended integer order polynomials of any linear uncertainty. $L_N^m(v,\mathbf{b})$ and $L_D^m(v,\mathbf{a})$ are extended integer order multiple linear polynomial families. Closed loop characteristic equation of this system can be written as,

$$\begin{aligned}\Delta_L^m(v,\mathbf{c}) &= C_N^m(s)P_{N1}^m(s,\mathbf{b_1})P_{N2}^m(s,\mathbf{b}_2)...P_{Nt}^m(s,\mathbf{b_t}) \\ &+C_D^m(s)P_{D1}^m(s,\mathbf{a_1})P_{D2}^m(s,\mathbf{a_2})...P_{Dt}^m(s,\mathbf{a_t})\end{aligned} \tag{19}$$

Considering the lower and upper limits of the uncertain parameters vector $\mathbf{c}$, the hypercube $Q_\mathbf{c}$ is defined in the following form.

$$\begin{aligned}Q_\mathbf{c} = \{\mathbf{c} : \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij} \ \ and \ \ \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \\ i = 1,2,\cdots,n \qquad j = 1,2,\cdots,k\}\end{aligned} \tag{20}$$

Hypercube $Q_\mathbf{c}$ stands for a polynomial family defined by perturbations of lower and upper limits of uncertain parameters b_{ij} and a_{ij}. Let us denote this polynomial family by $\Omega_\mathbf{c} \in R^n$. Then, the following corollary holds.

Corollary 2: The complex roots of the multiple linear characteristic equation $\Omega_\mathbf{c}$ forms a set of root points defined as,

$$R(\Omega_\mathbf{c}) = \{v_r : \Delta_L^m(\mathbf{c}, v_r) = 0,\ \forall \mathbf{c} \in Q_\mathbf{c},\ r \in 1,2,\cdots,\xi\} \tag{21}$$

where, $R(\Omega_\mathbf{c})$ is defined as the roots region of the fractional order polynomial in Eq. 8.

Proof: $\Omega_\mathbf{c}$ is a subset of interval polynomial family Ω. So, $R(\Omega_h)$ becomes a subset of $R(\Omega)$. Therefore, if all elements of $R(\Omega)$ are stable roots, all elements of $R(\Omega_\mathbf{c})$ are also stable roots. ¤

Then, stability investigation of fractional order systems with multiple linear uncertainties can be done as given in Appendix 1. Stability investigation processes can be better understood with the illustrative examples given in the next section.

5. ILLUSTRATIVE EXAMPLES

This section includes illustrative examples to verify the results given in the previous sections. The first example has two fractional order systems with interval uncertainty. Second example has two fractional order systems with affine linear uncertainty.

Example 1: Consider the fractional order systems with interval uncertainty and PI controller.

$$\begin{aligned}C(s) &= K_p + \frac{K_i}{s} \\ G_{11}(s,\mathbf{a}) &= \frac{1}{a_{11}s^{1.1} + a_{12}} \\ G_{12}(s,\mathbf{b}) &= \frac{1}{b_{11}s^{0.6} + b_{12}} \frac{n!}{r!(n-r)!}\end{aligned} \tag{22}$$

where, $K_p = 1$, $K_i = 2$ and the uncertain parameters are considered as $a_{11} \in [1, 1.2]$, $a_{12} \in [0.7, 0.9]$, $b_{11} \in [1.8, 2]$ and $b_{12} \in [0.9, 1]$. Connecting these systems in series will give the following transfer function.

$$\begin{aligned}G_1(s,\mathbf{a},\mathbf{b}) = C(s)G_{11}(s,\mathbf{a})G_{12}(s,\mathbf{b}) = \\ \frac{s+2}{a_{11}b_{11}s^{1.7} + a_{11}b_{12}s^{1.1} + a_{12}b_{11}s^{0.6} + a_{12}b_{12}}\end{aligned} \tag{23}$$

Thus, the characteristic equation of the system found in Eq. 23 can be written as,

$$\begin{aligned}\Delta(s) = a_{11}b_{11}s^{1.7} + a_{11}b_{12}s^{1.1} + \\ a_{12}b_{11}s^{0.6} + s + a_{12}b_{12} + 2 = 0\end{aligned} \tag{24}$$

Applying $s = v^m$ mapping for $m = 10$, one can rewrite the characteristic equation of the system as,

$$\begin{aligned}\Delta_u^{10}(v) = a_{11}b_{11}s^{17} + a_{11}b_{12}s^{11} + s^{10} + \\ a_{12}b_{11}s^{6} + a_{12}b_{12} + 2 = 0\end{aligned} \tag{25}$$

For this example, members of the characteristic equation which compose the corners of the hypercube can be found with the following limits of the uncertain parameters.

$$\begin{aligned}\Delta_{g_1} &= \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \underline{b}_{11}\ \underline{b}_{12}], v), \\ \Delta_{g_2} &= \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \underline{b}_{11}\ \overline{b}_{12}], v), \\ \Delta_{g_3} &= \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \overline{b}_{11}\ \underline{b}_{12}], v), \\ \Delta_{g_4} &= \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \overline{b}_{11}\ \overline{b}_{12}], v), \\ &\vdots \\ \Delta_{g_{13}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \underline{b}_{11}\ \underline{b}_{12}], v), \\ \Delta_{g_{14}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \underline{b}_{11}\ \overline{b}_{12}], v), \\ \Delta_{g_{15}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \overline{b}_{11}\ \underline{b}_{12}], v), \\ \Delta_{g_{16}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \overline{b}_{11}\ \overline{b}_{12}], v)\end{aligned} \tag{26}$$

Similarly, edge polynomials can be obtained for $h \in [0,1]$ as follows,

$$\begin{aligned}\Delta_{e_1} &= \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ \underline{b}_{11}\ S(b_{12},h)], v), \\ \Delta_{e_2} &= \Delta^m([\underline{a}_{11}\ \underline{a}_{12}\ S(b_{11},h)\ \underline{b}_{12}], v), \\ \Delta_{e_3} &= \Delta^m([\underline{a}_{11}\ S(a_{12},h)\ \underline{b}_{11}\ \underline{b}_{12}], v), \\ \Delta_{e_4} &= \Delta^m([S(a_{11},h)\ \underline{a}_{12}\ \underline{b}_{11}\ \underline{b}_{12}], v), \\ &\vdots \\ \Delta_{e_{29}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \underline{b}_{11}\ S(b_{12},h)], v), \\ \Delta_{e_{30}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ S(b_{11},h)\ \underline{b}_{12}], v), \\ \Delta_{e_{31}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ S(b_{11},h)\ \overline{b}_{12}], v), \\ \Delta_{e_{32}} &= \Delta^m([\overline{a}_{11}\ \overline{a}_{12}\ \overline{b}_{11}\ S(b_{12},h)], v).\end{aligned} \tag{27}$$

Fig. 3(a) shows the roots region $R(\Omega)$ of the system in Eq. 23. Roots in the region $R(\Omega)$ ensures the condition $\pi/20<|\arg\{R(\Omega)\}|<\pi/10$ and therefore the system is robustly stable for the given uncertain parameters. Fig. 3(b) shows the roots of corner and edge polynomials in the root spaces and indicates the robust stability of the polynomial family given by Eq. 23. Fig. 3(c) demonstrates the step-responses of corner polynomials and confirms robust stability of the system. Root region in Fig. 3(a) is built using 65.536 roots and exposed edge based root region in Fig. 3(b) is built using 1600 roots. Thus, the proposed method dramatically decreases the number of roots which are used to construct the exact root region. So, the computational cost for stability investigation is considerably reduced.

Example 2: Consider the fractional order systems with affine linear uncertainty.

$$\begin{aligned} G_{21}(s,\mathbf{a}) &= \frac{1}{\left(2a_{21}+3a_{22}\right)s^{2.3}+\left(a_{21}+a_{22}\right)} \\ G_{22}(s,\mathbf{b}) &= \frac{1}{\left(b_{21}+2b_{22}\right)s^{0.6}+\left(b_{21}+b_{22}\right)} \end{aligned} \tag{28}$$

where, the uncertain parameters are $a_{21}\in[1.5,2]$, $a_{22}\in[0.3,1.3]$, $b_{22}\in[0.3,1.4]$ and $b_{21}\in[1.4,2.5]$. Multiplication of these systems gives the following fractional order system with multiple affine linear uncertainty.

$$\begin{aligned} & G_2(s,\mathbf{a},\mathbf{b}) = G_{21}(s,\mathbf{a})G_{22}(s,\mathbf{b}) = \frac{N_2(s)}{D_2(s)} \\ & N_2(s)=1 \\ & D_2(s)=\left(2a_{21}b_{21}+4a_{21}b_{22}+3a_{22}b_{21}+6a_{22}b_{22}\right)s^{2.1}+ \\ & \left(2a_{21}b_{21}+2a_{21}b_{22}+3a_{22}b_{21}+3a_{22}b_{22}\right)s^{1.3}+ \\ & \left(a_{21}b_{21}+2a_{21}b_{22}+a_{22}b_{21}+2a_{22}b_{22}\right)s^{0.8}+ \\ & \left(a_{21}b_{21}+a_{21}b_{22}+a_{22}b_{21}+a_{22}b_{22}\right) \end{aligned} \tag{29}$$

Applying $s=v^m$ mapping for $m=10$, one can rewrite the characteristic equation of the system as,

$$\begin{aligned} & \Delta(s)=\left(2a_{21}b_{21}+4a_{21}b_{22}+3a_{22}b_{21}+6a_{22}b_{22}\right)s^{21}+ \\ & \left(2a_{21}b_{21}+2a_{21}b_{22}+3a_{22}b_{21}+3a_{22}b_{22}\right)s^{13}+ \\ & \left(a_{21}b_{21}+2a_{21}b_{22}+a_{22}b_{21}+2a_{22}b_{22}\right)s^{8}+ \\ & \left(a_{21}b_{21}+a_{21}b_{22}+a_{22}b_{21}+a_{22}b_{22}\right)+1=0 \end{aligned} \tag{30}$$

Fig. 4(a) shows the roots region of the system in Eq. 29. Roots in the region $R(\Omega)$ do not ensure the condition $\pi/20<|\arg\{R(\Omega)\}|<\pi/10$ and therefore the system is unstable for the given uncertain parameters. Fig. 4(b) shows the roots of corner and edge polynomials in the root spaces and Fig. 4(c) demonstrates the step-responses of corner polynomials. It is clear in Fig. 4 that the system in Eq. 29 is unstable.

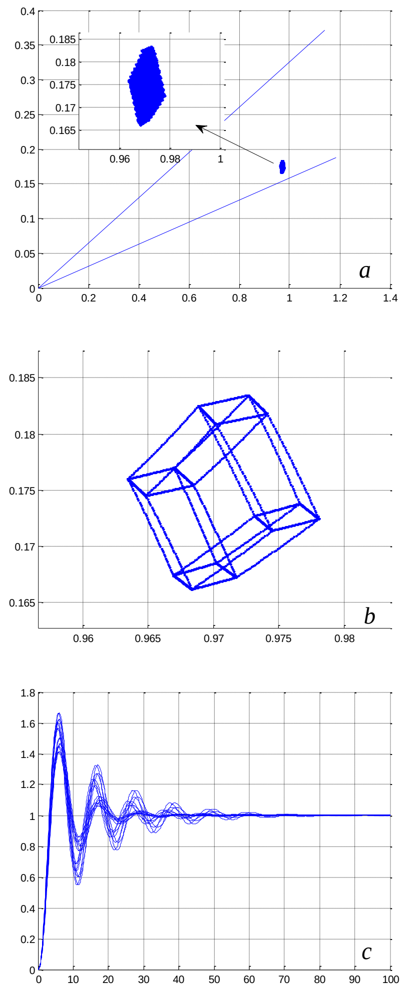
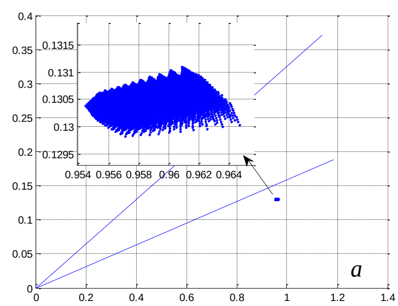


Figure 3. Stability test results for the uncertain fractional order control system given by Eq. 23.
(a) Root space obtained for region- sampled robust stability analysis, (b) Root space obtained for edge-sampled robust stability analysis, (c) Step responses of corner polynomials.

Root region given in Fig. 4(a) is built by 1.048.576 roots and exposed edge based root region given in Fig. 4(b) is built by 3.200 roots. Like the previous example, the propsed method reduces computation cost for root region investigation which is an important criteria for robust stability computation.



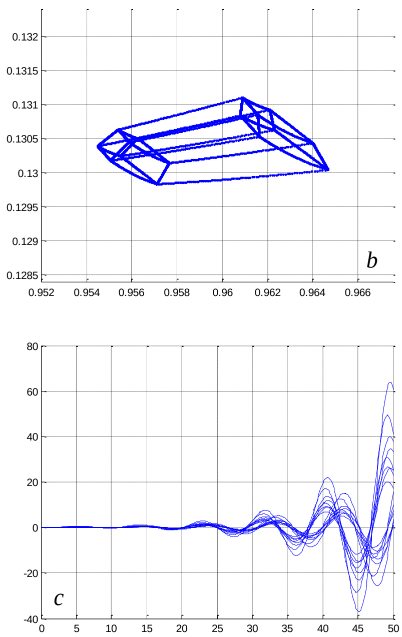


Figure 4. Stability test results for the uncertain fractional order control system given by Eq. 29.
(a) Root space obtained for region- sampled robust stability analysis, (b) Root space obtained for edge-sampled robust stability analysis, (c) Step responses of corner polynomials.

6. CONCLUSION

This study employs exposed edge based numerical robust stability analysis method for fractional order control systems with multiple linear uncertainties which contains two or more fractional order systems with any linear uncertainties. Linear uncertainty types and fractional order systems with multiple linear uncertainty structure are presented. A stability analysis method for fractional order polynomials which is based on roots locations is considered in this paper. The fractional order polynomial is transformed in to a polynomial of extended integer orders using a common multiplier. Then, stability is investigated considering the roots which are located in the first Riemann sheet. Advantages of the proposed method lay in simplification of the roots region computation of fractional order multiple linear uncertain systems. This simplification method brings easy investigation of the stability of a complicated system therefore, the paper contributes to the field of fractional order control systems.

REFERENCES

Ahn, H. S, Chen, Y. Q. and Podlubny, I., (2007). Robust stability test of a class of linear time invariant interval fractional-order system using Lyapunov inequality. *Applied Mathematics and Computation*. 187 (1), 27-34.

Arena, P., Caponetto, L., Fortuna, L. and Porto, D., (2000). *Nonlinear Non-integer Order Circuits and Systems - An Introduction*. World Scientic, Singapore.

Bhattacharyya, S. P., Chapellat, H. and Keel, L. H., (1995). *Robust Control: The Parametric Approach*. Prentice Hall.

Bhattacharyya, S. P., Datta, A. and Keel, L. H., (2010). *Linear control theory: structure, robustness, and optimization*. CRC press.

Caponetto, R., Dongola, G., Fortuna, L. and Petras, I., (2010). *Fractional Order Systems, Modeling and Control Applications*. World Scientific, Singapore.

Caponetto, R. and Dongola, G., (2013). A numerical approach for computing stability region of FO-PID controller. *Journal of the Franklin Institute*. 350 (4), 871-889.

Chen, Y. Q., Ahn, H. S. and Podlubny, I., (2006). Robust stability check of fractional order linear time invariant systems with interval uncertainties. *Signal Processing*. 86, 2611–2618.

Chen, Y. Q., Petras, I. and Dingyü, X., (2009). Fractional order control - A tutorial. *American Control Conference (ACC).*

Gutierrez, R. E., Rosario, J. M. and Machado, J. T., (2010). Fractional Order Calculus: Basic Concepts and Engineering Applications. *Mathematical Problems in Engineering*.

Hamelin, F. and Boukhobza, T., (2005). Geometric-based approach to fault detection for multilinear affine systems. *16th IFAC World Congress*, Czech Republic.

Hilfer, R., (2000). *Applications of fractional calculus in physics*, World Scientific, Singapore.

Lu, J. G. and Chen, G. R., (2009). Robust stability and stabilization of fractional-order interval systems: an LMI approach. *IEEE Transactions on Automatic Control*. 54 (6), 1294-1299.

Lu, J. G. and Chen, Y. Q., (2010a). Stability and stabilization of fractional-order linear systems with convex polytopic uncertainties. *Journal of Fractional Calculus and Applied Analysis*. 16 (1), 142-157.

Lu, J. G. and Chen, Y. Q., (2010b). Robust stability and stabilization of fractional-order interval systems with the fractional order α: the $0<\alpha<1$ case. *IEEE Transactions on Automatic Control*. 55 (1), 152–158.

Magin. R. L., (2012). Fractional calculus in bioengineering: A tool to model complex dynamics. *13th International Carpathian Control Conference (ICCC).*

Matusu, R. and Prokop, R., (2011). Graphical analysis of robust stability for systems with parametric uncertainty: an overview. *Transactions of the Institute of Measurement and Control*. 33 (2), 274–290.

Miller, K. S. and Ross, B., (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons.

Monje, C. A., Chen, Y. Q., Vinagre, B. M., Dingyü, X. and Feliu, V., (2010). *Fractional-order systems and controls*. Springer, New York.

Oldham, K. B. and Spanier, J., (1974). *Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order.* Academic Press, New York.

Parada, F. J. V., Tapia, J. A. O. and Ramirez, J. A., (2007). Effective medium equations for fractional Ficks law in porous media. *Physica*, 373, 339-53.

Petras, I., Podlubny I., O'leary, P., Dorcak, L. and Vinagre, B. M., (2002). *Analogue Realization of Fractional Order Controllers*. Technical University of Kosice, Kosice.

Petras, I., (2009). Stability of fractional-order systems with rational orders: a survey. *Fractional Calculus and Applied Analysis*, 12 (3).

Podlubny, I., (1999a). *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*. Academic Press, San Diego.

Podlubny, I., (1999b). Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers. *IEEE Transactions on Automatic Control*, 44 (1), 208–214.

Radwan, A. G., Soliman, A. M., Elwakil, A. S. and Sadeek, A., (2009). On the stability of linear systems with fractional-order elements. *Chaos, Solitons & Fractals*. 40 (5), 2317-2328.

Senol, B. and Yeroglu, C., (2012a). Computation of the value set of fractional order uncertain polynomials: A 2q convex parpolygonal approach. *2012 IEEE International Conference on Control Applications (CCA).*

Senol B. and Yeroglu, C., (2012b). Robust Stability Analysis of Fractional Order Uncertain Polynomials. *2012 International Conference on Fractional Differentiation and its Applications (ICFDA).*

Senol, B. and Yeroglu, C., (2013). Frequency boundary of fractional order systems with nonlinear uncertainties. *Journal of the Franklin Institute*. 350 (7), 1908-1925.

Senol, B., Ates, A., Alagoz, B. B. and Yeroglu, C., (2014). A numerical investigation for robust stability of fractional-order uncertain systems. *ISA Transactions*. 53 (2), 189-198.

Shaw, J. and Jayasuriya, S., (1996). A new algorithm for testing the stability of a polytope: a geometric approach for simplification. *Journal of Dynamic Systems, Measurement and Control*. 118, 611-614.

Tan, N. and Atherton, D. P., (1998). Magnitude and Phase Envelopes of Systems with Affine Linear Uncertainty. *International Conference on Control (UKACC)*.

Tan, N. and Atherton, D. P., (2000). Frequency response of uncertain systems: A *2q*-convex parpolygonal approach. *Control Theory and Applications*. 147, 547-555.

Tan, N., (2002). Computation of the frequency response of multilinear affine systems. *IEEE Transactions on Automatic Control*. 47 (10). 1691-1696.

Tan, N. and Atherton, D. P., (2002). Robust Stability of Multilinear Affine Polynomials. *IEEE International Conference on Control Applications*.

Tan, N., Ozguven, O. F. and Ozyetkin, M. M., (2009). Robust stability analysis of fractional order interval polynomials, *ISA Transactions*. 48, 166-172.

Vinagre, B. M., Chen, Y. Q. and Petras, I., (2003). Two direct Tustin discretization methods for fractional-order diferentiator/integrator. *Journal of the Franklin Institute*. 340, 349-362.

Yeroglu, C., Ozyetkin, M. M. and Tan, N., (2010). Frequency response computation of fractional order interval transfer functions. *International Journal of Control Automation and Systems*. 8 (5), 1009-1017.

Yeroglu, C. and Senol, B. (2013). Investigation of robust stability of fractional order multilinear affine systems: 2q-convex parpolygon approach. *Systems & Control Letters*. 62 (10), 845-855.

APPENDIX 1

This appendix is summarized from (Senol et al., 2014; Radwan et al., 2009) to present roots location based stability analysis for fractional order interval polynomials. Studies related to the stability analysis of expanded degree integer order polynomial in the root space can be found in the literature (Caponetto et al., 2010; Senol et al., 2014; Radwan et al., 2009). Radwan et al. put this analysis method in a systematic form (Radwan et al., 2009) and the Radwan procedure for the stability analysis was summarized as follows.

- Expand the orders of the fractional order polynomial with m expansion coefficient and calculate the root phases of the polynomial in the first Riemann sheet, $\varphi \in (-\pi / m, \pi / m)$.

- If the root phases are in the range of $\pi / 2m < |\varphi_r| < \pi / m$, the system is stable.

- If the root phases are equal to $\pi / 2m$, the system oscillates.

- Otherwise, the system is unstable.

For $m = 1$, the Riemann surface refers to the open left half of the complex plane and this region is known as the stability region for integer order polynomials. Thus, the stability condition for integer order systems can be written as $\pi / 2 < |\varphi_v| < \pi$. Stability regions for integer and fractional order systems are given in Fig. 5.

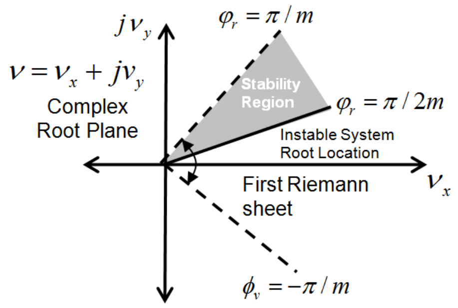


Figure 5. Stability region for an expanded integer order polynomial in the first Riemann surface after $s = v^m$ mapping for $m > 1$.