A New Approach of State estimation of Linear Discrete Systems

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Abstract: In this paper a new technique is developed to estimate the states of both deterministic and uncertain discrete-time stochastic linear systems. The proposed approach is based on pole placement technique in which a set of constraints is imposed on the estimated output. The stability of the estimation error is rigorously analyzed for both deterministic and stochastic cases. The developed approach is then applied to estimation problems of discrete-time deterministic linear system as well as uncertain stochastic systems for which the system model and the output measurements are assumed to be corrupted with Gaussian or non- Gaussian zero mean white noise sequences. Simulations results are presented to illustrate the effectiveness of the developed procedure. For the deterministic case, the application of the proposed approach gives much better results when compared with those using Luenberger observer. For the stochastic case, the developed state estimator gives better results when compared with Kalman filter if the statistics of the random signals and/or the system parameters are unknown and/or the noise signals are non-Gaussian.

Keywords: State estimation, discrete Systems, constrained estimator, stability analysis, Pole placement.

1. INTRODUCTION

Most, if not all real systems are represented by incomplete state measurements because some of the states are either very expensive and/or impossible to measure. Examples of these systems are pollution in streams (Mahmoud, Hassan, & Darwish, 1985), jet engines (Maggiore, Ordóñez, Passino, & Adibhatla, 2003) … etc. Knowledge of the performance of the state variables is necessary in order to know the performance of the system and/or to generate state feedback control strategies.

Luenberger observer is normally used to estimate the states of linear deterministic systems for which the systems dynamics and the measurement models are known. The estimated error resulting from this technique showed to be stable and goes to zero as the time goes to infinity (Phillips & Nagle, 2007). As an alternative approach, and due to the duality principle between estimation and control (Goodwin, Graebe, & Salgado, 2001), linear quadratic regulator (LQR) technique can be applied to design the observer gain (Zhong, 2004). For these types of estimators, the poles of the observer are chosen to be near the origin of the z-plane for discrete systemsit is to achieve fast response. However, this may lead to large overshoot of the estimation error in the transient period.

In real life systems are corrupted by both input noise and output measurement noise. Moreover, they can be uncertain due to the impact of the environment, aging … etc. (Williams, 2010), (Åström, 2012) and (Brittain, Corner, Robinson, & Bond, 2010). In such a situation, the observer gain designed with poles far from those of the original system lead to high overshoot in the estimated error which may cause saturation of the system if the estimated states are used to generate closed loop control strategies. On the other hand, for linear systems with certain parameters and uncorrelated input and output white Gaussian noise vectors with known statistics, Kalman filter (KF) is the optimal minimum variance state estimator (Kalman, 1960) (Hongpeng, Chao, Yi, & Qu, 2013) and (Welch, 2004). For uncertain linear stochastic systems, extended Kalman filter (EKF) is the most applicable approach to estimate the states as well as the parameters of the system (Marins, Yun, Bachmann, McGhee, Zyda, 2001), (Wang,& Papageorgiou, 2005), (Hovland,& Antoine, 2006). and (Bolognani, Tubiana, & Zigliotto , 2003). However, such an approach is suboptimal and in most of practical applications it leads to unobservable and undetectable and hence unstable estimator especially if the time horizon is long enough. On the other hand, the application of Kalman filter necessitates the knowledge of the covariance matrices of the input and output noise vectors. In most of the situations, the covariance matrices of the input and output noise vectors are either unknown or approximately known. Several techniques are implimented to handle this problem and showed to give good performance (Bos, Bombois, & Van den Hof, 2005). The adaptive robust extended Kalman filter (AREKF), is widely used in the case of unknown noise covariaces, and leads to stable state estimator. In this approach the design of the estimator is based on the stability analysis, and determines whether the error covariance matrix should be reset according to the magnitude of the innovation (Xiong, Zhang, & Liu, 2009). Another problem may occur where the noise of the system is non-Gaussian (*Saab*,1995) and (Nsour, Abdallah, & Zohdy, 2013) . In this case Kalman filter can be applied to estimate the state, where the noise is considered to be white Gaussian noise and the first and the second moments are used to develop the algorithm of the Kalman filter, but the results are suboptimal rather than optimal (Gustafsson, & Nordlund, 2005). Also, the particle filter (Hoteit, Pham, Triantafyllou, & Korres, 2008), (Chou & Nakajima, 2016) and (Won, Melek, & Golnaraghi, 2010) can be used to deal with non-Gaussian problems. This filter is easy to implement and tune, its main drawback is that it is quite computer intensive since with the computational complexity increases quickly with the state dimension.

In (Zhang, Mehr, & Shi, 2010) a new estimation algorithm, based on the Bayes's theorem, is developed where the filter is experimented by applying it observation data .

In this paper, we present a new state estimator for linear discrete-time dynamical systems. The design of the proposed filter is based on pole placement in which bounds are imposed on the output estimation error of the system. With this approach it is not needed to allocate the poles of the estimator far from those of the system to achieve fast response of the estimated states. The poles of the new filter will be allocated very near from the original system's poles. Therefore, we avoid the existence of excessive overshoot during the transient period of the estimation process, especially if the output measurements are corrupted by noise signals. Such a technique can also be applied to stochastic systems. In this case the main advantages of developed filter are:

a) It does not depend on knowledge of the covariance matrices of the corrupting noise signals.

b) It can handle estimation problem with Gaussian or non-Gaussian noise signals.

c) It leads to stable results with uncertain system parameters.

The only information needed is to have just a guess of the variances of the noise signals especially at the output. The convergence of the developed approach for both the deterministic and the stochastic cases are analyzed. Illustrative examples for both the deterministic and the stochastic cases are presented to show the effectiveness of the proposed filter. For the stochastic case, Monte Carlo simulations are used to compare the results with the well-known Kalman filter.

The rest of the paper is organized as follows. The problem formulation, the design of the developed estimator for deterministic discrete-time linear systems and the stability analysis of the new filter are given in section 2. In section 3 the case of discrete-time stochastic linear system is considered and the stabitity of the estimator is also analyzed. Illustrative examples for both the deterministic and stochastic systems are presented in section 4. The paper is concluded in section5.

2. THE DEVELOPED CONSTRAINTED ESTIMATOR

Consider the following uncertain dynamical discrete-time linear system:

$$\begin{aligned} \boldsymbol{x_{k+1}} &= \left(\boldsymbol{A}+\boldsymbol{\Delta A_k}\right)\boldsymbol{x_k}+\boldsymbol{w_k} \\ \boldsymbol{y_{k+1}} &= \boldsymbol{C}\boldsymbol{x_{k+1}}+\boldsymbol{v_{k+1}} \end{aligned} \tag{1}$$

where $\boldsymbol{x_k} \in \boldsymbol{R^n}$ is the state vector, $\boldsymbol{y_k} \in \boldsymbol{R^r}$ is the measured output vector, $\boldsymbol{A} \in \boldsymbol{R^{n\times n}}$ is the system matrix with the nominal values of the parameters, $\boldsymbol{\Delta A_k} \in \boldsymbol{R^{n\times n}}$ include the uncertainties of the parameters which are uncorrelated white Gaussian or non- Gaussian with zero means and standard deviation either known or unknown. The system (1) is assumed to be stable and the pair $(\boldsymbol{A},\boldsymbol{C})$ is observable, $\boldsymbol{C} \in \boldsymbol{R^{r\times n}}$ is the output matrix, $\boldsymbol{w_k} \in \boldsymbol{R^n}$ and $\boldsymbol{v_k} \in \boldsymbol{R^r}$ are respectively, two zero mean independent input and output noise vectors with covariance matrices $\boldsymbol{Q_k} = \boldsymbol{E}\left\{\boldsymbol{w_k w_k^T}\right\} \in \boldsymbol{R^{n\times n}}$, $\boldsymbol{R_k} = \boldsymbol{E}\left\{\boldsymbol{v_k v_k^T}\right\} \in \boldsymbol{R^{m\times m}}$ and $k \in \{1,2,...\}$ is the discrete time. System (1) satisfies the following model properties:

$\boldsymbol{E}\left\{\boldsymbol{x_k w_j^T}\right\}=0 \quad \forall\ \boldsymbol{j \ge k}$; $\boldsymbol{E}\left\{\boldsymbol{x_k v_j^T}\right\}=0 \quad \forall\ \boldsymbol{k},\boldsymbol{j}$;
$\boldsymbol{E}\left\{\boldsymbol{y_k v_j^T}\right\}=0 \quad \forall\ \boldsymbol{k < j}$; $\boldsymbol{E}\left\{\boldsymbol{y_k w_j^T}\right\}=0 \quad \forall\ \boldsymbol{j > k}$

and $\boldsymbol{E}\left\{\boldsymbol{\Delta A_k}\right\}=0$ the uncertain parameters of the system are uncorrelated with the input and output noise vectors, $\boldsymbol{x_k}$ is independent of $\boldsymbol{\Delta A_j} \quad \forall\ \boldsymbol{j \ge k}$, $\boldsymbol{y_k}$ is independent of $\boldsymbol{\Delta A_j} \quad \forall\ \boldsymbol{j > k}$.

Knowing the estimate $\hat{\boldsymbol{x}}_{\boldsymbol{k|k}} \in \boldsymbol{R^n}$ using the measurements $\{\boldsymbol{y_1},\boldsymbol{y_2}.....\boldsymbol{y_k}\}$, then as we receive the measurements $\boldsymbol{y_{k+1}}$, it is desired to estimate the state vector $\hat{\boldsymbol{x}}_{\boldsymbol{k+1|k+1}}$ such that:

$$\underline{\boldsymbol{\varepsilon}} \le \boldsymbol{y_{k+1}} - \hat{\boldsymbol{y}}_{\boldsymbol{k+1|k+1}} \le \overline{\boldsymbol{\varepsilon}} \tag{2}$$

where $\hat{\boldsymbol{y}}_{\boldsymbol{k+1|k+1}} \in \boldsymbol{R^r}$ is the estimated output, and $\underline{\boldsymbol{\varepsilon}} \in \boldsymbol{R^r}$, $\overline{\boldsymbol{\varepsilon}} \in \boldsymbol{R^r}$ are respectively the lower and upper bounds of the estimation error element by element.

For the deterministic case since the best estimation is achieved if $\boldsymbol{y_{k+1}} = \hat{\boldsymbol{y}}_{\boldsymbol{k+1|k+1}}$, the $\underline{\boldsymbol{\varepsilon}}$, $\overline{\boldsymbol{\varepsilon}}$ can be given a very small number. However, for the stochastic case, $\boldsymbol{y_{k+1}}$ is corrupted by the measurement noise. Therefore, the best estimate $\boldsymbol{y_{k+1|k+1}}$ is the expected value of $\hat{\boldsymbol{y}}_{\boldsymbol{k+1|k+1}}$. Since this is unknown, a reasonable value of the i^{th} element of $\underline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\varepsilon}}$ is in the range $[\boldsymbol{\sigma_i},2\boldsymbol{\sigma_i}]$ where $\boldsymbol{\sigma_i}$ is the standard deviation of the corresponding output noise signal. The proposed estimator is realized through the following three steps:

Step 1: Prediction

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} \\ \hat{y}_{k+1|k} &= C\hat{x}_{k+1|k} = CA\hat{x}_{k|k} \end{aligned} \tag{3}$$

where $\hat{x}_{k+1/k} \in R^n$ is the predicted estimate of the state vector, and $\hat{y}_{k+1/k} \in R^r$ is the predicted estimate of the output given the set of measurements $\{y_1, y_2 .....y_k\}$.

Step 2: Filtering

Once the measurement vector y_{k+1} is received, the filtered estimate of the state vector $\hat{x}_{k+1|k+1}$ and the corresponding output vector $\hat{y}_{k+1|k+1}$ are given in by:

$$\begin{aligned} \hat{x}_{k+1|k+1} &= A\hat{x}_{k|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right] \\ \hat{y}_{k+1|k+1} &= C\hat{x}_{k+1|k+1} \end{aligned} \tag{4}$$

where $L \in R^{n\times r}$ is the observer gain.

Using (1) and (4), the estimation error defined by $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$ is such that:

$$\begin{aligned} \hat{x}_{k+1|k+1} = A_c\tilde{x}_{k|k} + (I - LC)\,\Delta A_k x_{k|k} \\ + (I - LC)w_k - Lv_{k+1} \end{aligned} \tag{5}$$

where $A_c = (A - LCA)$ (6)

The observer gain L is designed such that the estimation error $\tilde{x}_{k+1|k+1} \to 0$ as $k \to \infty$. However, as the poles of A_c are allocated closer to the origin of the z-plane, the settling time of the estimation error is shorter but the estimator shows worst relative stability (greater overshoot) which may be undesirable if the estimated states are used to generate closed loop control policies. To overcome this drawback, the gain matrix L is chosen such that the poles of A_c matrix are near to those of the A matrix of the system. However, to achieve faster response of the estimation error we go to the third step of the proposed estimation technique.

Step 3: The Linear Update Estimator

Since the estimator is not orthogonal as in ideal case of Kalman filter in which the system is certain and the input and output noise vectors are white Gaussian, then it is possible to update the estimator resulting from step 2 above.

This is the main idea of the new developed filter to get a very fast response of the estimated states while avoiding the undesirable large overshoot during the transient period of the estimator. This idea is achieved by satisfying the set of the imposed inequality constraints (2). Such an objective is fulfilled in this step.

Defining the new estimated states vector by $\hat{\hat{x}}_{k+1|k+1}$. Now let:

$$\hat{\hat{x}}^0_{k+1|k+1} = \hat{x}_{k+1|k+1} \tag{7}$$

The estimation error is defined by:

$$\tilde{x}^0_{k+1|k+1} = x_{k+1} - \hat{\hat{x}}^0_{k+1|k+1} \tag{8}$$

Now, we have one of the following two cases which may arise:

<u>Case I:</u>

The estimated outputs satisfy the imposed set constraints in (2). In this case, we let:

$$\hat{x}_{k+1|k+1} = \hat{\hat{x}}^0_{k+1|k+1} \tag{9}$$

and we proceed to the next sampling instant of time.

<u>Case II:</u>

If a subset of the estimated outputs violates the imposed constraints, then this subset is identified and has to be saturated to the lower or the upper bounds which have been violated. Let us define the subset of the violated outputs by $z_{k+1} \in R^l$ where l is the number of the violated constraints and the vector z_{k+1} is given by:

$$z_{k+1|k+1} = Dx_{k+1} + v'_{k+1} \tag{10}$$

For this subset of violated constraints, it is desired to update the estimator (4) to satisfy the equality constraints given by:

$$z_{k+1} - \hat{\hat{z}}^i_{k+1|k+1} = \mu \tag{11}$$

where $\hat{\hat{z}}^i_{k+1|k+1} = D\hat{\hat{x}}^i_{k+1|k+1}$,the matrix $D \in R^{l\times n}$ contains the rows of the matrix C corresponding to the set of the violated outputs and v'_{k+1} is a noise vector containing the elements of the v_{k+1} corresponding to the set of the violated outputs, $\mu \in R^l$ contains the elements of $\underline{\varepsilon}$ or $\bar{\varepsilon}$ corresponding to the violated outputs. Equation (10) will be assumed as new received subset of measurements through which the estimator will be corrected since equation (4) will not lead to orthogonal projection on the innovation space.

The update estimator is proposed to take the form:

$$\begin{aligned} \hat{\hat{x}}^1_{k+1|k+1} &= \hat{\hat{x}}^0_{k+1|k+1} + H[z_{k+1} - \hat{\hat{z}}^0_{k+1|k+1}] \\ \hat{\hat{z}}^0_{k+1|k+1} &= D\hat{\hat{x}}^0_{k+1|k+1} \end{aligned}$$

Such a procedure has to be repeated till the equality constraints given by (2) are satisfied.

$$\hat{\hat{x}}^i_{k+1|k+1} = \hat{\hat{x}}^{i-1}_{k+1|k+1} + H[z_{k+1} - \hat{\hat{z}}^{i-1}_{k+1|k+1}] \tag{12}$$

where $\hat{\hat{x}}^0_{k+1|k+1}$ is as defined by (7) and $H \in R^{n\times l}$ is a gain matrix to be defined later. As $z_{k+1} - \hat{\hat{z}}^i_{k+1|k+1} = \mu$ element by element, we go to the next sampling instant of time.

After the first iteration, the updated error is given by:

$$\tilde{x}^1_{k+1|k+1} = x_{k+1} - \hat{\hat{x}}^1_{k+1|k+1} \tag{13}$$

Substituting (11) into (13) with the iteration number *i=1*, we get:

$$\begin{aligned} \tilde{x}^1_{k+1|k+1} = x_{k+1} \\ - \left\{\hat{\hat{x}}^0_{k+1|k+1)} + H\,(z_{k+1} - \hat{\hat{z}}^0_{k+1|k+1})\right\} \end{aligned} \tag{14}$$

Using (12) and (10) into (14), we have:

$$\begin{aligned} \tilde{x}^1_{k+1|k+1} &= (I - HD)\left[x_{k+1} - \hat{\hat{x}}^0_{k+1|k+1}\right] - Hv'_{k+1} \\ &= (I - HD)\tilde{x}^0_{k+1|k+1} - Hv'_{k+1} \end{aligned} \tag{15}$$

Using the same procedure it can be shown that the updated estimation error at $(i+1)^{th}$ iteration is given by:

$$\tilde{x}^{i+1}_{k+1|k+1} = (I - HD)\tilde{x}^i_{k+1|k+1} - Hv'_{k+1} \tag{16}$$

or:

$$\tilde{x}^{i+1}_{k+1|k+1} = (I - HD)^{i+1}\tilde{x}^{0}_{k+1|k+1} - \sum_{j=0}^{i}(I - HD)^{i-j} Hv'_{k+1} \qquad (17)$$

Using (5), equation (17) can be written in the following form:

$$\begin{aligned}\tilde{x}^{i+1}_{k+1|k+1} &= (I - HD)^{i+1} A_c \tilde{x}_{k|k} \\ &+ (I - HD)^{i+1}(I - LC)\Delta A_k x_{k|k} \\ &+ (I - HD)^{i+1}(I - LC)w_k - (I - HD)^{i+1} Lv_{k+1} \\ &- \sum_{j=0}^{i}(I - HD)^{i-j} Hv'_{j+1}\end{aligned}$$

To guarantee the stability of $\tilde{x}^{i+1}_{k+1|k+1}$, the gain matrix H is designed such that:

$$\left|\lambda\{(I - HD)A_c\}\right| < 1 \qquad (18)$$

The existence of the matrix H is guaranteed if the pair (A_c, D) is observable. It is worth mentioning that the calculation required by this observer is very moderate since firstly it is not required to apply update phase except for a very few points secondly the gain matrix H is calculated once at this phase and applied iteratively to satisfy the violated constraints where the iteration number is small as will be shown later on in the simulation section.

3. STABILITY OF THE LINEAR UPDATE ESTIMATOR

In this section we show firstly that the linear update estimator is equivalent to the least square quadratic estimator (LSQ) with special weighting matrices to allocate the poles at the desired values.

We also show that such an estimator is a Gauss-Newton method. Therefore, its convergence will be as that of Gauss-Newton method which belongs to the class of contraction mapping algorithms.

Since the estimator is not orthogonal and the update procedure is applied only at the sampling instant at which some of the constraints are violated and all the information are known, the linear update estimator is equivalent to the static estimation problem in which we correct the current estimated states iteratively using the same set of measurements.

3.1 Least Square Quadratic Estimator

Given the assumed new received vector of measurements z_{k+1}, let us define the following LSQ estimator problem to be used to update the estimator $\hat{x}^{1}_{k+1|k+1}$:

$$\min_{\hat{x}^{1}_{k+1|k+1}} . \; J = \frac{1}{2}\left\|\hat{x}^{1}_{k+1|k+1} - \hat{x}^{0}_{k+1|k+1}\right\|^2_{W^{-1}} + \frac{1}{2}\left\|z_{k+1} - \hat{z}^{1}_{k+1|k+1}\right\|^2_{M^{-1}} \qquad (19)$$

where $W \in R^{n\times n}$ and $M \in R^{l\times l}$ positive definite weighting matrices.

The necessary conditions of optimality are such that:

$$\frac{\partial J}{\partial \hat{x}^{1}_{k+1|k+1}} = W^{-1}[\hat{x}^{1}_{k+1|k+1} - \hat{x}^{0}_{k+1|k+1}] - D^T M^{-1}[z_{k+1} - \hat{z}^{1}_{k+1|k+1}] = 0 \qquad (20)$$

Let $\hat{x}^{1}_{k+1|k+1} = \hat{x}^{0}_{k+1|k+1} + \Delta\hat{x}^{0}_{k+1|k+1}$ (21)

Then $\hat{z}^{1}_{k+1|k+1} = D\hat{x}^{1}_{k+1|k+1}$ (22)

$$\hat{z}^{1}_{k+1|k+1} = D\hat{x}^{0}_{k+1|k+1} + D\Delta\hat{x}^{0}_{k+1|k+1} \qquad (23)$$

Now using (21), (22) and (23) into (20) while using (10) and after simple algebraic manipulation we get:

$$\begin{aligned}\hat{x}^{1}_{k+1|k+1} &= \hat{x}^{0}_{k+1|k+1} \\ &+ (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} Dx_{k+1} \\ &- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D\hat{x}^{0}_{k+1|k+1} \\ &+ (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned} \qquad (24)$$

The error between the actual and the estimated state $\tilde{x}^{1}_{k+1|k+1} = x_{k+1} - \hat{x}^{1}_{k+1|k+1}$ is given by:

$$\begin{aligned}\tilde{x}^{1}_{k+1|k+1} &= [I - (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D]\tilde{x}^{0}_{k+1|k+1} \\ &- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned} \qquad (25)$$

let $H = [W^{-1} + D^T M^{-1} D]^{-1} D^T M^{-1}$

Therefore, $\tilde{x}^{1}_{k+1|k+1}$ is such that:

$$\begin{aligned}\tilde{x}^{1}_{k+1|k+1} &= [I - (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D]\tilde{x}^{0}_{k+1|k+1} \\ &- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned}$$

$$\tilde{x}^{1}_{k+1|k+1} = [I - HD]\tilde{x}^{0}_{k+1|k+1} - Hv'_{k+1} \qquad (26)$$

Equation (26) is equivalent to equation (16) if the weighting matrices W and M as chosen such that the poles are allocated as desired by (18).

3.2 Gauss- Newton's Method

In this section we show that the linear update estimator is a Gauss- Newton's method. To fulfill this objective, consider the following nonlinear least square problem:

$$\min . \; f(\alpha) = \frac{1}{2} r(\alpha) r^T(\alpha) \qquad (27)$$

where:

$$r(\alpha) = \Gamma \begin{bmatrix} \hat{z}^{1}_{k+1|k+1} - z_{k+1} \\ \hat{x}^{1}_{k+1|k+1} - \hat{x}^{0}_{k+1|k+1} \end{bmatrix}, \quad \Gamma^T \Gamma = \begin{bmatrix} M^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \qquad (28)$$

let

$$\Delta\hat{z}^{0}_{k+1|k+1} = Dx_{k+1} - D\hat{x}^{0}_{k+1|k+1} + v'_{k+1} \qquad (29)$$

Then by using (21), (22) and (29), $r(\alpha)$ can be written as:

$$r(\alpha) = \Gamma \begin{bmatrix} -\Delta\hat{z}^{0}_{k+1|k+1} + D\Delta\hat{x}^{0}_{k+1|k+1} \\ \Delta\hat{x}^{0}_{k+1|k+1} \end{bmatrix} \qquad (30)$$

The necessary condition of optimality is such that

$$\frac{\partial f}{\partial \Delta\hat{x}^{0}_{k+1|k+1}} = D^T M^{-1}(-\Delta\hat{z}^{0}_{k+1|k+1} + D\Delta\hat{x}^{0}_{k+1|k+1})$$
$$+W^{-1}\Delta\hat{x}^{0}_{k+1|k+1} = 0 \tag{31}$$
$$= -D^T M^{-1}\Delta\hat{z}^{0}_{k+1|k+1}$$
$$+(W^{-1} + D^T M^{-1} D)\Delta\hat{x}^{0}_{k+1|k+1} = 0$$

from which one gets:

$$\Delta\hat{x}^{0}_{k+1\backslash k+1} = (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1}\Delta\hat{z}^{0}_{k+1|k+1} \tag{32}$$

Using (29) into (32) we have:

$$\Delta\hat{x}^{0}_{k+1|k+1} = (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D x_{k+1}$$
$$- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D\hat{x}^{0}_{k+1\backslash k+1} \tag{33}$$

Therefore $\hat{x}^{1}_{k+1|k+1}$ will be:

$$\hat{x}^{1}_{k+1|k+1} = \hat{x}^{0}_{k+1|k+1} + \Delta\hat{x}^{0}_{k+1|k+1}$$
$$= \hat{x}^{0}_{k+1\backslash k+1}$$
$$+ (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D x_{k+1} \tag{34}$$
$$- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D\hat{x}^{0}_{k+1|k+1}$$
$$+ (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}$$

Equation (34) is equivalent to equation (24) which proves that the linear update estimator is a Gauss- Newton's method.

To complete our analysis, the convergence of the mean value of the output error vector $\tilde{z}^{i}_{k+1|k+1} = z_{k+1} - \hat{z}^{i}_{k+1|k+1}$ is demonstrated. Using (5), (11) with $i=1$ $\tilde{z}^{1}_{k+1|k+1}$ is given by:

$$\tilde{z}^{1}_{k+1|k+1} = D\tilde{x}^{0}_{k+1|k+1} + v'_{k+1} - DHD\tilde{x}^{0}_{k+1|k+1} - DHv'_{k+1}$$
$$\tilde{z}^{1}_{k+1|k+1} = \left(I - DH\right) D\tilde{z}^{0}_{k+1|k+1} \tag{35}$$

In general, at the i^{th} iteration we get:

$$\tilde{z}^{i}_{k+1|k+1} = (I - DH)^{i}\, \tilde{z}^{0}_{k+1|k+1}$$

For which the norm is given by:

$$\left\|\tilde{z}^{i}_{k+1|k+1}\right\| = \left\|(I - DH)^{i}\, \tilde{z}^{0}_{k+1|k+1}\right\|$$
$$\leq \left\|(I - DH)^{i}\right\| \left\|\tilde{z}^{0}_{k+1|k+1}\right\| \tag{36}$$

If after i^{th} iteration, the norm of matrix $(I - DH)^{i}$ is less than one, then as $i \to N$ $\left\|\bar{\tilde{z}}^{i}_{k+1\backslash k+1}\right\| \to \|\mu\|$ and the mean value of the output of the system converges to the mean value of the desired output which satisfies the imposed constraints.s

4. STABILITY OF THE LINEAR STATE ESTIMATOR

In this section, we analyze the stability of the proposed estimator. Using the results of step (1), (2) and (3) given in section 3, with the iteration number $i=1$, while using (3) and (4) we get:

$$\hat{x}^{1}_{k+1|k+1} = \hat{x}^{0}_{k+1\backslash k+1} + H\left[z_{k+1} - \hat{z}^{0}_{k+1|k+1}\right]$$

$$\hat{x}^{1}_{k+1|k+1} = \hat{x}_{k+1|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right]$$
$$+ H\left[z_{k+1} - \hat{z}^{0}_{k+1|k+1}\right]$$

For the 2nd iteration, we have:

$$\hat{x}^{2}_{k+1|k+1} = \hat{x}^{1}_{k+1\backslash k+1} + H\left[z_{k+1} - \hat{z}^{1}_{k+1|k+1}\right]$$
$$= \hat{x}_{k+1|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right]$$
$$+ H\left[z_{k+1} - \hat{z}^{0}_{k+1|k+1}\right]$$
$$+ H\left[z_{k+1} - \hat{z}^{1}_{k+1|k+1}\right]$$

Since the linear update estimator is a Gauss-Newton method, and as shown above it need N number of iterations to converge to the required bounds as specified by the imposed constraints, filtered estimate at the $(k+1)^{th}$ sampling instant is given by:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right]$$
$$+ \sum_{j=1}^{\infty} H\left[z_{k+1} - \hat{z}^{j-1}_{k+1|k+1}\right] \tag{37}$$

At the end of the convergence, the last term in $\hat{x}_{k+1|k+1}$ has a finite value, and can be written in the form:

$$\alpha_{k+1} = \sum_{j=1}^{N} H\left[z_{k+1} - \hat{z}^{j-1}_{k+1|k+1}\right] \tag{38}$$

or $\alpha_{k+1} = K_{k+1}\left[y_{k+1} - \hat{y}_{k+1|k}\right]$ (39)

where K_{k+1} is a gain matrix of adjustable parameters such that:

$$K_{k+1}\left[y_{k+1} - \hat{y}_{k+1|k+1}\right]$$
$$= \sum_{j=1}^{N} H\left[z_{k+1} - \hat{z}^{j-1}_{k+1|k+1}\right] \tag{40}$$

Using (1), (37) and (40) it can be shown that:

$$\tilde{x}_{k+1|k+1} = A_c^{\,k+1}\tilde{x}_0 + \sum_{j=0}^{k} A_c^{\,k-j}\Big\{-K_{j+1}CA\ \tilde{x}_{j|j}$$
$$+\left(I - LC - K_{j+1}C\right)\Delta A_j x_j$$
$$+\left(I - LC - K_{j+1}C\right)w_j - \left(L + K_{j+1}\right)v_{j+1}\Big\}$$

let $L'_k = L + K_j$

then $\tilde{x}_{k+1|k+1}$ can be written as:

$$\tilde{x}_{k+1|k+1} = A_c^{\,k+1}\tilde{x}_0 + \sum_{j=0}^{k} A_c^{\,k-j}\Big\{-K_{j+1}CA\ \tilde{x}_{j|j}$$
$$+\left(I - L'_{j+1}C\right)\Delta A_j x_j \tag{41}$$
$$+\left(I - L'_{j+1}C\right)w_j - L'_{j+1}v_{j+1}\Big\}$$

Theorem 1:
With assumption that the equilibrium point is stable and observable or detectable, the violated constraints (2) take place during the transient period at the estimation process.

Therefore, as $k \to \infty$ $\left(A_c - K_k CA\right) \to A_c$. Then :

1- If all the solutions of the homogenous equation:

$$\tilde{x}_{k+1|k+1} = A_c \tilde{x}_{k|k} \qquad for \quad k \geq 0 \tag{42}$$

remains bounded as $k \to \infty$ then:

a-The same is true for all the solutions of the mean value of non-homogenous system:

$$\begin{aligned}\tilde{x}_{k+1|k+1} = &\left(A_c - K_{k+1}CA\right)\tilde{x}_{k|k} \\ &+\left(I - L'_{k+1}C\right)\Delta A_k x_k \qquad for \quad k \geq 0 \\ &+\left(I - L'_{k+1}C\right)w_k - \left(L'_{k+1}\right)v_{k+1}\end{aligned} \tag{43}$$

provided that $\sum_{k=0}^{\infty} \lambda_k < \infty \quad for \quad k \geq 0$ (44)

where $\left\| E\left\{ K_{k+1}CA\, \tilde{x}_{k|k} \right\}\right\| = \lambda_k \left\|\bar{\tilde{x}}_{k/k}\right\|$ (45)

and $E\left\{ \tilde{x}_{k|k} \right\} = \bar{\tilde{x}}_{k/k}$

b- If $\lim_{k\to\infty}\left(A_c - K_{k+1}CA\right) = A_c$ (46)

Then $\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \to 0$ as $k \to \infty$ and hence the mean value of both homogenous and non-homogenous system are exponentially stable.

2-If the expected value of the norm of all the solutions of the homogenous equation (42) remains bounded as $k \to \infty$ then the same is also true for the mean value of the norm of all the solutions of the non- homogenous system (43):

provided that $\sum_{k=0}^{\infty}\left\{\beta_k + \rho_k\right\} < \infty \quad for \quad k \geq 0$ (47)

where

$$\left\|\left(I - L'_{k+1}C\right)\right\|\sum_{r=1}^{n}\sigma_{w\,r} + \left\|\left(L'_{k+1}\right)\right\| \; E\left\{\left\|\tilde{x}_{k|k}\right\|\right\}\sum_{r=1}^{m}\sigma_{v\,r} \tag{48}$$

$$E\left\{\left\|\left(I - L'_{k+1}C\right)\Delta A_k x_k\right\|\right\} = \rho_k E\left\{\left\|\tilde{x}_{k|k}\right\|\right\}$$

and $\quad E\left\{\left\|K_{k+1}CA\right\|\right\} = \beta_k E\left\{\left\|\tilde{x}_{k|k}\right\|\right\}$ (49)

and $\sum_{r=1}^{n}\sigma_{w\,r}^{2}$, $\sum_{r=1}^{m}\sigma_{v\,r}^{2}$ are the summation of the square of the standard deviation of the input and output noise vector respectively.

Proof:

1a- Equation (42) can be written as:

$$\tilde{x}_{k+1|k+1} = A_c^{\,k+1}\tilde{x}_0 \qquad for \quad k \geq 0 \tag{50}$$

Since the solution of (50) is bounded as there exists c_1 such that for $k \geq 0$

$$\left\|A_c^{\,k+1}\right\| \leq c_1 \tag{51}$$

The expected value of (41) is given by:

$$\bar{\tilde{x}}_{k+1|k+1} = A_c^{\,k+1}\bar{\tilde{x}}_0 + \sum_{j=0}^{k} A_c^{\,k-j} E\left\{-K_{j+1}CA\tilde{x}_{j|j}\right\} \tag{52}$$

Taking the norm of equation (52) while using (51) we have:

$$\begin{aligned}\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq &\left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| \\ &+ \sum_{j=0}^{k}\left\|A_c^{\,k-j}\right\|\left\|E\left\{-K_{j+1}CA\tilde{x}_j\right\}\right\|\end{aligned}$$

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq c_1\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} c_1\lambda_j \left\|\bar{\tilde{x}}_{j|j}\right\| \tag{53}$$

Let $g_j = c_1\lambda_j$ (54)

Using (54) in (53), we have:

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq c_1\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} g_j \left\|\bar{\tilde{x}}_{j|j}\right\| \tag{55}$$

From (44) there exists c_2 such that:

$$\sum_{k=0}^{\infty}\lambda_k < c_2$$

By Gronwall lemma [44], equation (55) will be:

$$\begin{aligned}\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| &\leq c_1\left\|\bar{\tilde{x}}_0\right\|\prod_{j=0}^{k}(1+g_j) \\ &\leq c_1\left\|\bar{\tilde{x}}_0\right\|\exp\sum_{j=0}^{k} g_j \\ &\leq c_1\left\|\bar{\tilde{x}}_0\right\|\exp(c_1c_2)\end{aligned} \tag{56}$$

This shows that the mean value of the non- homogenous system is bounded.

1b- Since the mean value of the equilibrium is observable and L is designed such that the estimator is asymptotically stable then there exists $0 \leq \delta < 1$ such that:

$$\left\|A_c^{k+1}\right\| < \delta^k \quad for \quad k \geq K$$

Therefore, for both homogeneous and non- homogeneous system we have:

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k}\left\|A_c^{\,k-j}\right\|\lambda_j \left\|\bar{\tilde{x}}_{j|j}\right\|$$

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} c_1\lambda_j \left\|\bar{\tilde{x}}_{j|j}\right\|$$

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} g_j \left\|\bar{\tilde{x}}_0\right\| \tag{57}$$

Using Gronwall lemma, equation (57) will be:

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\|\prod_{j=0}^{k}(1+g_j)$$

$$\begin{aligned}\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| &\leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\|\exp\sum_{j=0}^{k} g_j \\ &\leq \delta^{k+1}\left\|\bar{\tilde{x}}_0\right\|\exp(c_1c_2)\end{aligned} \tag{58}$$

as $k \to \infty$, $\delta^{k+1} \to 0$. This proves that the mean value of the homogenous and non-homogenous system are exponentially stable.

2-From (41), the norm of the error $\tilde{x}_{k+1|k+1}$ will be:

$$\left\|\tilde{x}_{k+1\mid k+1}\right\| \le \left\|A_c^{\,k+1}\right\|\left\|\tilde{x}_0\right\| + \sum_{j=0}^{k}\left\|A_c^{\,k-j}\right\|\left\{\left\|-K_{j+1}CA\tilde{x}_{j\mid j}\right\| + \left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\| + \left\|\left(I - L'_{j+1}C\right)w_j\right\| + \left\|\left(L'_{j+1}\right)v_{j+1}\right\|\right\}$$

$$\left\|\tilde{x}_{k+1\mid k+1}\right\| \le c_1\left\|\tilde{x}_0\right\| + \left\{\sum_{j=0}^{k} c_1\left\|-K_{j+1}CA\tilde{x}_{j\mid j}\right\| + \left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\| + \left\|\left(I - L'_{j+1}C\right)w_j\right\| + \left\|\left(L'_{j+1}\right)v_{j+1}\right\|\right\} \tag{59}$$

The expected value of equation (59) will be:

$$E\left\{\left\|\tilde{x}_{k+1\mid k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} + \sum_{j=0}^{k} c_1\left\{E\left\{\left\|-K_{j+1}CA\tilde{x}_{j\mid j}\right\|\right\} + E\left\{\left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\|\right\} + \left\|\left(I - L'_{j+1}C\right)\right\| E\left\{\left\|w_j\right\|\right\} + \left\|\left(L'_{j+1}\right)\right\| E\left\{\left\|v_{j+1}\right\|\right\}\right\}$$

$$E\left\{\left\|\tilde{x}_{k+1\mid k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} + \sum_{j=0}^{k} c_1\left\{E\left\{\left\|-K_{j+1}CA\tilde{x}_{j\mid j}\right\|\right\} + E\left\{\left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\|\right\} + \left\|\left(I - L'_{j+1}C\right)\right\|\sum_{r=1}^{n}\sigma_{wr} + \left\|\left(L'_{j+1}\right)\right\|\sum_{r=1}^{m}\sigma_{vr}\right\}$$

Let $g'_j = c_1[\beta_j + \rho_j]$ (60)

$$E\left\{\left\|\tilde{x}_{k+1\mid k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} + \sum_{j=0}^{k} g'_j E\left\{\left\|\tilde{x}_{j\mid j}\right\|\right\} \tag{61}$$

Using Gronwall lemma, we have:

$$\begin{aligned} E\left\{\left\|\tilde{x}_{k+1\mid k+1}\right\|\right\} &\le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\}\prod_{j=0}^{k}(1+g'_j) \\ &\le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\}\exp\sum_{j=0}^{k} g'_j \end{aligned} \tag{62}$$

From (47) there exists c_3 such that $\sum_{k=0}^{\infty}\{\beta_k + \rho_k\} < c_3$.

Hence (62) is such as:

$$E\left\{\left\|\tilde{x}_{k+1\mid k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\}\exp(c_1 c_3) \tag{63}$$

This proves that the mean value of the norm of non-homogeneous system is bounded.

However, one has to notice that the mean value of the norm of the solution of the non-homogenous system depends on the standard deviations of the applied noise signals as well as the uncertain parameters. As the levels of the noise signal and uncertainties increase, especially the output noise, the algorithm will have the tendency to diverge. This fact is well known and it is now justified mathematically.

5. SIMULATION AND RESULTS

In this section, the developed algorithm will be applied to both deterministic and stochastic linear discrete-time estimation problems. For the stochastic case, since uncertain linear discrete-time stochastic systems will be considered, and for the purpose of comparison with other estimators, Monte Carlo simulation (Eckhardt, 1987) will be used as a vechile for this purpose. Two indicators will be applied in our analysis. They are defined as follows:

***The Root Mean Square Index (RMSI)*:**
This is a constant indicator to be calculated for each state. It is defined as:

$$RMSI_i = \sqrt{\frac{\sum_{j=1}^{NOMI}\sum_{k_0}^{k_f}\left[x_{i_j}(k) - \hat{x}_{i_j}(k \mid k)\right]^2}{NOMI * \left(k_f - k_0 + 1\right)}} \tag{64}$$

where $NOMI$ is the number of Monte Carlo iterations, k_0 is the initial time and k_f is the final time.

The Root Mean Square Estimation Error (RMS(k)):
This is time varying indicator to be estimated for each state and is given by:

$$RMS_i(k) = \sqrt{\frac{\sum_{j=1}^{NOMI}\left[x_{i_j}(k) - \hat{x}_{i_j}(k \mid k)\right]^2}{NOMI}} \tag{65}$$

Example 1:

Consider the following crane model (Solihin, Akmeliawati, Muhida, & Legowo, 2010):

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{33} & 0 \\ 0 & a_{42} & a_{43} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ b_{31} \\ b_{41} \end{bmatrix} u(t) + w(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} x(t) + v(t) \end{aligned} \tag{66}$$

where $a_{33} = -1/\tau$, $a_{42} = -g/l$, $a_{43} = 1/l\tau$, $b_{31} = k_a\tau$, $b_{41} = -k_a/\tau l$, $l = 0.4m$ is Payload cable length, $\tau = 0.09s$ is identified time constant, $g = 9.81m/s^2$ is gravitational acceleration and $k_a = 0.26m/s$ is identified speed constant. The state $x_1 = h$ is the horizontal trolley position and $x_2 = \theta$ is the angular swing, $x_3 = \dot{h}$ is the change in the horizontal trolley position and $x_4 = \dot{\theta}$ is the change in the angular swing. The controller $u(t)$ is considered to be zero and the system is discretized with sampling period $\Delta T = .01\,sec$. The initial condition of the state $x(0)$ and the estimated state $\hat{x}_{0|0}$ are given by $x(0) = [0.1 \quad 1 \quad 1 \quad 1]^T$, $\hat{x}_{0|0} = [0 \quad 0 \quad 0 \quad 0]^T$, the covariance of the input noise vector $w(k)$ is $Q = E\left\{w(k)w^T(k)\right\} = diag[q \;\; q \;\; q \;\; q]$ where q is given different values as shown in Table 1 and the covariance of the output noise vector $v(k)$ is $R = E\left\{v(k)v^T(k)\right\} = I$.

Case 1: The deterministic case

The estimated states using the developed filter are denoted by x_{ik}^{df}. The results are compared with those using Luenberger observer to be denoted by x_{ik}^{lo}. The gain matrix of Luenberger observer is adjusted to achieve almost the same settling time. The results are shown in Figs. 1-6. From these results it is clear that the Luenberger observer leads to a very high overshoot of the estimated states except the first estimated state while the developed filter leads to satisfactory results for all of the state variables.

Case 2: Stochastic case with certain parameters, white Gaussian noise vectors with known covariances

The results of the developed filter, denoted by DF, is compared with the Kalman filter to be denoted by KF. under the ideal operating conditions in which the system is certain, the input and the output corrupting noise signals are assumed to be white Gaussian with known covariance matrices. For all the stochastic cases, and for the purpose of comparison, 100 Monte Carlo simulations runs are performed and indicators defined by (64) and (65) are used. Simulations results are presented in Figs. 7,8 for this case the covariance of the input noise is given by $Q = diag[0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1]$. The results show that the Kalman filter produces the least ***RMS(k).*** The ***RMSI*** is calculated for the states while using different values of the covariance of the input noise as presented in Table 1. Again, it is obvious that KF is the best in all of these cases. Moreover, Fig. 13 shows the average estimation error for the developed filter.

Case 3: Stochastic with certain parameters, non-Gaussian noise vectors with known covariances

In this case the input and the outputs corrupting noise signals are assumed to be white Non-Gaussian with uniform noise distribution and known covariance matrices. Simulations are shown in Figs. 9,10 for $Q = diag[0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1]$ as well as Table 1 for different values of the input covariance matrix ***Q***. From these results it is clear that the results of the developed filter are better than those of KF.

Case 4: Stochastic with uncertain parameters, white Gaussian noise vectors with unknown covariances

In this case the input and the output corrupting noise signals are assumed to be white Gaussian with unknown covariance matrices and uncertain system parameters. First the covariance matrices of the input and the output noise signals are estimated using the method developed in (Xiong et al., 2009) and applied when needed. Firstly, the developed filter is compared with the Kalman filter while using the nominal values of the system parameters. The results are denoted by KF. Estimating the parameters of a linear system leads to a nonlinear estimation problem. In this case Extended Kalman filter is used and the results are denoted by EKF. Also the results are compared with iterated constrained state estimator (Hassan, 2012) to be denoted by ICE. Simulations are shown in Figs. 11,12 for $Q = diag[0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1]$. The results show that the developed filter produces the least ***RMS(k)*** for this case as well as for the ***RMSI*** different values of the covariance of the input noise as shown in Table 2.

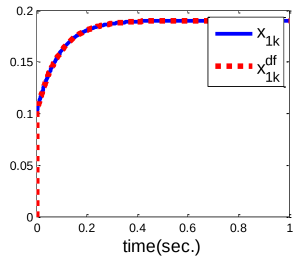

Fig. 1 x_{1k} and $\hat{x}_{1k}$ for the developed filter

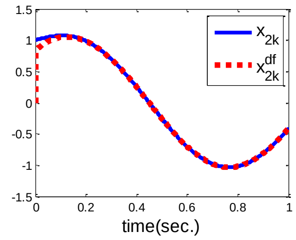

Fig. 2 x_{2k} and $\hat{x}_{2k}$ for the developed filter

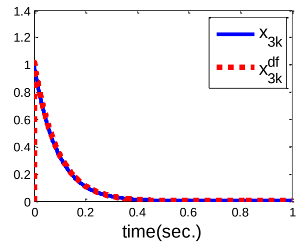

Fig. 3 x_{3k} and $\hat{x}_{3k}$ for the developed filter

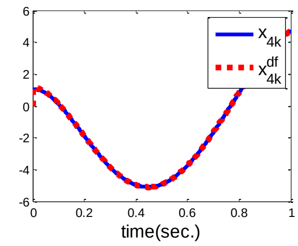

Fig. 4 x_{4k} and $\hat{x}_{4k}$ for the developed filter

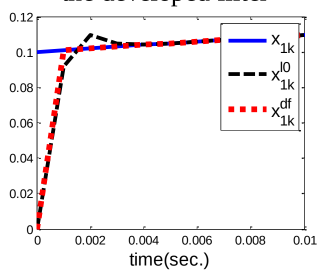

Fig. 5 x_{1k} and $\hat{x}_{1k}$ for luenberger observer and developed filter

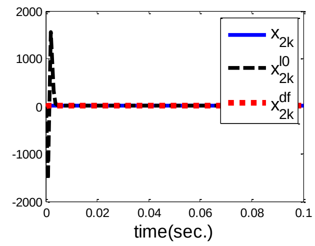

Fig. 6 x_{2k} and $\hat{x}_{2k}$ for luenberger observer and developed filter

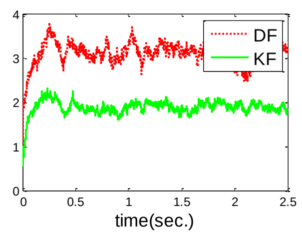

Fig. 7 RMS_3 for x_{3k} (Case2)

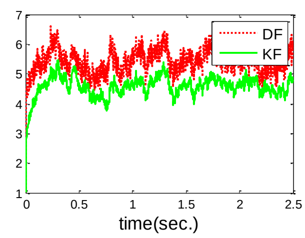

Fig. 8 RMS_4 for x_{4k} (Case2)

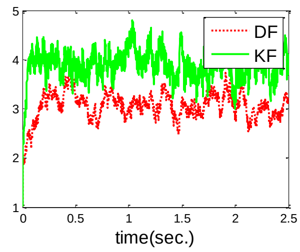

Fig. 9 $\boldsymbol{RMS_3}$ for $\boldsymbol{x_{3k}}$ (Case3)

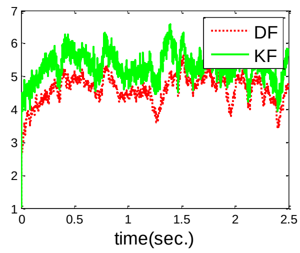

Fig. 10 $\boldsymbol{RMS_4}$ for $\boldsymbol{x_{4k}}$ (Case3)

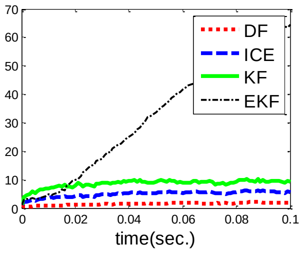

Fig. 11 $\boldsymbol{RMS_3}$ for $\boldsymbol{x_{3k}}$ (Case4)

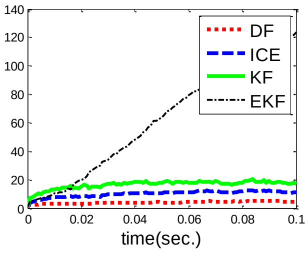

Fig. 12 $\boldsymbol{RMS_4}$ for $\boldsymbol{x_{4k}}$ (Case4)

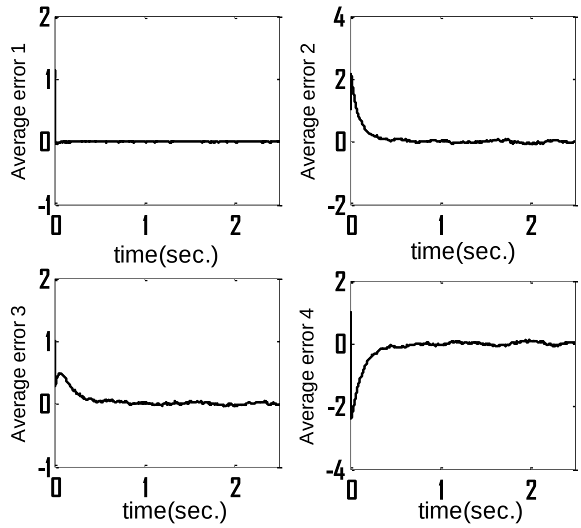

Fig. 13 The average error between the states and the estimated states for the developed filter

Table 1: Comparison between DF, KF for case 2 and case3

			Case2		Case3	
Q	R	Filter	DF	KF	DF	KF
0.01	1	x_1	0.32	0.27	0.36	0.37
		x_2	1.46	1.25	1.48	2.62
		x_3	2.77	0.65	2.68	3.10
		x_4	3.98	2.74	3.82	4.74
0.1	1	x_1	0.71	0.49	0.94	1.18
		x_2	3.49	3.36	3.5	4.81
		x_3	3.06	1.81	3.18	4.16
		x_4	5.34	4.49	4.62	5.37
1	1	x_1	1.56	0.72	1.56	1.84
		x_2	5.21	4.97	5.21	7.12
		x_3	4.98	2.10	4.98	6.57
		x_4	6.10	5.91	6.10	7.88

Table 2: Comparison between DF, KF and ICE for case4

		Case4		
Q	R	DF	KF	ICE
0.01	1	0.41	0.59	0.45
		2.30	6.27	3.02
		3.11	4.65	3.89
		4.28	7.81	5.95
0.1	1	0.81	1.03	0.86
		2.31	6.86	5.19
		2.16	7.72	5.47
		7.24	15.62	10.65
1	1	2.01	2.97	2.51
		7.44	13.62	10.48
		5.92	11.47	6.56
		9.09	18.91	12.19

Example 2:

Consider the following servo motor model (Ramli, Rahmat, & Najib, 2007):

$$\boldsymbol{\dot{x}(t)} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{J_m}\left(D_m + \dfrac{k_t k_b}{R_a}\right) \end{bmatrix} \boldsymbol{x(t)} + \begin{bmatrix} 0 \\ \dfrac{k_t}{R_a J_m} \end{bmatrix} \boldsymbol{u(t)} + \boldsymbol{w(t)} \qquad (67)$$

$$\boldsymbol{y(t)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x(t)} + \boldsymbol{v(t)}$$

where: J_m is motor inertia $= 30\times10^{-6}\,kgm^2$, k_b is back-emf constant= $60\times10^{-3}Vrad^{-1}$, k_t is motor torque constant $=17\times10^{-3}NmA^{-1}$, R_a is armature resistance = 3.2 Ω, D_m =equivalent viscous density by the motor = small (cannot be quoted). The state $x_1 = \theta$ is the angular displacement of the motor shaft and $x_2 = \dot{\theta}$ is the angular velocity of the motor shaft , The controller $\boldsymbol{u(t)}$ is considered to be zero and the system is discretized with sampling period $\Delta T = .01\ sec$. Again, the covariance of the input noise $\boldsymbol{w(k)}$ is $\boldsymbol{Q = E\left\{w(k)w^T(k)\right\} = diag}[q\ \ q\,]$ where the value of q=0.1 and the covariance of the output noise vector $\boldsymbol{v(k)}$ is $\boldsymbol{R = E\left\{v(k)v^T(k)\right\} = I}$.

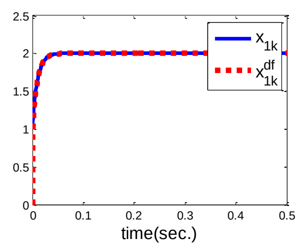

Fig. 14 $\boldsymbol{x_{1k}}$ and $\boldsymbol{\hat{x}_{1k}}$ forthe developed filter

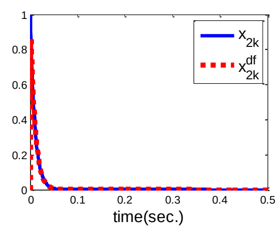

Fig. 15 $\boldsymbol{x_{2k}}$ and $\boldsymbol{\hat{x}_{2k}}$ for the developed filter

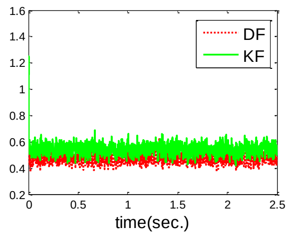

Fig. 16 $\boldsymbol{RMS_1}$ for $\boldsymbol{x_{1k}}$ (Case3)

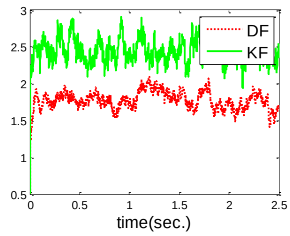

Fig. 17 $\boldsymbol{RMS_2}$ for $\boldsymbol{x_{2k}}$ (case3)

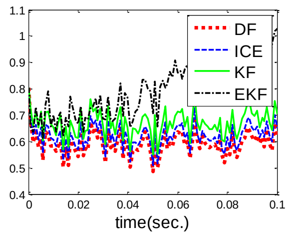

Fig. 18: $\boldsymbol{RMS_1}$ for $\boldsymbol{x_{1k}}$ (case4)

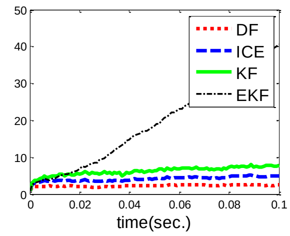

Fig. 19: $\boldsymbol{RMS_2}$ for $\boldsymbol{x_{2k}}$ (case4)

For case 1 Figs. 14,15 show respectively the results of the actual and the estimated states for the deterministic case , a 100 Monte Carlo simulations are executed for the three cases, namely 3 and 4 as in the first example for the value of $\boldsymbol{Q} = \boldsymbol{diag}[0.1\ \ 0.1]$ Figs. 16,17 show the results of case 3 and finally the results of case 4 are shown in Figs 18,19. Again, from these results it is clear that the developed filter gives better results for both case 3 and 4.

Discussion of the Results:

Based on simulation results, one can conclude the following:

1- From Table 1 and Table 2, it is clear that the least *RMSI* are achieved with the application of the developed estimator in cases 3 and 4.

2- The estimated states using the developed approach are stable either in cases (1, 2, 3, 4), for case 4 in which the EKF is unstable (see for Figs. 11,12).

3- In case1 where the Luenberger observer is designed such that the settling time is more or less the same as the developed filter, the overshoot is too high in the estimated states except for the measured states.

4- The average estimation error for the developed filter is stable.

5- Since the main objective is to insure that the estimated outputs are bounded and not tracking the actual measurements, the upper and lower bounds are chosen such as stated in section 2 in equation (2).

6. CONCLUSIONS

In this paper a new technique is developed for state estimation of either deterministic or uncertain stochastic discrete-time linear systems. For the stochastic case where the model parameters are assumed uncertain, the states of the system as well as the measured output vector are assumed to be corrupted by zero mean Gaussian and non-Gaussian noise signals. The statistical data of the parameters, input and the output noise vectors are assumed to be known or unknown. By imposing a set of inequality constraints on the error between the actual and the estimated outputs, an estimator is developed based on Luenberger observer on which a linear update phase is imposed. The stability of the presented approach is rigorously analyzed. This approach can be imbedded in the class of constrained estimation algorithms. Illustrative examples are presented to show the effectiveness of the developed technique. Simulation results show that the developed estimator leads to superior results than Luenberger observer for deterministic case and gives much better results than the well the known Kalman filter except for the ideal case in which the system is certain and the input and output noise signals are white Gaussian with known covariances.

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