Generalized Non-Switching Reaching Laws for SMC of Discrete Time Dynamical Systems

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Abstract: In this paper a framework for designing reaching laws for discrete time sliding mode control of dynamical systems is proposed. In this framework, the convergence to the sliding hyperplane is governed by a general function of the switching variable. The conditions that this function must satisfy, to ensure the existence of the quasi-sliding motion are demonstrated first for the nominal, and then for the perturbed system. This is a novel approach with respect to the solutions known in the literature, in which usually only specific, particular functions are considered. In the second part of the paper, the reaching laws derived using the proposed framework are applied to solve the supply chain control problem. Favorable properties of the obtained inventory management policies are proved analytically and demonstrated in a simulation example.

Keywords: Sliding-mode control, discrete-time systems, reaching law approach

1. INTRODUCTION

Sliding mode control (Edwards and Spurgeon, 1998) is a well-known technique that is well-suited for a large class of time-varying, uncertain, non-linear systems. It focuses on driving the controlled system representative point onto a properly selected hypersurface in the state space and maintaining the point on the surface for the rest of the control process. The main advantages of this method are its robustness (Draženović, 1969) and computational efficiency. The concept of sliding mode was at first considered for continuous time systems. However, as controllers are nowadays implemented almost exclusively in digital hardware, discrete time sliding mode control (Furuta, 1990; Golo and Milosavljević, 2000; Janardhanan and Bandyopadhyay, 2007; Mehta and Bandyopadhyay, 2009; Milosavljević, 1985; Sarpturk et al., 1987; Utkin and Drakunov, 1989) became an important research topic.

The design of a sliding mode controller begins by choosing the hypersurface so as to achieve the desired dynamics during the sliding phase. Then a control action that ensures the convergence to the hyperplane and the stability of the following sliding motion is derived. There are two ways in which this can be done. The first one begins by proposing a control law and then demonstrating, that with its application the properties mentioned above are ensured. The second method is the reaching law concept, and it is the one used throughout this work. The order of operations in this technique is in some sense reversed. First the desired evolution of the sliding variable is defined and then the control law that guarantees this evolution is calculated. This methodology has been first proposed for continuous time systems in (Gao and Hung, 1993). Then in (Gao et al., 1995) the authors proposed a ‘constant plus proportional’ reaching law for discrete time systems, and in (Bartoszewicz, 1996) the conditions necessary to ensure the existence of the quasi-sliding mode for the discrete time perturbed system were presented. This reaching law and its modifications are widely used by a number of researchers. In (Qu et al., 2014) this approach is extended by a disturbance compensator, which is derived under the assumption of limited rate of change of the disturbance. In (Niu et al., 2010) the authors remove both the proportional term and the switching term, and convergence to the sliding hyperplane is achieved only when the absolute value of the disturbance is smaller than its bound. In (Ren et al., 2013) the reaching law proposed in (Gao et al., 1995) is modified by making the constant switching term a function of the absolute value of the sliding variable. In (Veselic et al., 2010) the authors consider a reaching law in which the proportional term is not present, and the switching term switches between a large value when the distance from the hyperplane is big, and a smaller one in the neighborhood of the hyperplane. In (Zhao et al., 2013) the constant switching term is replaced by two power functions of the switching variable. This reaching law is then extended by a neural network that is used for disturbance estimation.

In this paper a general form of a non-switching reaching law is proposed. In contrast to (Gao et al., 1995), the representative point is not required to cross the hyperplane in each consecutive step of the quasi-sliding mode. It is sufficient, that it remains in a vicinity of the hyperplane, whose size is known a priori. In this way, the discontinuous switching term is removed from the reaching law, and also from the control signal. This leads to attenuating the undesirable chattering effect. Moreover, contrary to the previous works in this field, this paper does not concentrate on a single, specific reaching law. A reaching law, that is based on a general function of the sliding variable is proposed and analyzed. Then, the conditions, that this function must satisfy, in order to ensure the reachability and existence of the quasi-sliding mode are demonstrated, first for the nominal, and then for the perturbed system. One can use the framework proposed here to easily develop a reaching law that is best suited for a particular control problem.

In this work a logistic system is considered as an example of application of the proposed reaching law. The application of control theory methods for such systems was first proposed in (Simon, 1952) and (Vassian, 1954). It was demonstrated, that one cannot obtain stability, if the resupply policy is based only on the current stock level. This policy must also take into account the sizes of incoming shipments. One of the most important aspects in controlling inventory systems is the so-called bullwhip effect. The consumers' demand, inevitably, shows some variations. As one moves "up" the supply chain (from the retail store, through the wholesalers to the manufacturer) these variations in demand become amplified. This raises the requirements for storage space and lowers the efficiency of the supply chain. A good overview of this phenomenon is provided in (Miragliotta, 2006). The author points out the following main causes of the bullwhip effect: grouping orders to minimize transport costs, fluctuations of prices, temporary stock-outs that lead to overestimating the demand and "human" errors, namely the lack of knowledge of the dynamics of supply chains. Some of the authors use the continuous time approach, like in (Boukas et al., 2000) where the H_∞ methodology was used to design a resupply policy for the case of variable transport delays. However, as deliveries usually arrive at specific time instants (for most retail stores early in the morning), the discrete time approach seems more natural for this problem. In (Aggelogiannaki et al., 2008) a discrete time model predictive controller is proposed for dealing with smaller shipments caused by machine failures in factories. In (Gaalman, 2006) a Kalman filter is used to predict the consumers' demand during the next discretization period.

The remainder of this work is organized as follows. In sections 2 and 3 a general non-switching reaching law that uses a generic function of the sliding variable to govern its convergence to zero is proposed. The conditions, that this function must satisfy are demonstrated first for the nominal and then for the perturbed system. In Section 4 the application of the non-switching reaching law presented in sections 2 and 3 to the periodic review inventory management system is presented. The favorable properties of the obtained closed-loop system are shown. Section 5 comprises the results of computer simulations, which verify the analytical results. The last section presents the conclusions of this work.

2. NON-SWITCHING REACHING LAW FOR THE UNPERTURBED SYSTEM

In this section a reaching law, that will govern the convergence of the sliding variable to the vicinity of zero is proposed. A discrete time sliding mode controller, based on this reaching law, will also be designed. The definition of the quasi-sliding mode, that was proposed by Gao is modified. The representative point is no longer required to cross the sliding hyperplane in each step of the quasi-sliding motion. We only demand that it monotonically converges to some known vicinity of the sliding hyperplane $\sigma(kT) = 0$, and after entering this vicinity remains in it for the rest of the control process.

The unperturbed discrete time system, that can be expressed by

$$\boldsymbol{x}\left[(k+1)T\right] = \boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{b}u(kT), \tag{1}$$

is considered, where $\boldsymbol{x}(kT)$ is the $n \times 1$ state vector, $\boldsymbol{A}$ is the $n \times n$ state matrix, $\boldsymbol{b}$ is the input vector and $u(kT)$ is a scalar input. The demand state vector is denoted by $\boldsymbol{x}_d$. The sliding hyperplane is selected as

$$\sigma(kT) = \boldsymbol{c}^T\boldsymbol{e}(kT) = 0, \tag{2}$$

where $\boldsymbol{e}(kT) = \boldsymbol{x}_d - \boldsymbol{x}(kT)$ is the closed loop error vector, and parameter vector $\boldsymbol{c}$ is selected so that $\boldsymbol{c}^T\boldsymbol{b} \neq 0$ and that the closed-loop system exhibits the desired performance. The choice can be performed in a number of ways such as the quadratic optimization (Janardhanan and Kariwala, 2008), pole placement method (Bandyopadhyay and Janardhanan, 2006), or dead-beat design.

The following non-switching reaching law

$$\sigma\left[(k+1)T\right] = \sigma(kT) - f\left[\sigma(kT)\right] \tag{3}$$

is proposed, in which $f[\sigma(kT)]$ is a function of the sliding variable, that has the following properties

1. $f(0) = 0$ and $f[\sigma(kT) \neq 0] \neq 0$.

2. $f[\sigma(kT)]$ is a non-decreasing function of $\sigma(kT)$.

3. The function $f[\sigma(kT)]$ is Lipschitz continuous with Lipschitz constant $K \leq 1$.

The results presented in this paper are a major generalization of the previous work (Bartoszewicz and Leśniewski, 2014), in which only a single function $f[\sigma(kT)]$ was considered. Therefore, the contribution of this work is essentially different from (Bartoszewicz and Leśniewski, 2014) since in this paper a whole class of sliding mode controllers is considered, while in the previous article only a single, particular example is studied.

The first step is calculating the control law, that ensures, that the sliding variable will in fact evolve according to (3). From (1) the value of the sliding variable in the next time instant can be obtained as

$$\sigma\left[(k+1)T\right] = \boldsymbol{c}^T\boldsymbol{x}_d - \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(kT) - \boldsymbol{c}^T\boldsymbol{b}u(kT). \tag{4}$$

Now, by comparing the right hand sides of (3) and (4) one can arrive at

$$u(kT) = \left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\left\{f\left[\sigma(kT)\right] + \boldsymbol{c}^T\left(\boldsymbol{I}_n - \boldsymbol{A}\right)\boldsymbol{x}(kT)\right\}. \tag{5}$$

Remark 1. If the function $f[\sigma(kT)]$ is chosen so that its absolute value is upper bounded, namely

$$\left|f\left[\sigma\left(kT\right)\right]\right| \le f_{\max} \tag{6}$$

for any value of $\sigma(kT)$, then the application of the above controller ensures a bounded rate of change of the sliding variable. It follows from (3) and (6), that

$$\left|\sigma\left[\left(k+1\right)T\right]-\sigma\left(kT\right)\right| \le f_{\max} . \tag{7}$$

In the remainder of this section it will be demonstrated, that the proposed general switching reaching law ensures the existence and reachability of the quasi sliding mode.

Theorem 1. The proposed reaching law (3) ensures, that for any initial value $\sigma(0) \neq 0$ the representative point will converge to the hyperplane without ever crossing it and then remain on it for the rest of the control process.

Proof: From conditions 1 and 3 it can be observed, that the absolute value of $f[\sigma(kT)]$ never exceeds the absolute value of the sliding variable, namely $|f[\sigma(kT)]| \le |\sigma(kT)|$. Therefore, it follows from equation (3), that $\sigma[(k+1)T]$ can be either equal to zero, or have the same sign as $\sigma(kT)$. Therefore, the representative point will never cross the hyperplane. Furthermore, equation (3) can be rewritten as follows

$$\sigma\left[\left(k+1\right)T\right]-\sigma\left(kT\right) = -f\left[\sigma\left(kT\right)\right]. \tag{8}$$

Conditions 1 and 2 imply, that the right hand side of (8) is non-zero and has the opposite sign to $\sigma(kT)$ for any $\sigma(kT) \neq 0$. Therefore, if $\sigma(kT)$ were to converge to $\sigma(kT) = \delta$, where $\delta \neq 0$ the following relation would have to hold

$$\lim_{\sigma(kT)\to\delta} f\left[\sigma\left(kT\right)\right] = 0 . \tag{9}$$

Since function f is continuous, $\lim_{\sigma(kT)\to\delta} f\left[\sigma\left(kT\right)\right] = f\left(\delta\right)$. Therefore, condition 1 implies that (9) does not hold for any $\delta \neq 0$. This means, that $\sigma(kT)$ will converge to zero, from any initial value. Furthermore, once $\sigma(kT) = 0$ is satisfied, the right hand side of (8) will be equal to zero, which implies remaining on the hyperplane for the rest of the control process.

Remark 2. The above theorem implies only asymptotic convergence to the hyperplane. However, it is clear, that one can easily obtain finite time convergence by choosing function $f[\sigma(kT)]$ to satisfy the additional condition

$$\left|f\left[\sigma\left(kT\right)\right]\right| = \left|\sigma\left(kT\right)\right| \quad \text{for all} \quad \left|\sigma\left(kT\right)\right| \le \sigma_D , \tag{10}$$

where $\sigma_D > 0$ is a known constant. As $f[\sigma(kT)]$ is a non-decreasing function of $\sigma(kT)$ the above condition implies, that the absolute value of $\sigma(kT)$ will decrease at least by σ_D when $|\sigma(kT)| > \sigma_D$, and once $|\sigma(kT)| \le \sigma_D$, the sliding hyperplane will be reached in the next time instant. Therefore, if any function $f[\sigma(kT)]$ that satisfies (10) is chosen, the hyperplane will be reached in at most $\left\lceil \left|\sigma\left(0\right)\right| / \sigma_D \right\rceil$ steps.

3. NON-SWITCHING REACHING LAW FOR THE PERTURBED SYSTEM

In this section the reaching law presented above will be modified in order to apply it to the discrete time system subject to parameter uncertainties and disturbances, which can be expressed as

$$\boldsymbol{x}\left[\left(k+1\right)T\right] = \boldsymbol{A}\boldsymbol{x}\left(kT\right)+\boldsymbol{r}\left[\boldsymbol{x}\left(kT\right)\right]+\boldsymbol{b}u\left(kT\right)+\boldsymbol{d}\left(kT\right) , \tag{11}$$

where $\boldsymbol{r}$ is the $n \times 1$ vector corresponding to the influence of parameter uncertainties on the system state and $\boldsymbol{d}$ is the $n \times 1$ external disturbance vector. The sliding variable is again selected as (2). Due to the perturbations, the original reaching law cannot be satisfied. Therefore, it is replaced by the following one

$$\begin{aligned}\sigma&\left[\left(k+1\right)T\right] = \\ &= \sigma\left(kT\right)-f\left[\sigma\left(kT\right)\right]-\tilde{F}\left(kT\right)-\tilde{S}\left(kT\right)+F_1+S_1,\end{aligned} \tag{12}$$

where $\tilde{F}$ and $\tilde{S}$ correspond respectively to the impact of the external disturbance and the parameter uncertainty on the sliding variable

$$\tilde{S}\left(kT\right) = \tilde{S}\left[\boldsymbol{x}\left(kT\right)\right] = \boldsymbol{c}^T\boldsymbol{r}\left[\boldsymbol{x}\left(kT\right)\right], \;\; \tilde{F}\left(kT\right) = \boldsymbol{c}^T\boldsymbol{d}\left(kT\right) . \tag{13}$$

The terms S_1 and F_1 are the mean values of $\tilde{F}$ and $\tilde{S}$

$$S_1 = \left(S_U + S_L\right)/2, \qquad F_1 = \left(F_U + F_L\right)/2 , \tag{14}$$

where S_U, F_U are the upper bounds and S_L, F_L are the lower bounds of $\tilde{S}$ and $\tilde{F}$, i.e.

$$S_L \le \tilde{S}\left(kT\right) \le S_U, \quad F_L \le \tilde{F}\left(kT\right) \le F_U . \tag{15}$$

The particular bounds S_U and S_L are determined respectively by the maximum and minimum value of function $\boldsymbol{r}$ and the particular bounds F_U and F_L depend on the maximum and minimum value of disturbance $\boldsymbol{d}(kT)$ in the considered system. Moreover, the terms F_2 and S_2, that represent the greatest possible deviations from the mean values F_1, S_1 are introduced

$$S_2 = \left(S_U - S_L\right)/2, \qquad F_2 = \left(F_U - F_L\right)/2 . \tag{16}$$

For the unperturbed system, the three conditions presented in Section 2 were sufficient to ensure convergence of the system representative point to the sliding hyperplane. For the discrete time system ideal sliding motion on the hyperplane cannot be achieved, as it would require knowing in advance the value of the disturbance in the next time instant to generate the control signal $u(kT)$. Therefore, only a quasi-sliding motion in the vicinity of the hyperplane can be attained. Furthermore, in order to ensure the convergence to the neighborhood of the sliding hyperplane, the following additional condition on function $f[\sigma(kT)]$ is introduced.

For the perturbed system, there must exist such a $\sigma_{B+} > 0$ and $\sigma_{B-} < 0$, that the conditions

$$f\left[\sigma(kT)>\sigma_{B+}\right]>S_2+F_2,\quad f\left(\sigma_{B+}\right)=S_2+F_2,\\ f\left[\sigma(kT)<\sigma_{B-}\right]<-S_2-F_2,\quad f\left(\sigma_{B-}\right)=-S_2-F_2, \tag{17}$$

are all satisfied. Conditions (17) mean that if the system representative point is far off the hyperplane (2), then the absolute value of function f is big enough to ensure convergence to the vicinity of the hyperplane given by $\sigma_{B-}\le\sigma(kT)\le\sigma_{B+}$.

Using (2) and (11) the value of the control signal, that ensures that the sliding variable evolves according to (12) can be derived as

$$u(kT)=\left(\boldsymbol{c}^T\boldsymbol{b}\right)^{-1}\\ \left\{\boldsymbol{c}^T\left(\boldsymbol{I}_n-\boldsymbol{A}\right)\boldsymbol{x}(kT)+f\left[\sigma(kT)\right]-F_1-S_1\right\}. \tag{18}$$

Remark 3. If function $f[\sigma(kT)]$ is chosen to satisfy (6), then the application of the controller (18) ensures bounded rate of change of the sliding variable, i.e.

$$\left|\sigma\left[(k+1)T\right]-\sigma(kT)\right|\le f_{\max}+S_2+F_2. \tag{19}$$

In the two theorems that follow, it will be demonstrated, that the application of the controller (18) ensures the existence and reachability of the quasi sliding mode.

Theorem 2. The representative point will converge, in the worst case asymptotically, to the quasi-sliding mode band described by

$$\sigma_{B-}\le\sigma(kT)\le\sigma_{B+} \tag{20}$$

from any initial position.

Proof: In the proof the positive and negative values of $\sigma(kT)$ will be considered separately

Case 1. For $\sigma(kT)>\sigma_{B+}$ it can be observed from (12), that

$$\sigma\left[(k+1)T\right]-\sigma(kT)\le F_1+S_1-F_L-S_L-f\left[\sigma(kT)\right]=\\ =\left(F_U+F_L\right)/2+\left(S_U+S_L\right)/2-F_L-S_L-f\left[\sigma(kT)\right]=\\ =\left(F_U-F_L\right)/2+\left(S_U-S_L\right)/2-f\left[\sigma(kT)\right]=\\ =F_2+S_2-f\left[\sigma(kT)\right]. \tag{21}$$

From condition (17) and equation (21) it directly follows, that if $\sigma(kT)>\sigma_{B+}$ then $\sigma(kT)$ will strictly decrease.

Case 2. For $\sigma(kT)<\sigma_{B-}$ it follows from (12) that

$$\sigma\left[(k+1)T\right]-\sigma(kT)=F_1+S_1-f\left[\sigma(kT)\right]+\\ -\tilde{F}(kT)-\tilde{S}(kT)\ge\left(F_U+F_L\right)/2-F_U-f\left[\sigma(kT)\right]+\\ +\left(S_U+S_L\right)/2-S_U=\left(F_L-F_U\right)/2+\left(S_L-S_U\right)/2+\\ -f\left[\sigma(kT)\right]=-f\left[\sigma(kT)\right]-F_2-S_2. \tag{22}$$

Using condition (17) with the above equation it is evident that if $\sigma(kT)<\sigma_{B-}$ then $\sigma(kT)$ will strictly increase.

From the above analysis it follows, that the representative point will indeed converge to the band (20), in the worst case asymptotically, from any initial position. This conclusion ends the proof.

Theorem 3. Once the relation (20) is satisfied for some $k=k_0$, it will hold for the remainder of the control process.

Proof: It follows from conditions 2, 3 and equation (12) that $\sigma[(k+1)T]$ never decreases with the increase of $\sigma(kT)$. Therefore, in order to prove the theorem, it is sufficient to show that $\sigma(kT)=\sigma_{B-}$ implies $\sigma[(k+1)T]\ge\sigma_{B-}$ and $\sigma(kT)=\sigma_{B+}$ implies $\sigma[(k+1)T]\le\sigma_{B+}$.

Case 1. If $\sigma(kT)=\sigma_{B-}$ then it follows from (12) that

$$\sigma\left[(k+1)T\right]=\sigma_{B-}-f\left(\sigma_{B-}\right)-\tilde{F}(kT)-\tilde{S}(kT)+F_1+S_1\ge\\ \ge\sigma_{B-}-f\left(\sigma_{B-}\right)-F_2-S_2. \tag{23}$$

Taking into account conditions (17) one can observe, that the right hand side of inequality (23) is equal to σ_{B-}. Therefore, if the value of the sliding variable satisfies (20), it will not drop below σ_{B-} in the next time instant.

Case 2. If $\sigma(kT)=\sigma_{B+}$ then, using (12) one can arrive at

$$\sigma\left[(k+1)T\right]=\sigma_{B+}-f\left(\sigma_{B+}\right)-\tilde{F}(kT)-\tilde{S}(kT)+F_1+S_1\le\\ \le\sigma_{B+}-f\left(\sigma_{B+}\right)+F_2+S_2. \tag{24}$$

It follows from conditions (17), that the right hand side of (24) is equal to σ_{B+}. Therefore, if the value of the sliding variable satisfies (20), it will not exceed σ_{B+} in the next time instant.

Using the principle of mathematical induction with the results of both cases, it can be observed, that indeed once (20) becomes satisfied, it will hold for the remainder of the control process.

It is worth to stress, that control (18) is expressed in a closed form and it does not require excessive computational resources. This is an essential advantage in terms of control engineering and applied informatics.

4. INVENTORY SYSTEM CONTROL

In this section the periodic review inventory system (Bartoszewicz and Leśniewski, 2014; Leśniewski and Bartoszewicz, 2013) is analyzed. It will be used for testing the performance of the proposed reaching laws. There is a single warehouse in the system, which is resupplied by m remote suppliers. Each supplier p can send up to $u_{\max p}$ goods during a single review period, and the shipment arrives at the warehouse after the appropriate lead time L_p. Furthermore, as products can be damaged during transport, only a part of the goods sent by the supplier equal to α_p, where $\alpha_p\in(0,1]$ reach the warehouse. The on-hand stock in the warehouse is used to satisfy an *a priori* unknown consumers' demand. The resupply orders are generated by the controller, which is a part of the warehouse management policy. The control signal is denoted by u, and it determines the total amount of goods

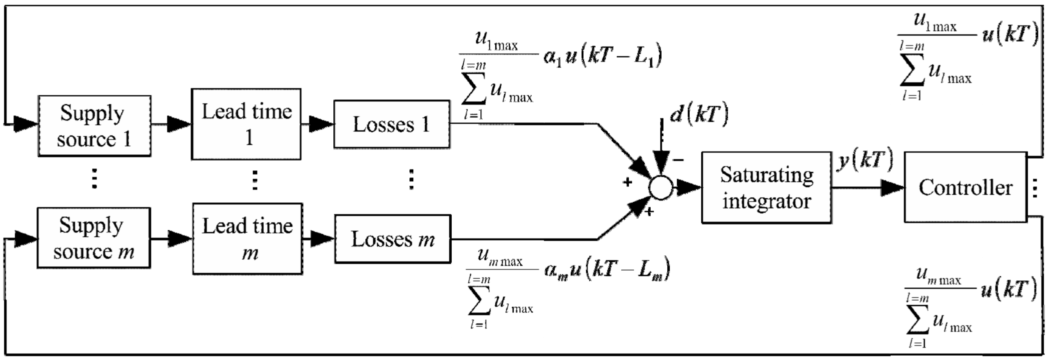

Fig. 1. Inventory system model.

ordered from all of the suppliers. This amount is distributed among the suppliers, in proportion to their output capabilities, namely supplier p receives an order for $u \cdot u_{\max p} \big/ \sum_{i=1}^{i=m} u_{\max i}$ goods. The scheme illustrating the system is depicted in Figure 1. It is assumed, that each lead time L_p is a multiple of the review period T, i.e. $L_p = \beta_p T$, where β_p is a positive integer. The stock level at time kT is represented by $y(kT)$. The consumers' demand is modelled as an *a priori* unknown non-negative function of time $d(kT)$, and only its maximum value, denoted by $d_{\max}$, is known in advance. At some times the amount of goods in the warehouse can be insufficient to completely satisfy this demand, therefore an additional function $h(kT)$ corresponding to the amount of goods that were sold during the k-th review period is introduced. Thus, for any $k \geq 0$

$$0 \leq h(kT) \leq d(kT) \leq d_{\max} . \tag{25}$$

The warehouse is empty before the start of the control process and the first supply order is generated at $kT = 0$, i.e. $y(kT < 0) = 0$ and $u(kT < 0) = 0$. Therefore, the warehouse stock level can be expressed as the following difference between the incoming and outgoing amounts of goods

$$y(kT) = \sum_{p=1}^{m} \sum_{j=0}^{k-1} \frac{u_{\max p}}{\sum_{i=1}^{m} u_{\max i}} \alpha_p u(jT - L_p) - \sum_{j=0}^{k-1} h(jT) . \tag{26}$$

In order to obtain a simpler notation all suppliers with equal lead times can be represented by a single, aggregate provider. An amount of goods equal to $a_i u$ will arrive at the warehouse from this provider, where the coefficient a_i is given by

$$a_i = \sum_{p:\beta_p = i} \alpha_p u_{\max p} \bigg/ \sum_{i=1}^{m} u_{\max i} \tag{27}$$

for $i = 1, 2, \ldots, n-1$, where $n = \max(\beta_p) + 1$. If there are no suppliers with a particular lead time iT, then the corresponding parameter a_i is equal to zero. This transformation allows to express the stock level in the following form

$$y(kT) = \sum_{i=1}^{n-1} \sum_{j=0}^{k-1} a_i u[(j-i)T] - \sum_{j=0}^{k-1} h(jT) . \tag{28}$$

Now the inventory system model will be expressed in the state space as

$$\begin{aligned} & \boldsymbol{x}[(k+1)T] = \boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{b}u(kT) + \boldsymbol{o}h(kT) \\ & y(kT) = \boldsymbol{r}^T \boldsymbol{x}(kT) \end{aligned} \tag{29}$$

where $\boldsymbol{A}$ is the $n \times n$ state matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & & a_1 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} \tag{30}$$

and $\boldsymbol{b}$, $\boldsymbol{o}$, and $\boldsymbol{r}$ are $n \times 1$ vectors

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{o} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \boldsymbol{r} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{31}$$

The first state variable is the current on-hand stock level, $x_1(kT) = y(kT)$, and the remaining variables are the delayed values of the control signal, i.e.

$$x_i(kT) = u[(k-n+i-1)T] . \tag{32}$$

The desired state of the system is $\boldsymbol{x_d} = [y_d \ 0 \ \ldots \ 0]$, where y_d is the demand value of the stock level.

Now the reaching law based sliding mode controller for the inventory system presented above will be designed. The first step is choosing the vector $\boldsymbol{c}$ so that the closed loop system exhibits dead-beat dynamics. The control signal that ensures $\sigma[(k+1)T] = 0$, can be obtained as $u(kT) = (\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{c}^T \ [\boldsymbol{x_d} - \boldsymbol{A}\boldsymbol{x}(kT)]$. Substituting this control signal into (29) leads to the following closed loop system matrix $\boldsymbol{A_c} = [\boldsymbol{I_n} - \boldsymbol{b}(\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{c}^T]\boldsymbol{A}$. This matrix has the following characteristic polynomial

$$\begin{aligned} \det(z\boldsymbol{I_n} - \boldsymbol{A_c}) = z^n + z^{n-1}(c_1 a_1 + c_{n-1} - c_n)/c_n + \\ + \cdots + z(c_1 a_{n-1} - c_2)/c_n . \end{aligned} \tag{33}$$

As was already stated, $\boldsymbol{c}$ must be selected to satisfy $\boldsymbol{c}^T\boldsymbol{b} \neq 0$. We find, using (31), that this condition implies $c_n \neq 0$. A discrete time system is asymptotically stable if all of its poles are located inside the unit circle. Moreover, in order to obtain finite time error convergence to zero, all of its poles must be located in the origin of the z-plane. Thus, the polynomial (33) must have the following form

$$\det\left(z\boldsymbol{I}_n - \boldsymbol{A}_c\right) = z^n . \tag{34}$$

It can be verified, that (33) simplifies to (34) when the following vector $\boldsymbol{c}$ is chosen

$$c_1 = 1, \quad c_i = \sum_{j=1}^{i-1} a_{n-j} \text{ for } i = 2,\cdots,n . \tag{35}$$

By analyzing the impact of the disturbance, the following values of parameters (14) and (16)

$$F_1 = -d_{\max}/2, \quad F_2 = d_{\max}/2, \quad S_1 = S_2 = 0 \tag{36}$$

are obtained. The non-switching reaching law based sliding mode controller can be derived by substituting (36) into (18)

$$u(kT) = \\ = \left\{ \boldsymbol{c}^T\left(\boldsymbol{I}_n - \boldsymbol{A}\right)\boldsymbol{x}(kT) + f\left[\sigma(kT)\right] + d_{\max}/2 \right\} \Big/ \sum_{i=1}^{n-1} a_i . \tag{37}$$

Moreover, it can be noticed, that by selecting $\boldsymbol{c}$ as (35) (Bartoszewicz and Leśniewski, 2014)

$$\boldsymbol{c}^T\left(\boldsymbol{I}_n - \boldsymbol{A}\right) = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} . \tag{38}$$

holds. Therefore, using equations (31), (35) and (38), control signal (37) actually simplifies to

$$u(kT) = \left\{ f\left[\sigma(kT)\right] + d_{\max}/2 \right\} \Big/ \sum_{i=1}^{n-1} a_i . \tag{39}$$

This concludes the design of the non-switching reaching law based sliding mode controller for the considered system. Next, the important properties that are guaranteed with its application will be demonstrated. In the following theorem it will be shown, that (39) is always non-negative and upper bounded. As this signal directly corresponds to replenishment orders sent to the suppliers, these properties are essential for application.

Theorem 4 The control signal (39) satisfies

$$0 \le u(kT) \le \left[f\left(y_d\right) + d_{\max}/2 \right] \Big/ \sum_{i=1}^{n-1} a_i \tag{40}$$

for any $k \ge 0$.

Proof: It follows from theorems 2 and 3, that the absolute value of the sliding variable will decrease in every period, until (20) gets satisfied. Furthermore, once (20) is true, it will hold for the rest of the control process. For the considered periodic review inventory system $\sigma(0) = \boldsymbol{c}^T\boldsymbol{x}_d = y_d$. Therefore, the sliding variable satisfies

$$\sigma(kT) \in \left[\sigma_{B-}, y_d\right] \tag{41}$$

for any $k \ge 0$. The value σ_{B-} in (41) depends on the choice of function $f[\sigma(kT)]$ and is defined by equation (17). The control signal (39) never decreases with the increase of the sliding variable. Therefore, by taking into account (17) and (36) and substituting the bounds (41) into (39) one can conclude that (40) indeed holds for any $k \ge 0$.

As already stated, the total replenishment order is distributed among the suppliers in proportion to their maximum supply rates. Thus, if function $f[\sigma(kT)]$ is selected in such a way, that the right hand side of (40) is equal to or smaller than $\sum_{l=1}^{m} u_{l\max}$, then no provider will ever be requested to send an amount of goods that exceeds its capabilities.

In logistic systems the costs of emergency storage are often quite high. Therefore, an efficient inventory management system should ensure, that the inventory level will never exceed some *a priori* known value. If the warehouse size is chosen as greater than or equal to this value, then all incoming shipments will be stored inside the warehouse, and the need for emergency storage will be completely eliminated. This issue is considered in the next theorem.

Theorem 5 The stock level will satisfy

$$y(kT) \le y_d - \sigma_{B-} , \tag{42}$$

for all $k \ge 0$.

Proof: From (41) it is evident, that

$$\sigma(kT) \ge \sigma_{B-} , \tag{43}$$

for all $k \ge 0$. By using (2) and (43) one obtains

$$y(kT) \le y_d - \sum_{i=2}^{n} c_i x_i(kT) - \sigma_{B-} . \tag{44}$$

All of the state variables, except for the first one, are the delayed values of the control signal. Thus, according to Theorem 4 they are always non-negative. The elements of vector $\boldsymbol{c}$ (35) are also non-negative. Therefore, one can conclude, that (44) implies (42) and this ends the proof.

In order to maximize profit, the inventory resupply policy should ensure 100% consumers' demand satisfaction. Therefore, in the final theorem the smallest possible value of the demand stock that guarantees, that after some initial time, the on-hand stock will never drop to zero will be derived. This property implies full satisfaction of the consumers' demand.

Theorem 6 If the demand stock level satisfies

$$y_d > \sigma_{B+} + \left(d_{\max} \sum_{i=1}^{n-1} i a_i \right) \Big/ \sum_{i=1}^{n-1} a_i , \tag{45}$$

then the stock level will be strictly positive for any $k > k_0 + n - 1$, where k_0 is the first time when (20) is satisfied. The term σ_{B+} on the right hand side of inequality (45) depends on the choice of function $f[\sigma(kT)]$ and is defined by equation (17).

Proof: From (20) it follows, that

$$y(kT) \geq y_d - \sum_{i=2}^{n} c_i x_i(kT) - \sigma_{B+} = \\ = y_d - \sum_{i=2}^{n} c_i u\big[(k-n+i-1)T\big] - \sigma_{B+} \tag{46}$$

for all $k \geq k_0$. Furthermore, as the control signal (39) never decreases with the increase of the sliding variable, from (20) one gets

$$u(kT) \leq d_{\max} \Big/ \sum_{i=1}^{n-1} a_i \tag{47}$$

also for all $k \geq k_0$. Inequality (47) can be combined with (46) to obtain

$$y(kT) \geq y_d - \left(d_{\max} \sum_{i=1}^{n-1} i a_i \right) \Big/ \sum_{i=1}^{n-1} a_i - \sigma_{B+} \tag{48}$$

for all $k > k_0 + n - 1$. If (45) holds, then the right hand side of (48) is strictly positive, and this ends the proof.

5. SIMULATION RESULTS

In this section the properties of the non-switching reaching law based sliding mode controller will be verified in computer simulations. A periodic review inventory system with three suppliers is considered. Their parameters are shown in Table 1.

Table 1. Suppliers' parameters.

p	L_p [days]	α_p	$u_{\max p}$ [items]
1	2	0.99	15
2	7	0.97	35
3	9	0.96	60

The review period is equal to one day. As one can observe from Table 1 the suppliers can send a total of 110 items in a single review period. The greatest lead time is equal to nine days, thus $\max(\beta_p) = 9$ and $n = 10$. Parameters a_i are $a_2 = 0.135$, $a_7 = 0.309$, $a_9 = 0.524$, and $a_1 = a_3 = a_4 = a_5 = a_6 = a_8 = 0$. Parameter $d_{\max} = 90$ items, and the actual evolution of function $d(kT)$ is shown in Fig. 2.

The properties of the proposed approach were demonstrated for any choice of function $f[\sigma(kT)]$ that satisfies the presented conditions. In order to perform the simulations, three different functions $f[\sigma(kT)]$ are selected to verify the analytical results. These functions are

$$\begin{aligned} f_1\big[\sigma(kT)\big] &= \frac{\sigma_0}{\sigma_0 + |\sigma(kT)|}\sigma(kT), \\ f_2\big[\sigma(kT)\big] &= r\tanh\big[\sigma(kT)/r\big], \\ f_3\big[\sigma(kT)\big] &= \begin{cases} \sigma(kT) & \text{for } |\sigma(kT)| < \sigma_D \\ \sigma_D \operatorname{sgn}\big[\sigma(kT)\big] & \text{for } |\sigma(kT)| \geq \sigma_D, \end{cases} \end{aligned} \tag{49}$$

where σ_0, r and σ_D are all design parameters. For each reaching law a combination of the appropriate parameter and the demand stock level y_d, is selected so that fully satisfying the consumers' demand and not exceeding the suppliers capabilities are both guaranteed. Furthermore, for each case using Theorem 2 the bounds of the quasi sliding mode σ_{B+}, σ_{B-} were calculated using conditions (17). Note that, choosing symmetric functions resulted in $\sigma_{B+} = -\sigma_{B-}$, which does not hold in general. The value $y_{\max}$, that the stock level will never exceed was derived using Theorem 5. The results are shown in Table 2.

The evolutions of the control signal, the on-hand stock and the sliding variable with the application of f_1 function are depicted in figures 3, 4 and 5. The results with application of f_2 are displayed in figures 6, 7, 8 and the results of simulations with the use of f_3 are shown in figures 9, 10 and 11.

As one can observe from figures 5, 8 and 11 all of the reaching laws ensure the existence and reachability of the quasi sliding mode. Furthermore, all of the controllers generate a non-negative and upper bounded control signal.

Table 2. Non-switching reaching law based controllers' parameters.

Function	Parameter [items]	$\sigma_{B+} = -\sigma_{B-}$ [items]	y_d [items]	$y_{\max}$ [items]
f_1	$s_0 = 66.7$	139	810	949
f_2	$r = 62.4$	57.4	750	807.4
f_3	$\sigma_D = 61.48$	45	738	783

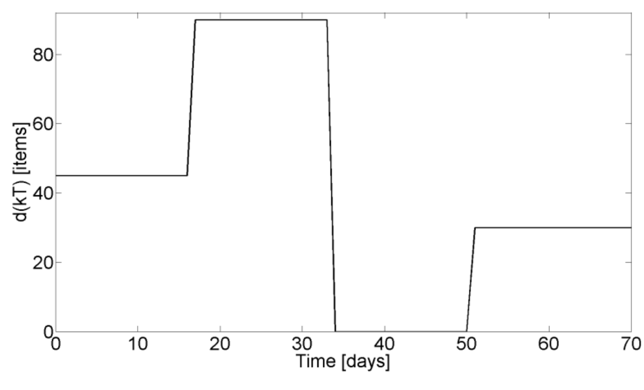

Fig. 2. The consumers' demand.

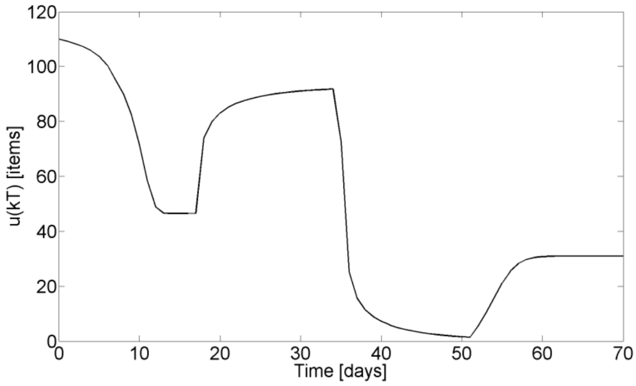

Fig. 3. The control signal generated with the application of function f_1.

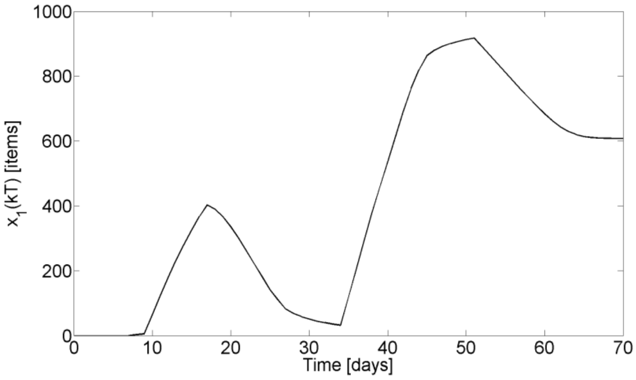

Fig. 4. The stock level with the application of function f_1.

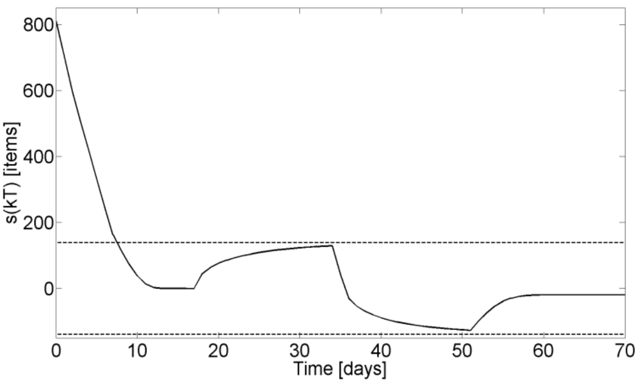

Fig. 5. The sliding variable with the application of function f_1 (the bounds σ_{B+}, σ_{B-} are marked with dashed lines).

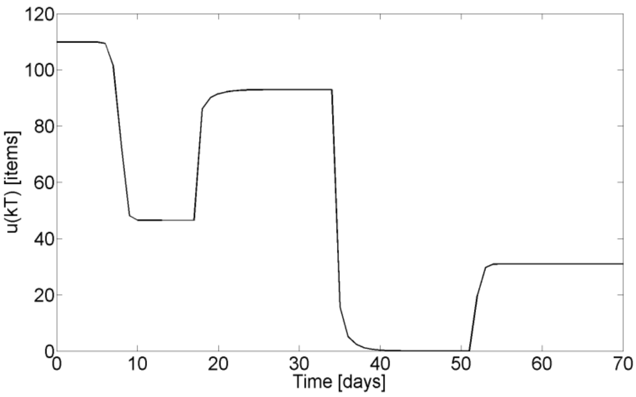

Fig. 6. The control signal generated with the application of function f_2.

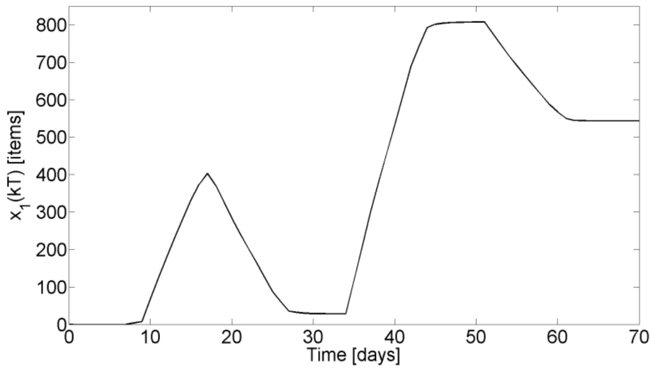

Fig. 7. The stock level with the application of function f_2.

For each of them, the full satisfaction of consumers' demand can be guaranteed by choosing the demand stock level as shown in the previous section. Comparing the results obtained with different reaching laws we notice that even when they are limited to the same maximum value of the control signal, they require different warehouse sizes to ensure full consumers' demand satisfaction. This can be seen from figures 4, 7 and 10. Function f_1 requires the largest warehouse size, and function f_3 the smallest. On the other hand, with the application of function f_3 we obtain a control signal, that after the reaching phase, directly follows the value of the consumers' demand. This means that the undesirable 'bullwhip' effect has not been diminished in this case. Such a rapidly changing replenishment order is hard to follow by the suppliers, and would require additional storage on their side. However, the choice of this function can be a preferred option in other applications which require fast system response. The advantage of function f_1 is that it leads to generating a significantly smoother control signal. The properties of function f_2 lie between the other two functions.

In a particular situation, the choice of function $f[\sigma(kT)]$ should therefore be motivated by a compromise between the cost of storage and the willingness of suppliers to fulfill rapidly changing orders. This choice is not limited to the functions f_1, f_2 and f_3, which were given as examples only.

The values of quadratic performance indices obtained with different functions f_i are shown in Table 3. The second performance index, in which $\delta u(kT) = u(kT) - u[(k-1)T]$, reflects the cost of generating a control signal that is rapidly varying with time, as for many control systems this can be strongly disadvantageous. As one can see from the table, selecting a particular function, the designer chooses a desirable tradeoff between unavoidable sliding imperfection and smoothness of the control signal.

In terms of control engineering the simulations show, that the proposed control algorithm ensures stability and good transient performance of the considered multiple time delay

Table 3. Control performance indices.

Function	$\sum_{k=0}^{70} s^2(kT)$	$\sum_{k=0}^{70} \delta u^2(kT)$
f_1	$2.54 \cdot 10^6$	$1.63 \cdot 10^4$
f_2	$1.79 \cdot 10^6$	$2.21 \cdot 10^4$
f_3	$1.69 \cdot 10^6$	$2.73 \cdot 10^4$

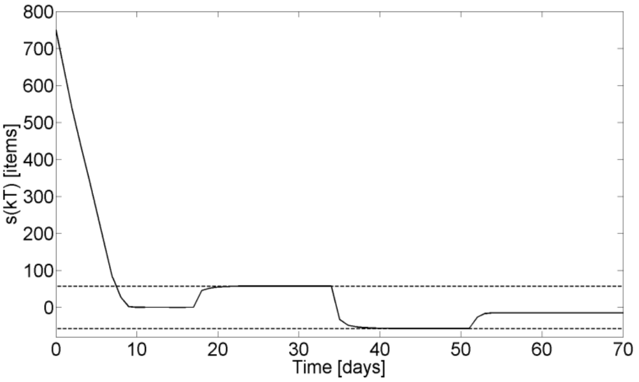

Fig. 8. The sliding variable with the application of function f_2 (the bounds σ_{B+}, σ_{B-} are marked with dashed lines).

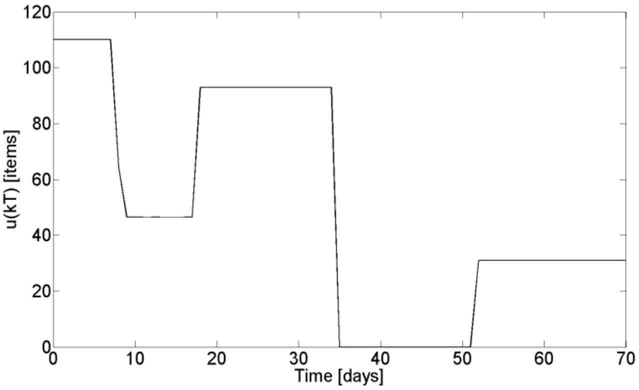

Fig. 9. The control signal generated with the application of function f_3.

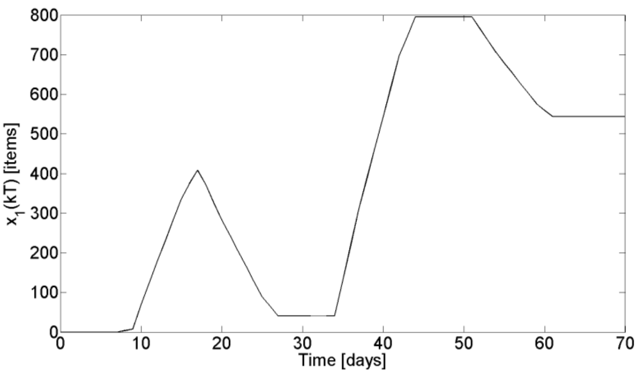

Fig. 10. The stock level with the application of function f_3.

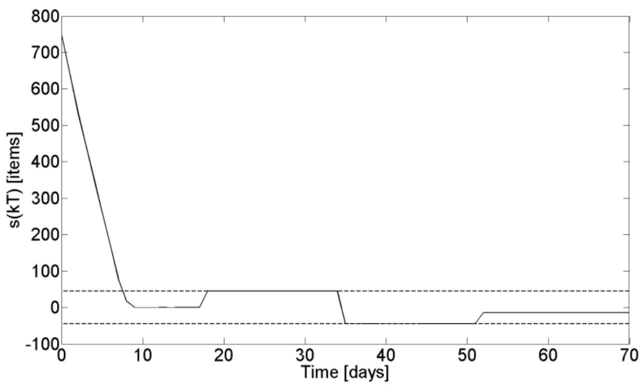

Fig. 11. The sliding variable with the application of function f_3 (the bounds σ_{B+}, σ_{B-} are marked with dashed lines).

system. Furthermore, robustness of the system with respect to strongly mismatched external disturbance has also been demonstrated.

6. CONCLUSIONS

In this work a non-switching reaching law has been presented. This reaching law is based on a general function of the sliding variable. The conditions, that this function must satisfy, in order to ensure reachability and existence of the sliding mode for the nominal system were presented. Then, the additional conditions, under which the quasi-sliding mode is ensured for systems subject to parameter uncertainties and external disturbances were demonstrated. The results presented here are a significant generalization of the work (Bartoszewicz and Leśniewski, 2014), in which only a single function $f[\sigma(kT)]$ was considered. The application of the proposed reaching law to the inventory resupply system has been considered. It has been shown that it ensures a limited value of the resupply orders and eliminates the need for costly emergency storage. The minimal warehouse size, which ensures full consumers' demand satisfaction has been calculated. These properties were verified in computer simulations. In terms of control engineering, the simulations demonstrate, that using the proposed approach ensures the system stability and good transient performance. The proposed framework for designing reaching laws can also easily incorporate constraints of the control signal. Furthermore, by selecting the function on which the reaching law is based, the designer can choose the best trade-off between robustness and the admissible rate of change of the control signal for a particular application.

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