Analytical solving of the storage location assignment problem in drug dispensing systems based on a Min-Plus control approach

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Abstract: Pharmacy automation is becoming a key technology to cope with the increase in the demand of healthcare services. In this context, the Automated Drug Dispensing Systems allow for a large volume of dispensing compared to manual dispensing. In this paper, we address the medication assignment problem for an Automated Dispensing System based on a Free-Fall Flow-rack. The current use of these systems adopts a random distribution of drugs inside the rack. The research gap to be filled is to integrate the mutual use of certain drugs together and their placement in neighboring compartments. First, we use a network of Conflicting Timed Event Graphs (CTEGs), a class of timed Petri nets with shared resources, to model the dispensing system. Second, we develop a new method for controlling CTEGs under Mutual Exclusion Constraints (MECs) to solve the problem of drug assignment, using a control approach based on Min-Plus algebra. Finally, a case study of assigning drugs is given to illustrate the proposed methodology and show the efficiency of the developed control laws.

Keywords: Automated Dispensing Systems, Drugs assignment, Mutual exclusion constraints, Discrete event systems, Conflicting timed event graphs, Min-Plus algebra, Control laws.

1. INTRODUCTION

Automated dispensing systems (ADS) have become essential in handling healthcare demands. These systems ensure rapid drug delivery near to the point of use. The study of Automated Dispensing Systems has received increased attention recently in the literature. The greater part of papers in the literature dealing with automated dispensing systems, have focused in evaluating the impact of these systems on reducing dispensing errors (Fanning et al., 2016; Hoffman and Proulx, 2003; Hyland et al., 2007; Pazour and Meller, 2012; Oswald and Caldwell, 2007; Pedersen et al., 2015). Other authors have studied the economic impact of the deploying of automated dispensing systems (Chapuis et al., 2015; Risør et al., 2018). Metahri and Hachemi in (Metahri and Hachemi, 2020) evaluated the saving in delivery time provided by ADS based on a Flow-rack compared to manual delivery.

A particular type of automated dispensing systems concerns automated dispensing cabinets (ADCs). The use of ADCs as the hospital's primary distribution model increased from 22% in 2002 to 70% in 2017 (Deng, 2018). Nonetheless, ADC delivery errors can persist, compromising patient safety and particularly vigilant drug delivery. Unfortunately, human factors such as inattentiveness and fatigue can lead to medication errors. The current health crisis associated with the COVID-19 pandemic is a living example of healthcare worker overload, which can lead to reduced vigilance and increased risk of medication errors. There are several types of distribution errors. For example, drug name confusion due to similarities in dosage, brand name, box color, design, etc. (Hoffman and Proulx, 2003). There are several techniques to reduce these errors. Reference can be made to the use of capital letters in pharmaceutical terminology (Otero López et al., 2011). Using barcodes to identify drugs is another method of preventing drug administration (Hassink et al., 2012). There are also several studies conducted in different hospitals to show the effect of using ADC to reduce error rate and save time (Serrano et al., 2012; Oswald and Caldwell, 2007).

The problem of dispensing errors is more relevant to ADCs than ADSs, because the picking of drugs from the latter is automated. However, given the large volume of drugs distributed in ADSs, the impact of the arrangement of drugs inside the rack on the delivery time becomes significant in the long term. This is why we are interested in the problem of the allocation of drugs in a flow-rack, taking into account the drugs whose mutual use is frequent. In this context, we considered the problem of assigning drugs to rack lanes with the constraint of placing frequently used drugs together, as a control problem of a discrete event system.

In the current publication, we aim to solve the problem of drug localization in ADSs based on flow-rack, considering lane capacity and mutual use constraints. To our knowledge, this problem has not been addressed in the literature. So, in this study we plan to develop a synthesized controller which can allow promoting the placement of frequently used drugs together, in close lanes.

The authors in (Hachemi and Alla, 2013) develop the first work to model the drug allocation problem in an automated dispensing cabinet using discrete event system theory. This was a an untimed Petri net model proposal to represent cabinets and introduce an approach to controller synthesis by

adapting existing techniques and concepts of place invariant marking (Yamalidou et al., 1996) In fact, most of the results proposed in the literature present formal approaches for computing control laws to guarantee marking constraints and forbidden states for untimed Petri nets (Yamalidou et al., 1996; Krogh and Holloway, 1991). In the same context, the study in (Giua et al., 1992) transformed the forbidden marking problem into the mutual exclusion problem using controllers identical to those in (Yamalidou et al., 1996), independently derived without the use of marking invariants method. Therefore, they defined the generalized mutual exclusion constraint (Basile et al., 2000), also called the mutual exclusion constraint, as a constraint that limits the weighted sum of tokens in a sequence of places. If this weight vector is less than or equal to 1, then we are talking about an unweighted generalized mutual exclusion constraint. In this work, this case is assumed as a mutual exclusion constraint.

In fact, in all of these control methods mentioned, time is not considered in the system model and synthesis. However, time parameters are very important, particularly in dynamic systems, and should be integrated into modeling and control approaches. In (Atli et al., 2011), the author linked time and place and used timed place marked graphs as a specific class of discrete event systems with time, to solve the control integration problem. They deal with the prohibited state situation specified by marking exclusion constraints. However, conflicts and resource sharing are not considered. Other studies have chosen to use timed event graphs (TEGs) in models that handle controls within marking constraints. This class associated with the dioid algebra is used to solve various control problems. For example, in (Tebani et al., 2017), control loops are proposed to meet the temporal constraints of networked automated producer-consumer systems modeled by TEGs. In this control strategy, the dynamic behavior was expressed in Max-Plus-algebra inequalities. In the same context, another method to control the TEGs using min-plus formalisms is proposed in (Amari, 2015).

Recently, the authors of (Tebani et al., 2019) proposed a new feedback control approach for observable timed event graphs subject to linear marking constraints using Min-Plus algebra formalism. The aim is to issue realizable control laws to ensure marking constraints on a place or a path of places. Then, they relaxed their assumptions for partially observable transitions (Tebani and Amari, 2021). In (Li et al., 2021), the authors proposed another technique based on Max-Plus algebra to obtain the optimal controller for the TEG under marking constraints. The computed controller can be represented by a series of marked and timed places. Nevertheless, these approaches with TEG do not take resource sharing into account and address specific token limits (maximum number of tokens that apply only to places connected by a path).

The current contribution extends existing approaches (Tebani et al., 2019; Tebani and Amari, 2021) to control timed event graphs under capacity constraints by a more complex case of TEGs, integrating the phenomena of resource sharing and conflict. These systems are modeled by Conflicting Timed Event Graphs (CTEGs).

In several papers interested in this class using dioid algebra, such as (Addad et al., 2012), the authors gave equivalent Max-Plus expressions applied to the job shop example. In (Schafaschek et al., 2020), the authors established optimal control of timed event graphs using shared resources to ensure just-in-time criteria. As in (Moradi et al., 2017), the same class of optimal controls with conflict is developed with the help of residuation theory. Nevertheless, marking constraints are not considered in these resource-sharing TEGs, and the synthesized control law is in open-loop form. In this work, we propose a new modeling framework based on a CTEGs network representing a pharmaceutical dispensing system. We also establish a state-feedback control approach to guarantee mutual exclusion constraints and cap the number of tokens needed in some places. Mutual exclusion constraints that we address are needed at unrelated places and different paths of TEGs with shared resources are considered. We find that the proposed work extends existing control strategies and goes beyond the limitations of approaches that deal with marking constraints imposed only on connected places in timed event graphs. This new contribution generalizes these methods using the property of competition and sharing among resource users and mutual exclusion constraints. This study does not require special tools or extensive computational resources to solve equations and calculate control laws. Moreover, the analytical methodology employed does not suffer from the combinatorial explosion of states problem, as is the case for methods using timed automata.

The rest of this paper is organized as follows. In section 2 a theoretical background of the study is given, mainly TEGs and CTEGs using Min-Plus formalisms. In Section 3, we expose the model of automated dispensing system with a conflicting timed event graphs network. Next, we propose in section 4, a sample method to calculate the number of mutual exclusion constraints and we control within Min-Plus laws the Petri nets network to ensure those constraints. Then, we apply the obtained results on an example of an ADS controlling, and we validate it with a simulation. Finally, section 5 presents conclusions and perspectives for future work.

2. THEORETICAL BACKGROUND OF DISCRETE EVENT SYSTEMS

2.1 Timed Event Graphs (TEGs)

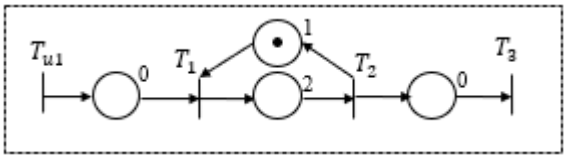

Fig. 1. A timed event graph.

Timed event graphs are a class of timed Petri nets where each place has exactly one upstream and one downstream transition (Murata, 1989). In this study, we consider the category of P-timed event graphs. In this case, a place relying transition T_j toT_i is characterized by a delay that represents the minimum stay duration of a token in this place. An example of a TEG is given by fig. 1.

In fact, the association of using TEGs with Min-Plus (or Max-Plus) is a well-known useful method to solve different problems of control (De Schutter et al., 2020), like legged locomotion via switching models (Lopes et al., 2014), disturbance decoupling problem (Shang et al., 2014), trajectory tracking (Cottenceau et al., 1999) and also representation of asymptotic behavior with variable resources (Lahaye et al., 1999). Unfortunately, timed event graphs are Petri nets with places displaying at most one upstream transition and one downstream transition. Thus, they are not suitable to represent the sharing of resources.

2.2 Conflicting Timed Event Graphs (CTEGs)

Conflicting timed event graphs [14] are a special class of timed Petri nets characterized by an ensemble $G=\{G_1,G_2,\ldots,G_N\}$ of TEGs attached between them by another ensemble $R=\{p_1, p_2, \ldots, p_m\}$ of shared places (resources). Each timed event graph noted, , $=\{1, \ldots, N\}$, has transitions T_j, $j=\{1,\ldots, q_i\}$, and q_i is the number of these transitions. The conflicting place $\tilde{p}$ is connected to a specific TEG G_i by one downstream transition and one upstream transition. The shared place is expended once by a unique graph at a time. In terms of Petri nets, when this place is not marked, there is just one live elementary circuit working with this resource. An example of this class of Petri nets is given by fig. 2.

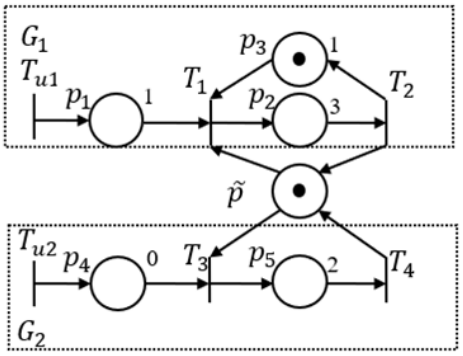

Fig. 2. Two conflicting TEGs.

2.3 Min-Plus algebra and TEGs

Min-Plus algebra is a mathematical formalism that authorizes a linear modeling of certain discrete event systems with a nonlinear representation, defined by differential or difference equations in conventional algebra. Some aspects related to the theory of dioid and Min-Plus algebra are recalled in this section. More details request the consultation of (Baccelli, 1992).

Generally, a dioid, noted $(D,\oplus,\otimes)$, is a set D required with an internal composition law $\oplus$, which is associative, commutative and has a neutral element, denoted ε so that: $\forall a \in D, a \oplus \varepsilon = \varepsilon \oplus a = a$. The second law, designated by $\otimes$, is associative, left- and right-distributive over the law $\oplus$. It concedes a neutral element, denoted$e$, and an absorbing element ε, so that: $\forall a \in D, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$. Notice also that the law $\oplus$ is idempotent, to be known: $\forall a \in D, a \oplus a = a$. If the second law $\otimes$ is commutative, a dioid is then called to be commutative.

We consider in these works the Min-Plus commutative dioid, also called Min-Plus algebra and denoted $\mathbb{R}_{min} = (\mathbb{R} \cup \{+\infty\},\oplus,\otimes)$. The law $\oplus$ in this case corresponds to the min application, i.e., $\forall a, b \in \mathbb{R}_{min}: a \oplus b = \min(a, b)$, with the neutral element$\varepsilon = +\infty$. The second law $\otimes$ corresponds to the classical addition, that is $\forall a, b \in \overline{\mathbb{R}}_{min}: a \otimes b = a + b$, with the neutral element$e = 0$. When it is a matrix dioid, the array of $n \times m$ matrices, where $n, m \in \mathbb{N}^2$and $\mathbb{R}_{min}$ is denoted $\mathbb{R}_{min}^{n\times m}$, the succeeding operations are given:

$$\forall A, B \in \mathbb{R}_{min}^{n\times m}, \forall C \in \mathbb{R}_{min}^{m\times p}:$$
$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}, \forall i = 1, \ldots, n, \forall j = 1, \ldots, m$$
$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{m} A_{ik} \otimes C_{kj}, \forall i = 1, \ldots, n,$$
$$\forall j = 1, \ldots, p$$

The equation of a timed event graph can be done in the time domain, where the system is written by dependent functions of time t. In this case, we are interested in the number of times the transitions are activated until the instant considered. Indeed, this representation consists in associating with each transition $T_i$of the graph, a function $x(t) \in \mathbb{R}_{min}$ called "counter" that allows to describe its dynamic behavior. Counters corresponding to the source transitions (input transitions) are components of the vector $u(t) \in \mathbb{R}_{min}$.

We denote by P_{ij}, the place which connects the transition $T_j$to T_i. The corresponding time delay is τ_{ij} and its marking is denoted M_{ij}. In general, we are interested in the early evolution of timed event graphs. That is, a transition is made as soon as it is feasible.

3. MODELLING OF THE AUTOMATED DISPENSING CABINET (ADC)

3.1 Description of the ADS

Before modeling, we expose the Automated Drug Dispensing System in a global view. Hence, we propose a simple

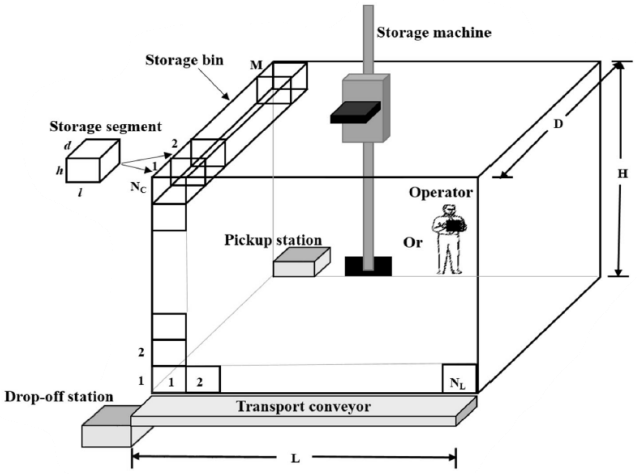

Fig. 3. The ADS description.

The system is storage system based on a deep rack composed of multitude sloping bins, which uses a gravitational conveyor

equipped with rolling wheels to allow the sliding of products by gravity from the storage to the picking side of the system. Each bin is composed of a multiple segments (locations) that contains several identical products (the same SKU: Stock Keeping Unit) placed one after the other. A pickup station is located on the storage face, and a drop-off station is located on the retrieval face. A transport conveyor used to connect the rack to the drop-off station. An operator or a storage machine allows the storage of items.

We consider that the rack is made up of n bins of m locations each. We assume a maximum speed between locations in the same bin, and one type of drug per bin.

3.2 CTEGs network model

For modelling, the definition of transitions and places of the Petri net modeling the ADS is provided in Table 1.

Table 1. Interpretation of Places and Transitions of the Petri net model.

Transition(T)/ Place (P)	Signification
T_{nm}	Transition of a drug to be assigned in the location m of the bin n
T_{n0}	Ejection of a drug from a bin n to the conveyor
P'_{nm}	The location m of the bin n is free
P_{nm}	The location m of the bin n is occupied by a drug

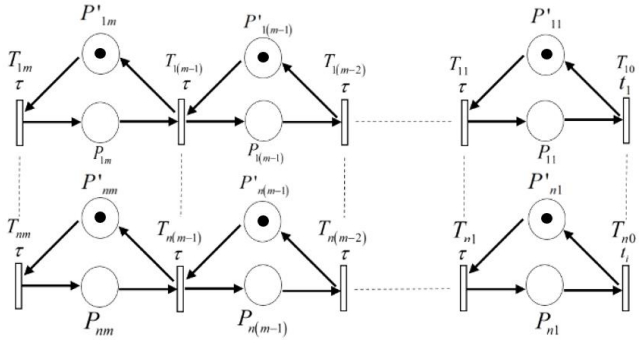

Fig. 4. Timed Petri net model of the Flow-rack.

4. DRUGS ASSIGNMENT AND ADCs CONTROL

The time parameter is a crucial issue, for this we will highlight this parameter. The fall time of the products (drugs) is negligible compared to the time of the movement of the products in the conveyor, The time of the crossing of the transitions noted τ (the movement of the drugs inside the bin) is constant in all the bins of the flow-rack with free fall and assumed to be very small. We consider t_i the delivery time of a drug from bin i. Note that the delivery times for bins in the same column are substantially identical. To determine the column number from the bin number, the Modulo is used, which corresponds to the remainder of a Euclidean division. Indeed, we divide the number of the bin by the number of columns of the bin and we retain the remainder of the division which gives the number of the column in which the drug is located.

The objective of the control approach of the FF flow-rack system is to place the most used drugs together in adjacent compartments close to the point of delivery (step by step); i.e. by first seeking to place them in the same column, if not in a close column.

We propose in the present paper a simple method, this method consists in developing a controller (min, +) which will privilege the placement of two drugs frequently used, in the same column or a close column.

Each combination of two drugs can have 2 types of constraints that must be satisfied: capacity constraints and mutual use constraints. We therefore speak of a Mutual Use of medications Matrix (MUM).

4.1 Control Problem Formulation

In this subsection, we propose a formal approach to control a multi-constrained Petri Net. The role of the computed controller is the satisfaction of the capacity constraints and the mutual use constraints imposed on the considered conflicting temporal event graph network, which describes the free-fall flow-rack system.

To illustrate the approach, we assume a FF flow-rack system containing 16 bins; $n = 16$ (4 rows and 4 columns), each one has 4 locations ($m = 4$). Let I_1, I_2, I_3 et I_4 be four different drugs. We then obtain the GETCs model of the process of figure 5 in accordance with the model of figure 4. In the Petri net assignment model, a bin is represented by a place; because we do not need to represent the movement of a drug inside a bin, since we are only interested in the face of the storage.

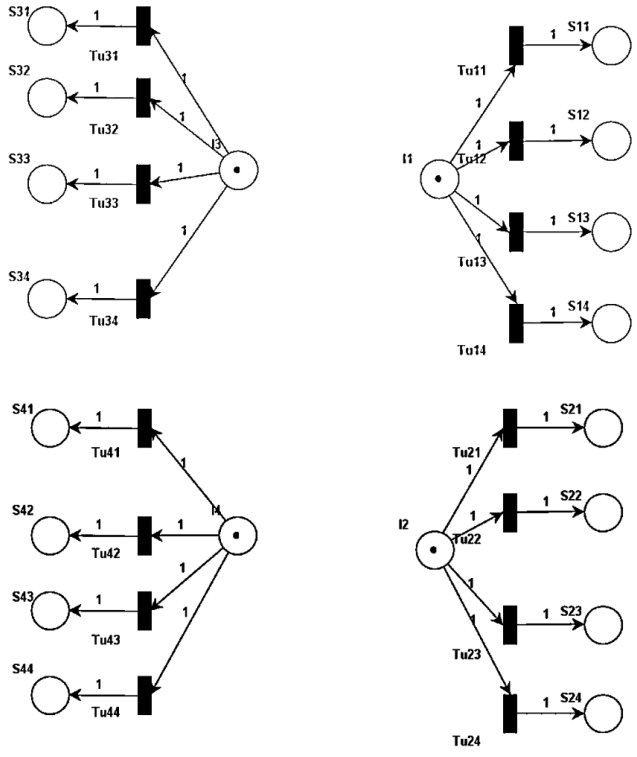

Fig. 5. Petri net model of the assignment process.

Let:

I_d : Place of drug d to be assigned (type of drug)

T_{dn} : Assignment of drug d to bin number n

4.2 Generalization of contraints

We have two types of constraints:

Capacity Constraint: This is the constraint that requires a bin to be reserved for a single drug.

The mutual use constraint: it requires that two frequently used drugs must be placed in neighboring compartments while respecting the first priority of our study (to put frequently used drugs in compartments of the same column near the conveyor)

A complete assignment consists of firing each of the transitions exactly once. Medications to be placed in nearby locations are expressed into constraints on the place markings, which are determined from the Mutual Medication Use Matrix (MUM).

4.3 Mutual drug Use Matrix (MUM)

Let assume that:

S: Mutual drug Use Matrix of dimension $d \times d$ such that $S_{jk} \in [0,1]$, a value of 0 indicates the lowest degree of mutual use between two drugs, while a value of 1 represents the highest degree of mutual use between two drugs. We assume that $S_{jj} = 0$.

I: binary matrix of S of dimension $d \times d$

$$I_{jk} \begin{cases} 1 \; if \; drugs \; j \; and \; k \; are \; in \; high \; demand \; together \\ \qquad\qquad (if \; S_{jk} \geq y) \\ 0 \qquad\qquad\qquad\qquad\qquad else \end{cases} \quad (1)$$

With j, k: drug indices to be allocated to compartments, $(j,k) \in [1,d]$

y: binarization threshold, $y \in]0,1]$

By choosing a reasonable binarization threshold, we can deduce the Mutual drug Use Matrix. In fact, data on mutual drug use can be extracted from the Pharmacy Information System (PIS), which can record drugs that are removed from the pharmacy inventory.

4.4 Description of constraints

Let $M_{dn}(t)$ be the marking of the place S_{dn} .

For each internal transition we associate a counter function denoted $x_{dn}(t)$ and $x'_{dn}(t)$, and for each source transition denoted T_{udn} , we assign a counter function $u_{dn}(t)$. Therefore, the marking of a place can be described by:

$$M_{dn}(t) = x_{dn}(t) - x'_{dn}(t) \quad (2)$$

According to the Petri Net model of the process, a mutual use constraint limits the sum of the marking of the constrained places to two, it can be expressed by:

$$\sum(d,n) \in [1,d] \times [1,n] M_{dn}(t) \leq 2 \quad (3)$$

According to (2), the inequality (3) can be rewritten as follows:

$$x_{dn}(t) - x'_{dn}(t) \leq 2 \quad (4)$$

Which is equivalent to the following form of Min-Pus algebra:

$$\otimes(d,n) \in [1,d] \times [1,n] x_{dn}(t) \leq 2 \otimes (d,n) \in [1,d] \times [1,n] x'_{dn}(t) \quad (5)$$

We consider that the marking of each constrained place has exactly one path with the control transition T_{udn} of the corresponding CTEG compartment model. The counter $x_{dn}(t)$ can be expressed by the inequality:

$$x_{dn}(t) \leq u_{dn}(t) \quad (6)$$

In Min-Plus algebra, this inequality implies:

$$\otimes(d,n) \in [1,d] \times [1,n] x_{dn}(t) \leq 2 \otimes (\otimes(d,n) \in [1,d] \times [1,n] u_{dn}(t)) \quad (7)$$

Due to (7), inequality (5) (which represents the mutual use constraint) holds if the following condition is true:

$$\otimes(d,n) \in [1,d] \times [1,n] u_{dn}(t) \leq 2 \otimes (\otimes(d,n) \in [1,d] \times [1,n] x'_{dn}(t) \quad (8)$$

Consequently, all the control laws of the form $u_{dn}(t)$ which satisfy the condition (8), represent a solution which respects the constraint of mutual use imposed. Therefore, for each constraint, this condition must be verified in order to control the whole FF flow-rack system.

As long as we want to avoid solutions that block input transitions from the start ($u_{dn}(t) = 0$) and that we have a feasible solution to the allocation problem, The control law $u_{dn}(t)$ should have three possible shapes: $u_{dn}(t) = x'_{dn}(t))$, $u_{dn}(t) = 1 \otimes x'_{dn}(t)$ or $u_{dn}(t) = 2 \otimes x'_{dn}(t)$.

The first is represented by an unmarked control square and the second is represented by a marked control square with 1 token, the third is represented by a marked control square with 2 tokens.

If the control law $u_{dn}(t)$ is shared between two (or more) constraints, we obviously consider the one that is minimal.

4.5 Case study

We consider an example consisting of assigning four (4) different drugs in a Free-Fall Flow-rack containing four (4) bins ($n = 4$). Figure 6 shows the case study.

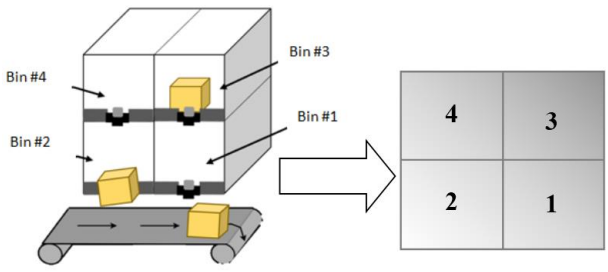

Fig. 6. The case study rack.

We take an example, with a binary mutual use matrix I between 4 drugs, such as:

$$I_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Mutually heavily used drugs are: $[(1,2),(3,4)]$

We also assume that there are four (4) different drugs, so we have 4 capacity constraints and 4 mutual use constraints. The expressions for the capacity constraints are given as follows:

$$(A_1): M'_{11} + M'_{12} + M'_{13} + M'_{14} \leq 1$$
$$(A_2): M'_{21} + M'_{22} + M'_{23} + M'_{24} \leq 1$$
$$(A_3): M'_{31} + M'_{32} + M'_{33} + M'_{34} \leq 1$$
$$(A_4): M'_{41} + M'_{42} + M'_{43} + M'_{44} \leq 1$$

We also have the following representation of mutual use constraints:

$$(A_5): M'_{11} + M'_{23} \leq 2$$
$$(A_6): M'_{12} + M'_{24} \leq 2$$
$$(A_7): M'_{31} + M'_{43} \leq 2$$
$$(A_8): M'_{32} + M'_{44} \leq 2$$

4.5.1 Design of Control Laws

Condition (8) implies that this constraint must satisfy the following inequalities. Therefore, the control laws that satisfy this condition represent a solution that respects the constraint. Let us choose the following control law as a solution.

$$(A_1)\ u_{11}(t) \otimes u_{12}(t) \otimes u_{13}(t) \otimes u_{14}(t) \leq 1 \otimes x'_{11}(t) \otimes x'_{12}(t) \otimes x'_{13}(t) \otimes x'_{14}(t)$$

With:

$$\begin{cases} u_{11}(t) = x'_{11}(t) \\ u_{12}(t) = x'_{12}(t) \\ u_{13}(t) = 1 \otimes x'_{13}(t) \\ u_{14}(t) = x'_{14}(t) \end{cases}$$

$$(A_2)\ u_{21}(t) \otimes u_{22}(t) \otimes u_{23}(t) \otimes u_{24}(t) \leq 1 \otimes x'_{21}(t) \otimes x'_{22}(t) \otimes x'_{23}(t) \otimes x'_{24}(t)$$

With:

$$\begin{cases} u_{21}(t) = 1 \otimes x'_{21}(t) \\ u_{22}(t) = x'_{22}(t) \\ u_{23}(t) = x'_{23}(t) \\ u_{24}(t) = x'_{44}(t) \end{cases}$$

$$(A_3)\ u_{31}(t) \otimes u_{32}(t) \otimes u_{33}(t) \otimes u_{34}(t) \leq 1 \otimes x'_{31}(t) \otimes x'_{32}(t) \otimes x'_{33}(t) \otimes x'_{34}(t)$$

With:

$$\begin{cases} u_{31}(t) = x'_{31}(t) \\ u_{32}(t) = x'_{32}(t) \\ u_{33}(t) = x'_{33}(t) \\ u_{34}(t) = 1 \otimes x'_{34}(t) \end{cases}$$

$$(A_4)\ u_{41}(t) \otimes u_{42}(t) \otimes u_{43}(t) \otimes u_{44}(t) \leq 1 \otimes x'_{41}(t) \otimes x'_{42}(t) \otimes x'_{43}(t) \otimes x'_{44}(t)$$

With:

$$\begin{cases} u_{41}(t) = x'_{41}(t) \\ u_{42}(t) = 1 \otimes x'_{42}(t) \\ u_{43}(t) = x'_{43}(t) \\ u_{44}(t) = x'_{44}(t) \end{cases}$$

$$(A_5)\ u_{11}(t) \otimes u_{23}(t) \leq 2 \otimes x'_{11}(t) \otimes x'_{23}(t)$$

With:

$$\begin{cases} u_{11}(t) = 2 \otimes x'_{23}(t) \\ u_{23}(t) = x'_{11}(t) \\ \square \end{cases}$$

$$(A_6)\ u_{12}(t) \otimes u_{24}(t) \leq 2 \otimes x'_{12}(t) \otimes x'_{24}(t)$$

With:

$$\begin{cases} u_{12}(t) = x'_{24}(t) \\ u_{24}(t) = 2 \otimes x'_{12}(t) \\ \square \end{cases}$$

$$(A_7)\ u_{31}(t) \otimes u_{43}(t) \leq 2 \otimes x'_{31}(t) \otimes x'_{43}(t)$$

With:

$$\begin{cases} u_{31}(t) = 2 \otimes x'_{31}(t) \\ u_{43}(t) = x'_{43}(t) \\ \square \end{cases}$$

$$(A_8)\ u_{32}(t) \otimes u_{44}(t) \leq 2 \otimes x'_{32}(t) \otimes x'_{44}(t)$$

With:

$$\begin{cases} u_{32}(t) = 2 \otimes x'_{44}(t) \\ u_{44}(t) = x'_{32}(t) \\ \square \end{cases}$$

We select for each constraint the following control laws as solutions:

$$u_{11}(t) = 2 \otimes x'_{23}(t) \oplus x'_{11}(t)$$
$$u_{12}(t) = x'_{24}(t) \oplus x'_{12}(t)$$
$$u_{13}(t) = 1 \otimes x'_{13}(t)$$
$$u_{14}(t) = x'_{14}(t)$$

$$u_{21}(t) = 1 \otimes x'_{31}(t)$$
$$u_{22}(t) = x'_{22}(t)$$
$$u_{23}(t) = x'_{23}(t) \oplus x'_{11}(t)$$
$$u_{24}(t) = 2 \otimes x'_{12}(t) \oplus x'_{24}(t)$$

$$u_{31}(t) = 2 \otimes x'_{31}(t) \oplus x'_{31}(t)$$
$$u_{32}(t) = 2 \otimes x'_{44}(t) \oplus x'_{32}(t)$$
$$u_{33}(t) = x'_{33}(t)$$

$$u_{34}(t) = 1 \otimes x'_{34}(t)$$

$$u_{41}(t) = x'_{41}(t)$$
$$u_{42}(t) = 1 \otimes x'_{42}(t)$$
$$u_{43}(t) = x'_{43}(t)$$
$$u_{44}(t) = x'_{32}(t) \oplus x'_{44}(t)$$

These selected solutions led to a drug assignment in the FF flow-rack system that respects the imposed mutual drug use constraints. This control is represented by 22 control places as shown in Fig. 7.

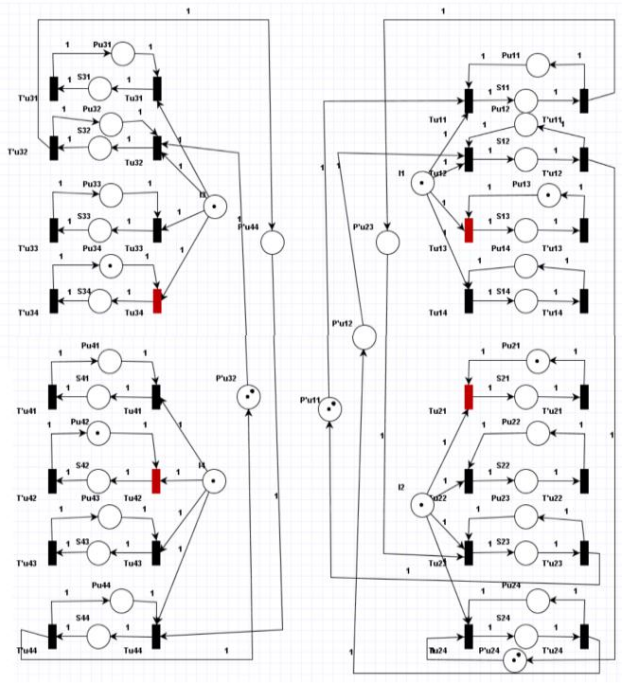

Fig. 7. The controlled CTEGs network model of the case study.

The interpretation of this result gives that drug 1 is assigned in bin 3 (S_{13}), drug 2 is assigned in bin 1 (S_{21}), drug 3 is assigned in bin 4 (S_{34}) and finally, the drug 4 is assigned to bin 2 (S_{42}).

This assignment gives a feasible solution, which respects the constraints of capacities and the constraints of mutual use and shows well that the four different drugs respect the combinations given by the matrix I of mutual use of drugs.

4.5.2 Comparison study

In this subsection, we will evaluate the different delivery times of several drug location configurations. For the case study taken here, with a 4-bin rack, we have 24 possible configurations, which corresponds to permutations 4 among 4. Indeed, the number of permutations of r items among n items is given by the formula: $P(n,r) = \frac{n!}{(n-r)!}$.

To calculate the delivery times of the different bin locations, we define an (x, y) plane as shown in Figure 8. The bins are marked by their centers, and the delivery point is assumed in the coordinates(0,0). For example, bin 2 corresponds to the coordinates(2,1).

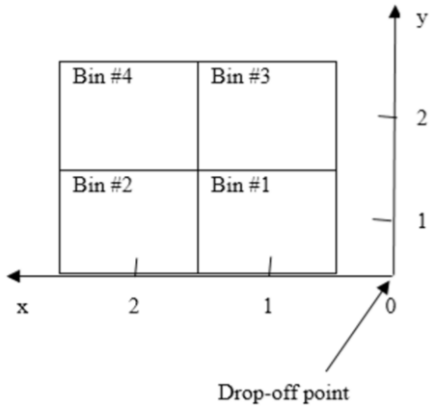

Fig. 8. Coordinates of bins and delivery point.

Note that the delivery time of a drug is the sum of the free fall time and the horizontal travel time on the conveyor to the drop-off point. To carry out the calculations, we refer to the article of (Metahri and Hachemi, 2018).

The following notations are introduced:

d_i: The bin location of the drug i.

t'_h: The horizontal travel time from the impact point to the drop-off point.

t'_v: The vertical travel time from a bin to the impact point on the transport conveyor.

t'_g: The total travel time from a bin to the drop-off point.

$$t'_t(j,k)$$

Thus:

$$t'_g = t'_v + t'_h \quad (9)$$

$$t'_v = C\sqrt{y} \quad (10)$$

where $C = \sqrt{\frac{2}{g}}$ and g is the acceleration due to gravity.

$$t'_h = \frac{x}{V_c} \quad (11)$$

where V_c is the speed of the transport conveyor.

Then $t'_g(x,y) = C\sqrt{y} + \frac{x}{V_c}$ (12)

From expression (12), we can deduce the total travel time of a pair of drugs (j,k) (picked simultaneously) denoted $t'_t(j,k)$, located respectively at the coordinates (xj, yj) and (xk, yk) :

$$t'_t(j,k) = Max\,(t'_g(xj,yj),\ t'_g(xk,yk))$$
$$= Max\,(C\sqrt{yj} + \frac{xj}{V_c},\ C\sqrt{yk} + \frac{xk}{V_c}) \quad (13)$$

For the numerical experiment, we choose a conveyor speed of $Vc = 1m/s$, and the coordinates (x, y) are expressed in meters (m). We will focus on the total transport time (delivery time) of the order of the two drugs 1 and 2 which are widely used mutually.

Table 1 summarizes the total travel time for each configuration; corresponding to an assignment of the 4 drugs

to the 4 bins. The total travel times are highlighted in the histogram of the Figure 9.

It can be seen that the configuration found by the controller corresponds to configuration 14, which gives the minimum total transport time. We can also see that there are other configurations, which give the same optimal time, in this case configurations 3, 4 and 13.

We see that the controller favors the placement of heavily mutually used drugs, close to each other. On the other hand, the distant placement of one medication relative to the other penalizes delivery time.

We can see in Figure 7 that in the controlled Petri net, there are only 4 firable transitions Tu13, Tu21, Tu34 and Tu42 (in red). Their firing gives a mark in places S_{13}, S_{21}, S_{34} and S_{42} respectively. This sequence of marked places gives the locations of the drugs. Note that the given solution is optimal, in the sense that the difference between the travel times of heavily mutually used drugs is minimal, due to the fact that the travel time is the maximum between the two travel times of the two medicines according to relation (13). In other words, the distant placement of one drug relative to the other penalizes delivery times, since the transport time of the two drugs follows a MAX function.

Table 1. Total travel time according to the configuration.

Configuration	d_1	d_2	d_3	d_4	t_t'(s)
1	1	2	3	4	2,45
2	1	2	4	3	2,45
3	1	3	2	4	1,64
4	1	3	4	2	1,64
5	1	4	2	3	2,64
6	1	4	3	2	2,64
7	2	1	3	4	2,45
8	2	1	4	3	2,45
9	2	3	1	4	2,45
10	2	3	4	1	2,45
11	2	4	1	3	2,64
12	2	4	3	1	2,64
13	3	1	2	4	1,64
14	3	1	4	2	1,64
15	3	2	1	4	2,45
16	3	2	4	1	2,45
17	3	4	1	2	2,64
18	3	4	2	1	2,64
19	4	1	2	3	2,64
20	4	1	3	2	2,64
21	4	2	1	3	2,64
22	4	2	3	1	2,64
23	4	3	1	2	2,64
24	4	3	2	1	2,64

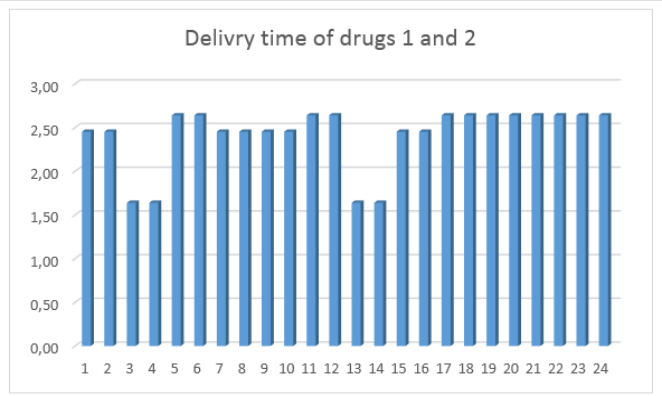

Fig. 9. Histogram of delivery times for the different configurations.

We notice that in general, the controller tends to choose locations in the same column of the rack for drugs that are heavily mutually used. This is due to the fact that the vertical movement time (corresponding to the free fall of the drug) is negligible compared to the horizontal transport time by the conveyor.

Finally, note that the controller synthesis approach remains valid and scalable for a real-size rack, it is just sufficient to formulate all the constraints, and solving these linear equations. In addition, calculating control laws does not require specific tools and heavy computing resources.

5. CONCLUSIONS

In this paper, we have developed a new control framework for the storage location assignment problem in drug dispensing systems using a network of conflicting timed event graphs. An elegant and efficient method based on Min-Plus formalisms is designed to solve the problem of drug assignment subject to two types of mutual exclusion constraints for the automated dispensing system: capacity constraints and mutual drug use constraints. The developed control laws are state feedbacks and can be represented by control places that ensure the system to respect these constraints. The proposed approach for control laws synthesis shows clearly the feasibility of this methodology and the ability of its application to real-world sized dispensing systems. A case study is conducted to illustrate the analytical approach and to check the efficiency of the developed controllers.

For further work, it would be interesting to extend the results of this study to treat more complex CTEGs network cases, since other mutual exclusion constraints guided by good practices dedicated to ADSs can be integrated. Indeed, as a future perspective to this work, we will be able to include in the controller synthesis, other factors such as the expiration date of drugs, drugs requiring controlled refrigeration, or a higher level of security (such as psychotropic drugs). In this case, we must express these requirements in the form of constraints to be imposed in the choice of storage locations.

Data Availability Statement: All data used to support the findings of this study are included within the manuscript.

The authors declare that they have no conflict of interest.

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