Robust Active FTC based on Sliding Mode Observer for Switching Systems with Actuator Faults: Application to HiMAT Vehicle

M. Elouni [*,***], **M. Rodrigues** [**], **H. Hamdi** [*], **B. Rabaoui** [*], **N. BenHadj Braiek** [*]

[*] *Advanced Systems Laboratory, Polytechnic School of Tunisia, University of Carthage, La Marsa 2078, Tunisia. (e-mail: mohamed.elouni@isetb.rnu.tn; habibhemdi@gmail.com; rabaaouibouali@gmail.com; naceur.benhadj@ept.rnu.tn).*
[**] *Automation and Process Engineering Laboratory, University of Lyon, Lyon, Villeurbanne, F-69622, France. (e-mail: mickael.rodrigues@univ-lyon1.fr).*
[***] *National Higher Engineering School of Tunis (ENSIT), University of Tunis, 5 Avenue Taha Hussein, Tunis BP 56-1008, Tunisia.*

Abstract: This paper presents a robust active fault tolerant control (FTC) scheme for a class of continuous-time switched systems in the presence of external disturbances and actuator failures. In this FTC strategy, a sliding mode observer (SMO) is developed so that the estimations of the system state and actuator fault can be obtained simultaneously. With the obtained estimation informations, a switched state feedback fault-tolerant controller is constructed for both compensating the effect of actuator faults and stabilizing the closed-loop system. By using the mode-dependent average dwell time (MDADT) concept and the multiple Lyapunov function sufficient conditions for the existence of the proposed SMO and fault tolerant controller are derived in terms of linear matrix inequalities (LMIs) optimization. Finally, an application to a highly maneuverable aircraft technology (HiMAT) vehicle and a comparative results are considered to illustrate the applicability and performance of the developed methods.

Keywords: Switched Systems, Fault Tolerant Control, Sliding Mode Observer, Mode-Dependent Average Dwell Time, LMIs Constraints.

1. INTRODUCTION

Switched systems represent an important class of hybrid systems, consisting of a collection of linear subsystems and a controlled or autonomous switching sequence specifying the active mode at each time instant. A wide range of physical and engineering systems can be represented by switched models such as mechanical control systems (Sakthivel et al., 2015), power systems (Johnson et al., 2018), automotive systems (Ifqir et al., 2018), smart energy systems (Yuan et al., 2018), flight control systems (Sang and Tao, 2011), fluid mixing (Elouni et al., 2023a) and some other areas.

Regarding the stability and stabilization problems of switched systems, important theoretical results have already been achieved using a common Lyapunov function to guarantee stability under arbitrary switching (Shorten and Narendra, 1998), which usually leads to a conservative result. To reduce the conservatism, multiple Lyapunov functions are proposed for guaranteeing stability under constrained switching with dwell time (DT) (Allerhand and Shaked, 2012) and average dwell time (ADT) (Hespanha and Morse, 1999). ADT switching strategy, a form of constrained switching signals, mandates that the average time for each subsystem's operation within a defined time interval exceeds a constant value. This approach is broader and less conservative compared to the conventional dwell time concept. Thereafter, the mode-dependent average dwell time (MDADT) concept (Zhao et al., 2011) is introduced as an extension of the ADT. This novel concept offers greater flexibility by enabling the association of an ADT with each mode. Moreover, the MDADT technique has garnered extensive utilization in the scholarly literature for addressing fault diagnosis (FD), fault estimation (FE) and implementing fault tolerant control (FTC) for switched linear systems (Hu et al., 2016, 2021; Ladel et al., 2020, 2021).

During the past decades, FD and FTC have become increasingly important in ensuring higher safety and reliability of modern industrial processes. The goal of FD is to detect, isolate, and identify the occurrence of faults or anomalies in a system. Various FD approaches for continuous-time switched systems have been developed in the literature, such as the adaptive observer (AO) method (Liu and Zhao, 2020), unknown input observer (UIO) method (Du et al., 2019), proportional integarl observer (PIO) method (Benzaouia and Telbissi, 2019), descriptor observer method (Elouni et al., 2023b), interval observer method (Zammali et al., 2022), and sliding mode observer (SMO) method (Li et al., 2019; Elouni et al., 2024a). On the other hand, FTC focuses on developing control strategies that can accommodate and mitigate the effects of faults or stability properties despite the presence of faults, ensuring that the system continues to operate within specified operational limits and meets desired performance criteria. In the field of FTC, two main categories can be distinguished: the passive approach and the active approach. The passive approach treats faults as uncertainties and aims to design a controller that can tolerate these uncertainties to some extent. It considers fault occurrence as a form of uncertainty and employs robust control techniques and fault accommodation strategies. While the passive approach is relatively simple in terms of design and implementation, it may not guarantee satisfactory performance or system stability when faced with non-predefined faults. It lacks the capability to specifically detect and isolate faults, which can lead to compromised performance or even system failures. By contrast, the active FTC (AFTC) approach involves the use of fault detection and isolation (FDI) techniques. FDI techniques play a crucial role in detecting the onset of faults and identifying their source. They are designed to analyze sensor measurements, residual signals, or system behavior to detect deviations caused by faults and pinpoint the faulty components or subsystems. By actively detecting and isolating faults, the active approach provides more accurate and timely information about the fault condition, enabling the system to take appropriate fault-tolerant actions. The idea behind AFTC resides in designing a control unit able to cancel the bad effect of such faults automatically and rapidly. The controller should be designed based on FDI techniques and consequent explicit controller redesign (Yang et al., 2010; Yu and Jiang, 2015; Amin and Hasan, 2019; Abbaspour et al., 2020).

In terms of AFTC, several researchers have proposed various FTC approaches for switched systems to address the challenges posed by fault occurrences. For instance, the authors in (Rodrigues et al., 2006) studied AFTC applied to discrete-time switched hybrid systems, and a linear output feedback control strategy employing linear matrix inequalities (LMIs) techniques to guarantee the stability of the closed-loop system, even in the presence of actuator failures. In (Du and Jiang, 2016), a fault-tolerant controller with switched dynamic output feedback was created for a specific category of discrete-time switched systems with state delay. Employing a reduced-order observer, the controller was developed to address faults by stabilizing the closed-loop systems, guided by information obtained from the FE. The study outlined in (Yang et al., 2018) introduced a generalized separation principle for switched systems. This principle facilitates the design of the controller, observer, and switching signal, allowing for independent optimization of each component. In (Zhu et al., 2020), the authors addressed the problem of stabilizing uncertain switched systems subject to external disturbances, actuator, and sensor faults. To accomplish this, they developed a reduced-order observer to robustly estimate system states, actuator and sensor faults, and proficiently reject disturbances in the context of FDI. Additionally, they designed a state-fault-feedback controller with the goal of mitigating the impact of faults and achieving stabilization in the closed-loop systems. Authors in (Telbissi et al., 2023) focused on developing an FTC for a class of continuous-time switched systems with actuator fault. The switched PIO was employed to estimate the system states and the actuator fault simultaneously. Leveraging the information obtained from this observer, a state feedback fault-tolerant controller was developed. The asymptotic stability of the entire system was thoroughly investigated, and this problem was addressed through LMIs by employing a multiple Lyapunov–Krasovskii functional, while also considering the constraints related to the ADT.

In the works mentioned earlier, the observer and controller are developed separately to avoid a bilinear matrix inequality (BMI) formalization of the FTC design strategy. Literature has already addressed the integrated design of the observer and controller for continuous-time switched systems. For example, an integrated sensor FTC scheme was developed for continuous-time switched systems in (Ladel et al., 2020), where sensor faults were estimated according to the MDADT concept. This approach involved the simultaneous synthesis of gains in the switched observer and the switched static feedback controller to guarantee the closed-loop system stability and to compensate for the effect of sensor faults automatically. Besides, an integrated LMI approach was used by (Ladel et al., 2021) to design a robust AFTC law against actuator fault for continuous-time switched systems. Finally, in (Elouni et al., 2023c), an integrated FD and FTC strategy was presented for switched systems subject to external disturbances and actuator faults. This strategy involves the utilization of a switched AO to simultaneously estimate both the actuator faults and the state variables. Based on this observer a state-feedback fault tolerant controller is developed to maintain asymptotic stability of the closed-loop system using the ADT concept. The strategy for solving the integrated FE and FTC design is in general a BMI problem. The challenge arises from the interdependence between the controller and the observer, making their design a complex task. This coupling establishes a relationship between the closed-loop state dynamics and the estimation error adding considerable complexity to the design process. Consequently, the Lyapunov stability condition becomes a set of BMIs, presenting a formidable challenge. Effectively dealing with BMIs is particularly demanding due to their dependence on optimization algorithms within BMI solvers.

Based on the above discussion, in the present paper we aim to address the state/actuator fault estimation and robust AFTC problem for continuous-time switched systems subject to external disturbances and actuator failures. Generally, the primary contributions of this manuscript stem from three key aspects. Firstly, we have designed a SMO for continuous-time switched systems subject to actuator fault and external disturbance under the MDADT switching concept and multiple Lyapunov functions, which is able to estimate both system states and actuator faults simultaneously. Secondly, in contrast to the existing results, such as the work presented in (Salwa et al., 2023) which didn't take into account the disturbances when analyzing FTC systems, the proposed AFTC is robust to external disturbances. Thus, robust stability for both, observer and controller, is developed by using a so-called weighted H_∞ control method to minimize the effect of disturbances not only on the system output but also on state estimation error and fault estimation error. At last, a sufficient conditions are derived in terms of LMIs to guarantee the existence of the SMO and the robust active fault tolerant controller. The gains of the observer and controller are computed separately to avoid the coupling and reduce the computation complexity.

The remainder of this paper is organized as follows. The problem formulation and some preliminaries are addressed in Section 2. Section 3 delineates the design procedure of the proposed SMO and formulates the sufficient condition for its existence. Section 4 details the proposed robust AFTC strategy. In Section 5, two examples are given to demonstrate the effectiveness of the developed approach. Finally, the conclusion is drawn in Section 6.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Preliminaries on LMIs and notations

A large number of problems concerning dynamical systems can be put in the form of convex optimization problems of a particular type called semi-definite programming (SDPs). The main interest of SDPs is the possibility of computing the global minimum, these SDPs are also known as LMIs. The book of (Boyd et al., 1994) has popularized the use of these techniques based on convex optimization for the analysis and control of dynamical systems. This method has been very popular among control engineers in recent years. This is because a wide variety of control problems can be formulated as LMI problems.
An LMI has the following form:

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{1}$$

where $x \in \mathbb{R}^m$ is a vector of the decision variables and $F_i \in \mathbb{R}^{m\times m}$, $i = 1, ..., m$ are symmetric matrices. Writing inequalities in LMI form often requires the use of special transformations. The results below present some of the most used ones.
Lemma 1. (Schur complement) (Boyd et al., 1994) Let be two symmetric matrices $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ and let be a matrix $X \in \mathbb{R}^{m\times n}$. Then the following expressions are equivalent:

$$\begin{bmatrix} Q & X^T \\ X & P \end{bmatrix} > 0, \begin{cases} Q > 0 \\ P - XQ^{-1}X^T > 0 \end{cases}, \begin{cases} P > 0 \\ Q - XP^{-1}X^T > 0 \end{cases} \tag{2}$$

Lemma 2. (Moodi and Farrokhi, 2014) Given any matrices X and Y with proper dimensions, and a scalar $\zeta \in \mathbb{R}^+$, we have the following inequality:

$$X^TY + Y^TX \leqslant \zeta^{-1}X^TX + \zeta Y^TY \tag{3}$$

The notations used throughout the paper are standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote an $n-$ dimensional and $n \times m-$ dimensional Euclidean space, respectively. I_n represents the $n \times n$ identity matrix; 0 denotes the zero matrix with appropriate dimensions. $\|.\|$ represents the 2-norm of a vector or matrix. For a real symmetric matrix P, $P > 0$, or $P < 0$ means that P is positive definite or negative definite, respectively. In a block matrix, the notation $*$ stands for the terms induced by symmetry. The superscript T denotes matrix transpose and $diag\{...\}$ stands for a block-diagonal matrix. M^+ denotes the pseudo-inverse of M.

2.2 Problem statement

In this part, the problem formulation and some preliminaries are presented. Consider the continuous-time switched system affected by actuator faults and unknown external disturbances, which is formulated as follows:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + F_{\sigma(t)}f_a(t) + Rd(t) \\ y(t) = Cx(t) \end{cases} \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^m$ are the state vector, the FTC law to be designed and the measurable output vector of the system, respectively. $f_a(t) \in \mathbb{R}^q$ represents the actuator fault vector and $d(t) \in \mathbb{R}^l$ is the unknown external disturbance vector. The switching signal $\sigma(t) : \mathbb{R}^+ \to \Theta = \{1, 2, ..., N\}$ is a piecewise right-continuous function which allows selecting the active subsystem in continuous time among the N possible modes. When $\sigma(t) = i$, is the index of the active subsystem and assumed to be known. In this case, the system (4) can be rewritten in the form below:

$$\begin{cases} \dot{x}(t) = A_ix(t) + B_iu(t) + F_if_a(t) + Rd(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

Indeed, in the above system representation, the parameter matrices are considered constant real matrices with appropriate dimensions.

Remark 1. The system model provided in (5) demonstrates increased versatility in comparison to the model presented in (Salwa et al., 2023). Our focus here encompasses the simultaneous consideration of actuator fault $f_a(t)$ and disturbance $d(t)$, a comprehensive treatment that sets it apart from previous literature. It is noteworthy that earlier works only partially tackled these aspects, as previously noted.

Before starting the design of the switched SMO, we make the assumption that:

Assumption 1.

$$rank(CF_i) = rank(F_i) = q$$

Assumption 2. It is assumed that the condition for the number of actuator faults and disturbances is met:

$$q + l \leqslant m$$

Assumption 3. The actuator fault $f_a(t)$ satisfies $\|f_a(t)\| \leqslant \alpha_1$, where α_1 is a non-negative real number.

Assumption 4.

$$rank \begin{bmatrix} I_n & R \\ C & 0 \end{bmatrix} = n + rank\,(R)$$

Assumption 5. The pair (A_i, C) is observable $\forall i \in \Theta$.

$$rank \begin{bmatrix} C \\ CA_i \\ \vdots \\ CA_i^{n-1} \end{bmatrix} = n$$

Remark 2. Assumptions 1 and 2 are commonly employed in FE-based FTC systems when addressing actuator faults. Assumption 3 implies that the actuator faults is norm bounded with unknown upper bound, which is a standard assumption in FE theory for practical systems. Assumption 5 guarantees the observability of the switched systems in order to prove the proposed observer existence.

Initially, we introduce a definition and a lemma that establish the basis for defining the MDADT concept and for presenting corresponding stability results associated with this concept.

Definition 1. (Zhao et al., 2011) For a switching signal $\sigma(t)$, let $N_{\sigma i}(t_1,t_2), \forall t_2 \geq t_1 \geq 0, i \in \Theta$, be the number of times that subsystem i is activated over the interval $[t_1,t_2]$ and let $T_i(t_1,t_2)$ denote the total running time of subsystem i over the interval $[t_1,t_2]$, $i \in \Theta$. We say that $\sigma(t)$ has a MDADT τ_{ai} if for any $i \in \Theta$ there exist positive numbers N_{0i} and τ_{ai} such that

$$N_{\sigma i}(t_1,t_2) \leq N_{0i} + \frac{T_i(t_1,t_2)}{\tau_{ai}}, \quad \forall t_2 \geq t_1 \geq 0$$

where N_{0i} are called mode dependent chatter bounds.

Lemma 3. (Zhao et al., 2011)

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \sigma(t) \in \Theta \tag{6}$$

and let $\lambda_i > 0$, $\mu_i > 1$ be given constants. Suppose that there exist C^1 functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$ and two class K_∞ functions κ_{1i} and κ_{2i} such that

$$\kappa_{1i}(\|x(t)\|) \leq V_i(x(t)) \leq \kappa_{2i}(\|x(t)\|) \tag{7}$$

$$\dot{V}_i(x(t)) \leq -\lambda_i V_i(x(t)) \tag{8}$$

and $\forall (\sigma(t_l) = i, \sigma(t_{l-1}) = j) \in \Theta \times \Theta, i \neq j$

$$V_i(x(t_l)) \leq \mu_i V_j(x(t_l)) \tag{9}$$

then the switched system (6) is globally uniformly asymptotically stable (GUAS) for any switching signal with MDADT:

$$\tau_{ai} \geq \tau_{ai}^* = \frac{\ln \mu_i}{\lambda_i} \tag{10}$$

3. SWITCHING SLIDING MODE OBSERVER DESIGN

In this section, a switched SMO is designed so that the estimations of system states and actuator fault can be obtained simultaneously. Based on (Elouni et al., 2024b), a switched SMO corresponding to the switched system model (5) is constructed as follows:

$$\begin{cases} \dot{z}(t) = N_i z(t) + G_i u(t) + L_i y(t) - \Upsilon_i \nu(t) \\ \hat{x}(t) = z(t) + T_2 y(t) \\ \hat{y}(t) = C\hat{x}(t) \\ r(t) = y(t) - \hat{y}(t) \end{cases} \tag{11}$$

where $z(t) \in \mathbb{R}^n$ is the observer state vector, $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state vector, $\hat{y}(t) \in \mathbb{R}^m$ is the observer output and $\nu(t)$ is the discontinuous vector to induce a sliding motion (Edwards et al., 2000) and to be designed. $r(t)$ is the so-called residual signal generated by comparison of the measured and estimated outputs. The gain matrix $\Upsilon_i \in \mathbb{R}^{n\times m}$ is the feedforward injection map. $N_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n\times p}$, $L_i \in \mathbb{R}^{n\times m}$and $T_2 \in \mathbb{R}^{n\times m}$ are the unknown observer gain matrices of appropriate dimensions to be determined.

Define the state estimation error $e_x(t) = x(t) - \hat{x}(t)$. Based on (5) and (11), we get:

$$e_x(t) = x(t) - \hat{x}(t) = (I_n - T_2 C)x(t) - z(t) \tag{12}$$

Since that for $rank \begin{bmatrix} I_n \\ C \end{bmatrix} = n$, there exists nonsingular matrices $T_1 \in \mathbb{R}^{n\times n}$ and $T_2 \in \mathbb{R}^{n\times m}$ matrices be defined as follows:

$$T_1 + T_2 C = I_n \tag{13}$$

Therefore, based on (13), the state estimation error can be reduced into

$$e_x(t) = T_1 x(t) - z(t) \tag{14}$$

According to (14), it follows that the dynamic equation of the state estimation error is

$$\dot{e}_x(t) = T_1 \dot{x}(t) - \dot{z}(t) \tag{15}$$

By substituting the expressions for $\dot{x}(t)$ and $\dot{z}(t)$, derived from equations (5) and (11) respectively, and performing some calculations, the equation (15) transforms into:

$$\begin{aligned} \dot{e}_x(t) = {} & N_i e_x(t) + T_1 R d(t) + T_1 F_i f_a(t) + \Upsilon_i \nu(t) \\ & + (T_1 A_i + N_i T_2 C - L_i C - N_i)x(t) \\ & + (T_1 B_i - G_i)u(t) \end{aligned} \tag{16}$$

By applying the following relations:

$$G_i = T_1 B_i \tag{17}$$

$$E_i = N_i T_2 - L_i \tag{18}$$

$$N_i = T_1 A_i + E_i C \tag{19}$$

$$T_1 R = 0 \tag{20}$$

The dynamic of the estimation error can be reduced as:

$$\dot{e}_x(t) = N_i e_x(t) + T_1 F_i f_a(t) + \Upsilon_i \nu(t) \tag{21}$$

Hence, to design simultaneously gain matrices T_1 and T_2 such that constraints (13) and (20) are satisfied, one can write the following relationship:

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_n & R \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \tag{22}$$

Under Assumption 4, a general solution of T_1 and T_2 using the generalized inverse matrix is expressed as below:

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & R \\ C & 0 \end{bmatrix}^+ \tag{23}$$

In the remaining, the dynamic of the output estimation error $e_y(t)$ can be described as follows:

$$\dot{e}_y(t) = [CN_i e_x(t) + CT_1 F_i f_a(t) + C\Upsilon_i \nu(t)] \tag{24}$$

The equation of ideal sliding mode may be obtained from conditions such as $e_y(t) = 0$, $\dot{e}_y(t) = 0$. In (11), $\nu(t)$ is considers an external feedforward compensation signal, on the other side in (24) it acts as an input signal. From (24), a virtual equivalent feedforward signal (Yeu et al., 2005) may be described for each $C\Upsilon_i \neq 0$ as:

$$\nu(t) = -(C\Upsilon_i)^{-1}[CN_i e_x(t) + CT_1 F_i f_a(t)] \tag{25}$$

Therefore, the discontinuous term $\nu(t)$ is defined as follows:

$$\nu(t) = \psi \,\mathrm{sgn}(e_y(t)) = \psi \frac{e_y(t)}{\|e_y(t)\|}, e_y(t) \neq 0 \tag{26}$$

where ψ is a positive real number with

$$\psi \geqslant \frac{\|T_1\| \, \|F_i\|}{\|\Upsilon_i\|}\alpha_1 \tag{27}$$

3.1 Stability analysis

For obtaining the matrices of gains and the feedforward compensation signal $\nu(t)$ that ensure the stability of the SMO expressed in (11), the following theorem is presented for designing the proposed SMO.

Theorem 1. Consider constant scalars $\lambda_i > 0$, $\mu_i > 1$, β, ϵ and $\gamma > 0$. The error system (21) is GUAS with an H_∞ performance index γ for any switching signal $\sigma(t)$ with MDADT satisfying constraint (10), if there exist symmetric and positive definite matrices $P_{1i} \in \mathbb{R}^{n\times n}$ and matrices $W_{1i} \in \mathbb{R}^{n\times m}$, such that $\forall(i,j) \in \Theta \times \Theta, i \neq j$, the following LMIs hold:

$$\begin{bmatrix} \mu_i P_{1j} & P_{1i} \\ P_{1i} & P_{1i} \end{bmatrix} \geqslant 0 \tag{28}$$

$$\begin{bmatrix} \Pi_i & P_{1i}(T_1F_i) & P_{1i}\Upsilon_i \\ * & -\frac{1}{\varepsilon}I_p & 0 \\ * & * & -I_m \end{bmatrix} \leqslant 0 \tag{29}$$

where

$$\begin{aligned} \Pi_i = {} & (T_1A_i)^T P_{1i} + P_{1i}(T_1A_i) + \lambda_i P_{1i} \\ & + \beta I_n + C^T {W_{1i}}^T + W_{1i}C + \gamma^2 I_n \end{aligned} \tag{30}$$

$$W_{1i} = P_{1i}E_i \tag{31}$$

Proof. In order to prove the stability and convergence of the state estimations errors, let us consider the multiple Lyapunov function:

$$V_i(e_x(t)) = e_x^T(t)P_{1i}e_x(t) \tag{32}$$

where $P_{1i} = {P_{1i}}^T > 0$.
Differentiating $V_i(e_x(t))$ with respect to time t and considering (21), it leads to:

$$\begin{aligned} \dot{V}_i(e_x(t)) = {} & e_x^T(t)\left[N_i^T P_{1i} + P_{1i}N_i\right] e_x(t) \\ & + 2e_x^T(t)P_{1i}T_1F_if_a(t) + 2e_x^T(t)P_{1i}\Upsilon_i\nu(t) \end{aligned} \tag{33}$$

If $\|f_a(t)\| \leqslant \alpha_1$, then

$$\begin{aligned} \dot{V}_i(e_x(t)) \leqslant {} & e_x^T(t)\left[N_i^T P_{1i} + P_{1i}N_i\right] e_x(t) \\ & + 2\alpha_1 \left\|e_x^T(t)P_{1i}T_1F_i\right\| + 2e_x^T(t)P_{1i}\Upsilon_i\nu(t) \end{aligned} \tag{34}$$

For any positive scalar β, we have the following inequality:

$$2\alpha_1 \left\|e_x^T(t)P_{1i}(T_1F_i)\right\| \leqslant \beta^{-1}{\alpha_1}^2\left\|e_x^T(t)P_{1i}(T_1F_i)\right\|^2 + \beta \tag{35}$$

Hence,

$$\begin{aligned} \dot{V}_i(e_x(t)) \leqslant {} & e_x^T(t)\left[N_i^T P_{1i} + P_{1i}N_i\right] e_x(t) \\ & + \beta^{-1}{\alpha_1}^2\left\|e_x^T(t)P_{1i}(T_1F_i)\right\|^2 + \beta \\ & + e_x^T(t)P_{1i}\Upsilon_i\nu(t) + \nu^T(t)\Upsilon_i^T P_{1i}e_x(t) \end{aligned} \tag{36}$$

By taking $\varepsilon = \beta^{-1}{\alpha_1}^2$, we can obtain :

$$\begin{aligned} \dot{V}_i(e_x(t)) \leqslant {} & e_x^T(t)\left[N_i^T P_{1i} + P_{1i}N_i\right] e_x(t) \\ & + \varepsilon\left\|e_x^T(t)P_{1i}(T_1F_i)\right\|^2 + \beta \\ & + e_x^T(t)P_{1i}\Upsilon_i\nu(t) + \nu^T(t)\Upsilon_i^T P_{1i}e_x(t) \end{aligned} \tag{37}$$

For $\|\nu(t)\| \leqslant \gamma\|e_x(t)\|$, the state estimation error converges asymptotically to zero and the gain from $\nu(t)$ to $e_x(t)$ is bounded by γ if:

$$\dot{V}_i(e_x(t)) + \lambda_i V_i(e_x(t)) - \nu^T(t)\nu(t) + \gamma^2 e_x^T(t)e_x(t) \leqslant 0 \tag{38}$$

Taking into account inequality (37), we obtain

$$\begin{aligned} & e_x^T(t)\left[N_i^T P_{1i} + P_{1i}N_i\right] e_x(t) + \lambda_i e_x^T(t)P_{1i}e_x(t) \\ & + \varepsilon\left\|e_x^T(t)P_{1i}(T_1F_i)\right\|^2 + \beta + e_x^T(t)P_i\Upsilon_i\nu(t) \\ & + \nu^T(t)\Upsilon_i^T P_{1i}e_x(t) + \gamma^2 e_x^T(t)e_x(t) - \nu^T(t)\nu(t) \leqslant 0 \end{aligned} \tag{39}$$

The inequality (39) can be reformulated as follows

$$\left[\, e_x^T(t) \;\; \nu^T(t) \,\right] \Xi_i \begin{bmatrix} e_x(t) \\ \nu(t) \end{bmatrix} \leqslant 0 \tag{40}$$

where

$$\Xi_i = \begin{bmatrix} \tilde{\Delta}_i & P_{1i}\Upsilon_i \\ \Upsilon_i^T P_i & -I_m \end{bmatrix}$$

with

$$\begin{aligned} \tilde{\Delta}_i = {} & N_i^T P_{1i} + P_{1i}N_i + \varepsilon P_{1i}\left(T_1F_i\right)\left(T_1F_i\right)^T P_{1i} \\ & + \lambda_i P_{1i} + \gamma^2 I_n + \beta I_n \end{aligned} \tag{41}$$

Then, the inequality (40) is satisfied if the following condition holds $\forall i \in \Theta$

$$\Xi_i = \begin{bmatrix} \tilde{\Delta}_i & P_{1i}\Upsilon_i \\ \Upsilon_i^T P_{1i} & -I_m \end{bmatrix} \leqslant 0 \tag{42}$$

By substituting N_i with its expression in (19) and defining $W_{1i} = P_{1i}E_i$, and applying the Schur complement, the inequality (42) becomes

$$\begin{bmatrix} \Pi_i & P_{1i}(T_1F_i) & P_{1i}\Upsilon_i \\ * & -\frac{1}{\varepsilon}I_p & 0 \\ * & * & -I_m \end{bmatrix} \leqslant 0 \tag{43}$$

where

$$\begin{aligned} \Pi_i = {} & (T_1A_i)^T P_{1i} + P_{1i}(T_1A_i) + \lambda_i P_{1i} \\ & + \beta I_n + C^T {W_{1i}}^T + W_{1i}C + \gamma^2 I_n \end{aligned}$$

This implies that $e_x(t)$ converges to zero based on Lyapunov stability theory in the presence of faults and the signal $\nu(t)$. By selecting $\Upsilon_i = P_{1i}^{-1}C^T$ (Yeu et al., 2005), the linearity of inequality (43) with respect to P_{1i} and W_{1i} allows for a numerical solution within the Linear Matrix Inequality framework.

The following part is devoted to formalizing the condition (9) in the form of the inequalities (28). We have:

$$V_i(e_x(t_l)) - \mu_i V_j(e_x(t_l)) = e_x^T(t_l)P_{1i}e_x(t_l) - \mu_i e_x^T(t_l)P_{1j}e_x(t_l) \tag{44}$$

Therefore, condition (9) is satisfied if the following condition is met:

$$P_{1i} \leqslant \mu_i P_{1j} \tag{45}$$

The previous inequality can be wtitten:

$$\mu_i P_{1j} - P_{1i} \geqslant 0 \tag{46}$$

Then, by using the Schur complement, (46) is equivalent to:

$$\begin{bmatrix} \mu_i P_{1j} & I_n \\ I_n & P_{1i}^{-1} \end{bmatrix} \geqslant 0 \tag{47}$$

We multiply both sides by $\begin{bmatrix} I_n & 0 \\ 0 & P_{1i} \end{bmatrix}$, thus

$$\begin{bmatrix} \mu_i P_{1j} & P_{1i} \\ P_{1i} & P_{1i} \end{bmatrix} \geqslant 0$$

is obtained. This concludes the proof.

3.2 Estimation of actuator faults

In this part, assuming that the designed SMO gain matrices in (11) are well-proposed, an efficient technique is developed for an actuator fault reconstruction procedure. Relying on the results of Theorem 1, one obtains that $e_y(t) = \dot{e}_y(t) = 0$ as $t \to \infty$ then the equation (24) becomes:

$$CT_1 F_i f_a(t) + C\Upsilon_i \nu(t) = 0 \tag{48}$$

Consequently,

$$CT_1 F_i f_a(t) = -C\Upsilon_i \nu(t) \tag{49}$$

Suppose that the discontinuous component $\nu(t)$ is substituted with a continuous approximation through an alternative method, and by utilizing equations (26) and (27), we will have:

$$\nu_\delta(t) = \alpha_1 \frac{\|T_1\| \, \|F_i\|}{\|\Upsilon_i\|} \frac{e_y(t)}{\|e_y(t)\| + \delta} \tag{50}$$

where δ is a small positive scalar chosen to reduce the chattering effect in the sliding motion. It can be shown that the discontinuous component $\nu(t)$ can be approximated to any degree of accuracy by (50) for a small enough choice of δ. Since $rank(CT_1F_i) = q$ it follows from (49) that actuator faults can be estimated by

$$\hat{f}_a(t) \approx -(CT_1F_i)^+ C\Upsilon_i \nu_\delta(t) \tag{51}$$

4. ACTIVE FTC CLOSED-LOOP SYSTEM DESIGN

Subsequently, once the fault information is obtained, we will address the fault-tolerant control design issue for the system described by equation (5). This aims to compensate the impact of actuator faults and stabilize the resulting closed-loop systems. The approach involves implementing an active FTC law with state feedback, as described below:

$$u(t) = -\Gamma_i \hat{x}(t) - q_{fi} \hat{f}_a(t) \tag{52}$$

where matrix $q_{fi} = B_i^+ F_i$ is supposed to compensate the actuator fault and B_i^+ is a pseudo-inverse of B_i.

Assumption 6.

$$rank(B_i, F_i) = rank(B_i) \tag{53}$$

Remark 3. Assumption 6 implies that the actuator fault should occur in the same channel as the controller. Mathematically speaking, it also ensures that there exists a matrix solution X such that $BX = F$.

Lemma 4. (Jiang et al., 2006) Under Assumption 6, there exists a matrix $B_i^+ \in \mathbb{R}^{p\times n}$ such that:

$$(I - B_i B_i^+)F_i = 0 \tag{54}$$

Substituting (52) into (5), the dynamic equation of the closed-loop system is obtained as follows:

$$\dot{x}(t) = (A_i - B_i\Gamma_i)x(t) + B_i\Gamma_i e_x(t) + F_i e_f(t) + Rd(t) \tag{55}$$

where $e_f(t) = f_a(t) - \hat{f}_a(t)$ is the fault estimtion error.

The closed loop of the continuous-time switched system with external disturbances becomes:

$$\begin{cases} \dot{x}(t) = (A_i - B_i\Gamma_i)x(t) + \bar{B}_i \bar{d}(t) \\ y(t) = Cx(t) \end{cases} \tag{56}$$

where $\bar{B}_i = [\, B_i\Gamma_i \;\; F_i \;\; R \,]$ and $\bar{d}(t) = \left[\, e_x^T(t) \;\; e_f^T(t) \;\; d^T(t) \,\right]^T$ can be considered as an external disturbance, and the boundedness of $e_x(t)$ and $e_f(t)$ can be guaranteed by Section 3.

In order to satisfy the stabilization condition derived in LMI terms for system (56) and to ensure robustness with respect to external disturbances, we propose the following theorem:

Theorem 2. Let Assumptions 1–6 hold. Consider constant scalars $\lambda_i > 0$, $\mu_i > 1$ and $\gamma > 0$. Assume there exist symmetric and positive definite matrices $X_{2i} \in \mathbb{R}^{n\times n}$, $X_{2j} \in \mathbb{R}^{n\times n}$ and matrices $Q_i \in \mathbb{R}^{p\times n}$ such that $\forall(i,j) \in \Theta\times\Theta, i \neq j$ the following LMIs hold:

$$\begin{bmatrix} \Omega_i & B_iQ_i & F_i & R & Q_iC^T \\ * & -2X_{2i} + \frac{1}{\gamma^2}I_n & 0 & 0 & 0 \\ * & * & -\gamma^2 I_q & 0 & 0 \\ * & * & * & -\gamma^2 I_r & 0 \\ * & * & * & * & -I_m \end{bmatrix} \leqslant 0 \tag{57}$$

$$\begin{bmatrix} \mu_i X_{2i} & X_{2j} \\ X_{2j} & X_{2j} \end{bmatrix} \geqslant 0 \tag{58}$$

where

$$X_{2i} = P_{2i}^{-1} \tag{59}$$

$$Q_i = \Gamma_i X_{2i} \tag{60}$$

$$\Omega_i = A_iX_{2i} + X_{2i}A_i^T - B_iQ_i - Q_i^T B_i^T + \lambda_i X_{2i} \tag{61}$$

Then, the closed-loop switched system (56) is GUAS with a weighted H_∞ performance level γ for any switching signal $\sigma(t)$ with MDADT satisfies the following conditions $\tau_{ai} \geqslant \tau_{ai}^* = \frac{\ln \mu_i}{\lambda_i}$. The controller gains are given by $\Gamma_i = Q_i X_{2i}^{-1}$.

Proof. In order to satisfy the stabilization condition of the closed loop system, we consider the following multiple Lyapunov function:

$$V_i(x(t)) = x^T(t)P_{2i}x(t), \forall i \in \Theta \tag{62}$$

By taking into account the closed-loop system (56), the time derivative of $V_i(x(t))$ satisfies

$$\dot{V}_i(x(t)) = x^T(t)\Pi_i x(t) + 2x^T(t)P_{2i}\bar{B}_i\bar{d}(t) \tag{63}$$

where

$$\Pi_i = (A_i - B_i\Gamma_i)^T P_{2i} + P_{2i}\left(A_i - B_i\Gamma_i\right) \tag{64}$$

The objective is to guarantee the stability of the closed loop system such that:

$$\|y(t)\|_{L2}^2 \leqslant \gamma^2 \left\|\bar{d}(t)\right\|_{L2}^2 \tag{65}$$

According to the MDADT concept, the closed-loop system (56) is GUAS with a weighted H_∞ performance γ if the following conditions are met $\forall(i,j) \in \Theta\times\Theta, i \neq j$:

$$\dot{V}_i(x(t)) \leqslant -\lambda_i V_i(x(t)) + \gamma^2 \bar{d}^T(t)\bar{d}(t) - y^T(t)y(t) \tag{66}$$

$$V_i(x(t_l)) \leqslant \mu_i V_j(x(t_l)) \tag{67}$$

Inequality (66) implies that

$$\begin{bmatrix} \Pi_i + \lambda_i P_{2i} + C^TC & P_{2i}\bar{B}_i \\ * & -\gamma^2 I \end{bmatrix} \leqslant 0 \tag{68}$$

Before and after multiplying (68) by $diag\left(P_{2i}^{-1}, P_{2i}^{-1}, I, I, I\right)$ and its transpose, respectively and using the Schur complement, we obtain the following inequality:

$$\begin{bmatrix} \Xi_{11_i} & B_i\Gamma_i P_{2i}^{-1} & F_i & R & P_{2i}^{-1}C^T \\ * & -P_{2i}^{-1}\gamma^2 P_{2i}^{-1} & 0 & 0 & 0 \\ * & * & -\gamma^2 I_q & 0 & 0 \\ * & * & & -\gamma^2 I_r & 0 \\ * & * & * & * & -I_m \end{bmatrix} \leqslant 0 \quad (69)$$

where

$$\Xi_{11_i} = P_{2i}^{-1}\Pi_i P_{2i}^{-1} + \lambda_i P_{2i}^{-1} \quad (70)$$

Using Lemma 2, we can express the following relation:

$$P_{2i}^{-1} + P_{2i}^{-1} \leqslant P_{2i}^{-1}\gamma^2 P_{2i}^{-1} + \frac{1}{\gamma^2} I_n \quad (71)$$

Or equivalently

$$-P_{2i}^{-1}\gamma^2 P_{2i}^{-1} \leqslant -2P_{2i}^{-1} + \frac{1}{\gamma^2} I_n \quad (72)$$

From this, inequality (69) is reformulated into

$$\begin{bmatrix} \Omega_i & B_i Q_i & F_i & R & \mathrm{X}_{2i}C^T \\ * & -2\mathrm{X}_{2i} + \frac{1}{\gamma^2} I_n & 0 & 0 & 0 \\ * & * & -\gamma^2 I_q & 0 & 0 \\ * & * & * & -\gamma^2 I_r & 0 \\ * & * & * & * & -I_m \end{bmatrix} \leqslant 0 \quad (73)$$

with

$$\mathrm{X}_{2i} = P_{2i}^{-1} \quad (74)$$

$$Q_i = \Gamma_i \mathrm{X}_{2i} \quad (75)$$

$$\Omega_i = A_i\mathrm{X}_{2i} + \mathrm{X}_{2i}A_i^T - B_iQ_i - Q_i^T B_i^T + \lambda_i \mathrm{X}_{2i} \quad (76)$$

The next step in the proof involves reformulating condition (67) into the form of LMI. This condition can be elaborated as follows:

$$P_{2i} \leqslant \mu_i P_{2j} \quad (77)$$

Inequality (77) is modified by substituting P_{2i} with X_{2i}^{-1} and P_{2j} with X_{2j}^{-1}. After applying the Schur complement twice, this inequality transforms into:

$$X_{2j} \leqslant \mu_i X_{2i} \quad (78)$$

The previous inequality can be wtitten:

$$\mu_i X_{2i} - X_{2j} \geqslant 0 \quad (79)$$

Then, by using the Schur complement, (79) is equivalent to:

$$\begin{bmatrix} \mu_i X_{2i} & I_n \\ I_n & X_{2j}^{-1} \end{bmatrix} \geqslant 0 \quad (80)$$

We multiply both sides by $\begin{bmatrix} I_n & 0 \\ 0 & X_{2j} \end{bmatrix}$, thus

$$\begin{bmatrix} \mu_i X_{2i} & X_{2j} \\ X_{2j} & X_{2j} \end{bmatrix} \geqslant 0$$

is obtained. This completes the proof.

Remark 4. In (Salwa et al., 2023), the authors utilize the common Lyapunov functional approach to examine the stability of the closed-loop system. While this method is commonly used, it has some limitations, particularly when it comes to finding a common Lyapunov function that applies to all subsystems. In contrast, our approach in this study ensures global stability of the augmented system by integrating the multiple Lyapunov-Krasovskii functional approach with the MDADT concept. This combination allows us to achieve less conservative results compared to using a common Lyapunov function.

Remark 5. According to the MDADT method, the continuous time switched system (56) is GUAS for any switching signal with average dwell times satisfying the conditions $\tau_{ai} \geqslant \tau_{ai}^* = \frac{\ln \mu_i}{\lambda_i}$. As a result, in Theorem 2, the parameters λ_i and μ_i are selected in a way that ensures the feasibility of the LMIs and satisfies the desired minimum average dwell times, denoted τ_{ai}^*.

5. ILLUSTRATIVE EXAMPLES

In this section, two examples are presented to illustrate the validity and effectiveness of the proposed FTC strategy. The first is an application to a HiMAT vehicle. The second compares our method with the one proposed in (Chen et al., 2019).

5.1 Example 1. Application to a HiMAT vehicle

In this part, we will apply the different obtained results to a HiMAT vehicle presented in (Xu et al., 2012) to show the validity and practicability of the proposed approach. The HiMAT vehicle can be viewed as a switching system, where each mode represents the system dynamics at a specific operating point within the entire envelope, as discussed in (Hou et al., 2010). For the sake of simplicity and conciseness, we opt for two specific operating points within the flight envelope, as depicted in Table 2, and acquire the corresponding data from (Li and Yang, 2013).

Table 1. Two Operating points of HiMAT vehicle.

Op. points	Mach number	Altitude (m)	Angle of attack (°)
1	0.6	6100	1.48
2	0.9	7625	4.27

The state variables of the linearized HiMAT dynamical model are $x(t) = [\, x_\alpha \; x_q \,]^T$, where x_α and x_q denote the angle of attack and the pitch rate, respectively. The control input $u(t) = [\, \xi_e \; \xi_c \,]^T$, with ξ_e and ξ_c denote elevon and canard input respectively.

The linearization model for each operating point can be modeled by a continuous-time switched system in (5) where the resulting matrices for both the modes of switched systems are parameterized as

Mode 1

$$A_1 = \begin{bmatrix} -1.35 & -0.98 \\ 17.1 & -1.85 \end{bmatrix},\; B_1 = \begin{bmatrix} -0.13 & -0.013 \\ -14.2 & 10.75 \end{bmatrix}$$

$$F_1 = B_1,\; R_1 = \begin{bmatrix} 0.4 \\ 0.7 \end{bmatrix},\; C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Mode 2

$$A_2 = \begin{bmatrix} -1.87 & -0.98 \\ 12.6 & -2.63 \end{bmatrix},\; B_2 = \begin{bmatrix} -0.16 & -0.005 \\ -29.2 & 21.3 \end{bmatrix}$$

$$F_2 = B_2,\; R_2 = \begin{bmatrix} 0.4 \\ 0.7 \end{bmatrix},\; C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assumptions 1-6 are satisfied and by selection of $\beta = 0.5$, $\varepsilon = 200$, $\gamma = 0.1$, $(\lambda_1 = 15, \lambda_2 = 18)$ and $(\mu_1 = 5, \mu_2 = 7)$ sloving the LMIs given in Theorems 1 and 2 under the LMI Toolbox of MATLAB, the observer and controller gains and the minimum average dwell times are derived as follows:

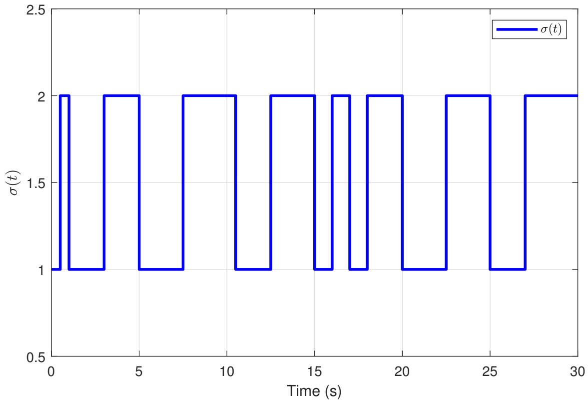

Fig. 1. The switching law $\sigma(t)$

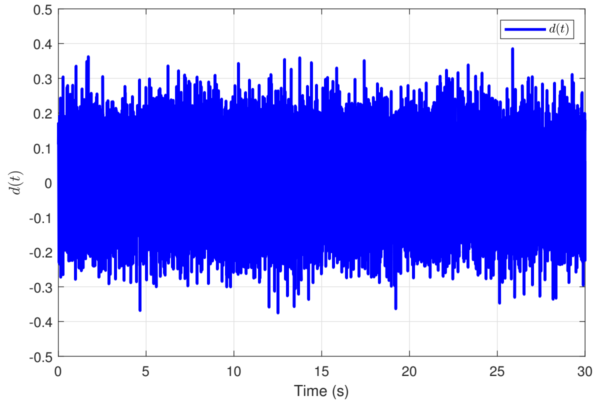

Fig. 2. The external disturbance $d(t)$

For $i = 1$

$$N_1 = \begin{bmatrix} -112.98 & -5.68 \\ 64.33 & 1.52 \end{bmatrix}, G_1 = \begin{bmatrix} 3.00 & -2.32 \\ -1.71 & 1.32 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 37.17 & -23.60 \\ -21.52 & 13.65 \end{bmatrix}, \Upsilon_1 = \begin{bmatrix} 12.10 & -6.84 \\ -6.84 & 4.04 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} -19.3323 & -0.8160 \\ -25.5179 & -0.8823 \end{bmatrix}$$

$$\tau_{a1}^* = \frac{\ln \mu_1}{\lambda_1} = 0.1073$$

For $i = 2$

$$N_2 = \begin{bmatrix} -201.94 & 32.65 \\ 115.37 & -19.44 \end{bmatrix}, G_2 = \begin{bmatrix} 6.22 & -4.58 \\ -3.55 & 2.62 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 79.73 & -47.31 \\ -45.71 & 27.12 \end{bmatrix}, \Upsilon_2 = \begin{bmatrix} 12.10 & -6.84 \\ -6.84 & 4.04 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} -13.5418 & -0.3304 \\ -18.5573 & -0.3560 \end{bmatrix}$$

$$\tau_{a2}^* = \frac{\ln \mu_2}{\lambda_2} = 0.1081$$

For the simulation study, suppose that the initial conditions of the SMO and the system state are chosen as $\hat{x}(0) = [\,0 \;\; 0\,]^T$ and $x(0) = [\,0.3 \;\; -0.2\,]^T$.

Applying the above gains, the switched closed-loop system (56) is GUAS with a weighted H_∞ performance index $\gamma = 0.1$ for any switching signal with MDADT satisfying the following conditions $\tau_{ai} \geqslant \tau_{ai}^*$. For simulation purposes, we apply the switching signal presented in Fig. 1 and specifies the switching between the two subsystems. It is evident that the constrained MDADT condition (10) is satisfied.

The considered actuator faults affecting the switched system (5) is defined by

$$f_a(t) = \begin{bmatrix} f_{a1}(t) \\ f_{a2}(t) \end{bmatrix}$$

and we suppose that

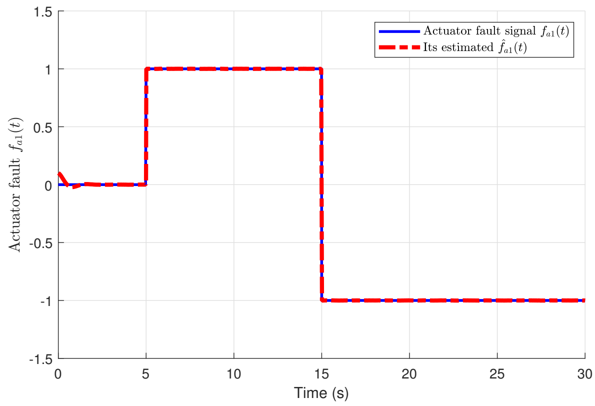

Fig. 3. The constant actuator fault $f_{a1}(t)$ and its estimate $\hat{f}_{a1}(t)$

$$f_{a1}(t) = \begin{cases} 0, & 0s \leqslant t < 5s \\ 1, & 5s \leqslant t < 15s \\ -1, & 15s \leqslant t \leqslant 30s \end{cases}$$

$$f_{a2}(t) = \begin{cases} 0, & 0s \leq t < 5s \\ 1.8\cos(1.6t + 3.25), & 5s \leq t < 23s \\ 0, & 23s \leq t \leq 30s \end{cases}$$

To demonstrate the robustness of the developed method, the unknown external disturbances is a permanent arbitrary signal with the values between -0.3 and 0.3 as shown in Fig. 2.

To simulate the achieved results, a model is constructed using the MATLAB and SIMULINK environment provided by MathWorks, Inc. The simulation results are depicted in the following figures. Figures 3 and 4 depict the signal of actuator faults and its corresponding estimate. It is evident that the fault estimation accurately converges to the actual fault. These figures highlight the effectiveness of the proposed switched SMO in achieving high-quality estimation for both constant and time-varying actuator faults, even in the presence of unknown disturbances.

Figs. 5 and 6 show the plots of both system states and their estimates. It can be deduced from these results that the state of the system is estimated correctly despite the actuator faults and the external disturbances.

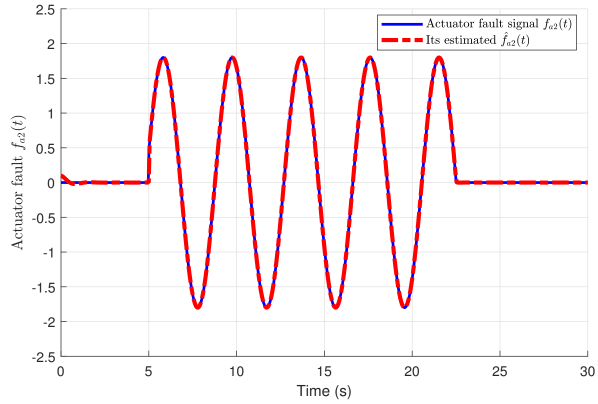

Fig. 4. The time-varying actuator fault $f_{a2}(t)$ and its estimate $\hat{f}_{a2}(t)$

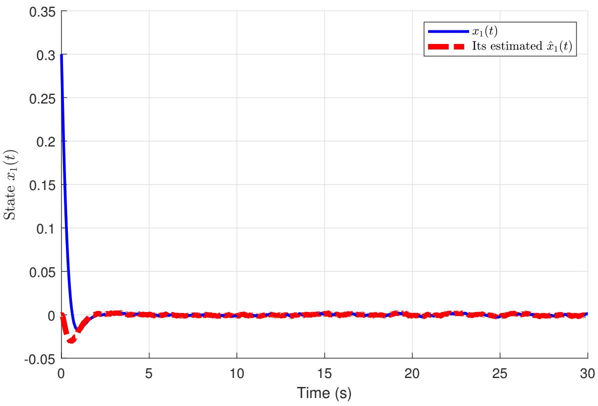

Fig. 5. The state $x_1(t)$ and its estimation $\hat{x}_1(t)$

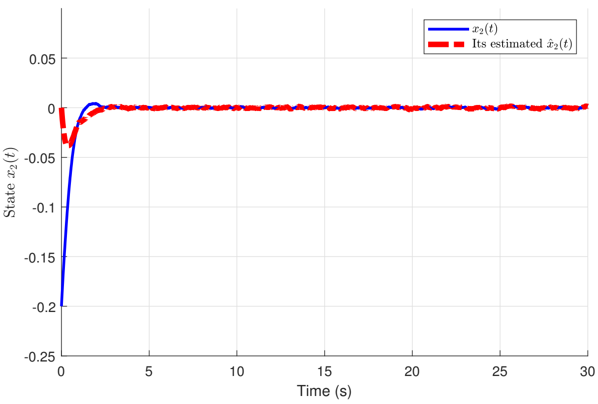

Fig. 6. The state $x_2(t)$ and its estimation $\hat{x}_2(t)$

Therefore, the proposed robust active FTC design achieves the performance under the actuator fault and the stability of the closed-loop system is guaranted.

5.2 Example 2. Comparison results

In this part, a practical example of the two-tank liquid level control system from (Mahmoudi et al., 2000) will be simulated to validate the feasibility of the developed procedure, which is illustrated in Fig. 7. The liquid level control system comprises two tanks, a single flow source, two outlet pipes, and a connecting pipe. The pipes are equipped with valves that can be controlled externally to either open or close. Although there exist eight potential system modes contingent upon the valve status, this paper focuses only on three specific valve configurations:

Model 1: R_2 ON, R_1 and R_3 OFF
Model 2: R_1 and R_2 ON, R_3 OFF
Model 3: R_2 and R_3 ON, R_1 OFF

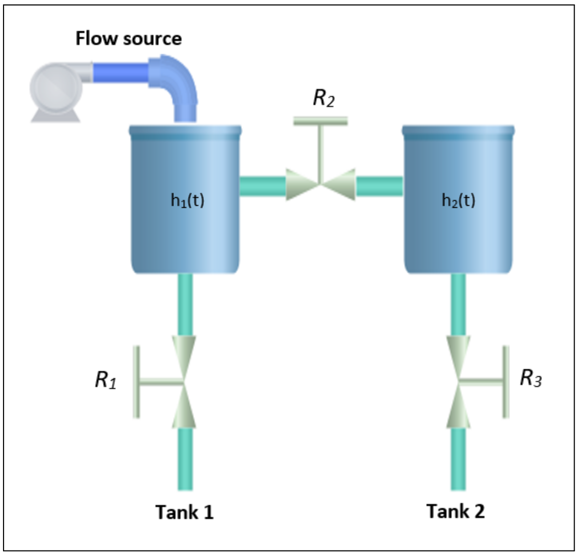

Fig. 7. The two-tank system.

The assumption made is that the flow through the valves adheres to laminar flow, thereby establishing a linear relationship between the flow rate in the valves and the liquid's height. The dynamics of the two-tank system are contingent on parameters like the tank capacity C_T and the pipe resistance R, and they can be represented by switched system (4)-(5). Here, the state vector, denoted as $x(t) = \left[h_1^T(t) \; h_2^T(t) \right]^T$ represents the liquid heights in the tanks. The input $u(t)$ corresponds to the flow source into Tank 1, and the output $y(t)$ represents the liquid levels in each tank. Additionally, the valve configuration is subject to transitions between the three specified modes.

Similar to Du et al. (2019), the parameter matrices are listed as follows:

$$A_1 = \begin{bmatrix} -0.0007 & 0.0007 \\ 0.0011 & -0.0011 \end{bmatrix}, A_2 = \begin{bmatrix} -0.0013 & 0.0007 \\ 0.0011 & -0.0011 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.0007 & 0.0007 \\ 0.0011 & -0.0044 \end{bmatrix}, B_i = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, F_i = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$R_i = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = 1, 2, 3.$$

Assumptions 1-6 are satisfied and by selection of $\beta = 0.5$, $\varepsilon = 100$, $\gamma = 0.01$, $(\mu_1 = 2, \mu_2 = 3, \mu_3 = 2)$ and $(\lambda_1 = 1.2, \lambda_2 = 1.4, \lambda_3 = 1.5)$ sloving the LMIs given in Theorems 1 and 2 under the LMI Toolbox of MATLAB, the observer and controller gains and the minimum average

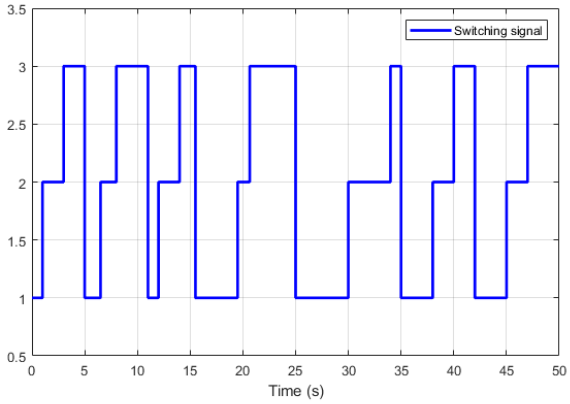

Fig. 8. The switching law $\sigma(t)$

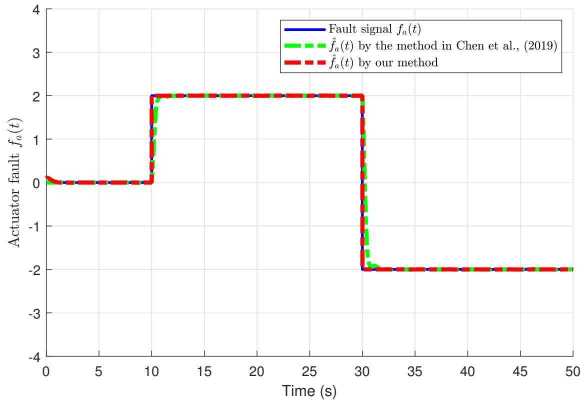

Fig. 9. Actuator fault $f_a(t)$ and its estimate $\hat{f}_a(t)$

dwell times are derived as follows:
For $i = 1$

$$N_1 = 10^4 \times \begin{bmatrix} -3.6091 & 3.6090 \\ 3.6090 & -3.6091 \end{bmatrix}, G_1 = \begin{bmatrix} 0.025 \\ -0.025 \end{bmatrix}$$

$$L_1 = 10^4 \times \begin{bmatrix} 1.8045 & -1.8045 \\ -1.8045 & 1.8045 \end{bmatrix}$$

$$\Upsilon_1 = \begin{bmatrix} 0.186 & -0.166 \\ -0.167 & 0.166 \end{bmatrix}, \Gamma_1 = [\, 7.5775 \;\; 4.7870 \,]$$

and $\tau^*_{a1} = 0.5776$

For $i = 2$

$$N_2 = 10^4 \times \begin{bmatrix} -3.6091 & 3.6090 \\ 3.6090 & -3.6091 \end{bmatrix}, G_2 = \begin{bmatrix} 0.025 \\ -0.025 \end{bmatrix}$$

$$L_2 = 10^4 \times \begin{bmatrix} 1.8045 & -1.8045 \\ -1.8045 & 1.8045 \end{bmatrix}$$

$$\Upsilon_2 = \begin{bmatrix} 0.186 & -0.166 \\ -0.167 & 0.166 \end{bmatrix}, \Gamma_2 = [\, 7.4702 \;\; 5.0009 \,]$$

and $\tau^*_{a2} = 0.7847$

For $i = 3$

$$N_3 = 10^4 \times \begin{bmatrix} -3.6091 & 3.6090 \\ 3.6090 & -3.6091 \end{bmatrix}, G_3 = \begin{bmatrix} 0.025 \\ -0.025 \end{bmatrix}$$

$$L_3 = 10^4 \times \begin{bmatrix} 1.8045 & -1.8045 \\ -1.8045 & 1.8045 \end{bmatrix}$$

$$\Upsilon_3 = \begin{bmatrix} 0.186 & -0.166 \\ -0.167 & 0.166 \end{bmatrix}, \Gamma_3 = [\, 8.1617 \;\; 3.6196 \,]$$

and $\tau^*_{a3} = 0.4621$

Fig. 8 plots the evolution of the switching signal, revealing the active mode of the switched system. It is evident that the constrained MDADT condition (10) is satisfied.

For the simulation purpose, the initial conditions of the SMO and the system state are chosen as $\hat{x}(0) = [\, 0 \;\; 0 \,]^T$ and $x(0) = [\, 0.8 \;\; 0.5 \,]^T$ and the external disturbance $d(t) = 0.1\sin(10t)$.

Now, we consider the situation that the continuous-time switched system (5) is exposed to the following actuator faults:

$$f_a(t) = \begin{cases} 0, & 0s \le t < 10s \\ 2, & 10s \le t < 30s \\ -2, & 30s \le t \le 50s \end{cases}$$

Fig. 9 illustrates the evolution of the actuator fault and its estimated value. It is evident that the fault estimation $\hat{f}_a(t)$ converges with a high degree of accuracy to the actual fault $f_a(t)$. This figure clearly demonstrates that the method presented in this paper offers superior fault estimation performance compared to the method proposed by (Chen et al., 2019).

To prove the efficiency of the proposed SMO, we conduct a comparison with the UIO-based fault estimation method designed in (Chen et al., 2019), assessing several key factors, including the H_∞ performance index, the mean square errors (MSE) and the minimum average dwell times.

Table 2. Comparison with the existing results

References	$\gamma_{\min}$	MSE	The minimum ADT
Chen et al. (2019)	0.1	0.1197	1.998
This paper	0.01	0.0352	0.4621

The comparison provided in Table 2 shows that the result obtained with the developed method in this work is less conservative than the one of (Chen et al., 2019).

From Figs. 10 and 11, we can observe that the system states are stabilized, and the impact of actuator faults is completely compensated.

To sum it up, the acheived outcomes illustrate that the designed SMO and active FTC strategy guarantee the robust stability of the closed-loop system and successfully fulfills the specified H_∞ performance criteria, even when disturbances and actuator faults are present.

Remark 6. The authors in (Chen et al., 2019) have considered the FE algorithm suitable for estimating only actuator faults of the form $f^{(q)}(t) = 0, q \in \mathbb{N}$. In contrast, this work considers time-varying actuator faults.
Remark 7. In the stability analysis of the closed-loop system (56), we employed the MDADT switching strategy, which allows each subsystem to have its own ADT. This strategy permits individualized ADTs for each mode, rather than requiring uniform ADTs across all modes. This approach offers more options for switching rules, providing

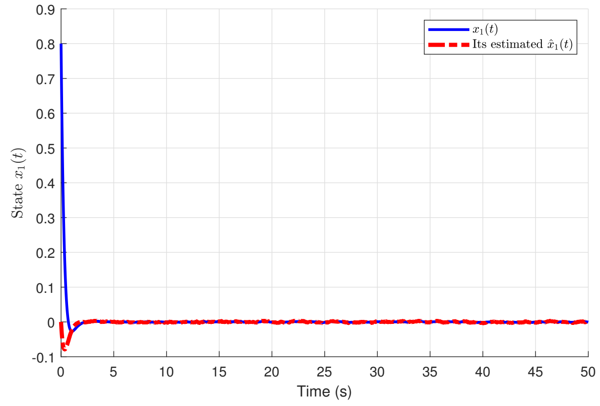

Fig. 10. The state $x_1(t)$ and its estimation $\hat{x}_1(t)$

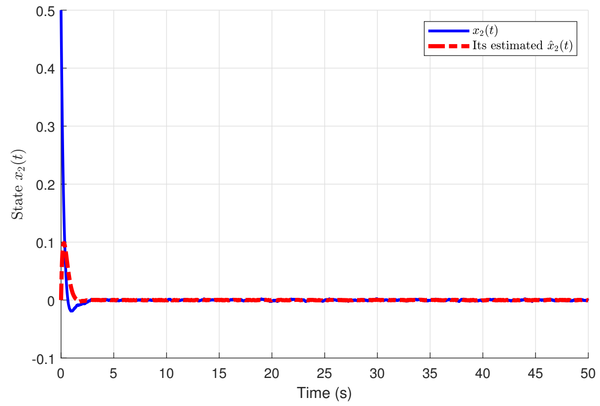

Fig. 11. The state $x_2(t)$ and its estimation $\hat{x}_2(t)$

greater flexibility compared to the ADT switching strategy used in (Chen et al., 2019).

6. CONCLUSIONS

In this work, we investigate the robust AFTC problem for a class of continuous-time switched systems subject to external disturbances and actuator faults. An SMO is developed to achieve simultaneous estimations of system states and actuator faults. Based on the obtained FE, a corresponding FTC law is constructed to compensate the faults and to stabilize the closed-loop system. The multiple Lyapunov functional method and MDADT technique are combined to ensure the stability of the closed-loop system, leading to less conservative results. Sufficient stability conditions are provided in terms of LMIs. Finally, two illustrative examples are used to demonstrate the developed techniques. As future work, we will consider the problem of robust fault tolerant control for switched nonlinear systems.

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