A finite-time control law synthesis method for a class of nonlinear systems based on differential geometry approach

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Abstract: In this paper, a finite-time control synthesis method is presented for a class of nonlinear systems. The proposed method is based on a differential geometric approach, in which a virtual system that is asymptotically stable in finite time with a strict-feedback form is constructed and is diffeomorphic to the system under consideration (i.e., the real system). The control law is derived via a transformation from the real system to the virtual system. This control law ensures that the original system has the same dynamic characteristics as the virtual system and is stable in finite time. The stability in finite-time control of the virtual system is analyzed based on the Lyapunov function. Theoretical analyses and numerical simulations are conducted on three systems and compared with existing methods to demonstrate the effectiveness of the proposed control law.

Keywords: Finite-time control; Fixed-time control; Differential geometry; Nonlinear control systems; Lyapunov-based analysis, Backstepping control

1. INTRODUCTION

In modern control theory, for different types of nonlinear systems, various control methods have been developed based on strict-feedback form and output feedback. Various approaches have been proposed, such as feedback inverse linearization, state/output feedback linearization, dynamic inversion, higher-order sliding modes, Lyapunov-based synthesis, adaptive control, sliding mode control, and fuzzy neural network methods, etc. Among them, a large number of studies apply Lyapunov function in finding control laws for nonlinear systems (Barbashin, E. A. (1967), Khalil, H. K. (2003), Kim, D. P. (2004)). The study (Barbashin, E. A. (1967)) has given some results on the settling time to evaluate the control quality of the system. The studies using backstepping methods, sliding mode control based on strict-feedback systems are presented in the study (Zhang et al. (2000)). For non-strict feedback systems, an effective tool used is to decompose the system into subsystems proposed in (Yu, J., Shi, P., & Zhao, L. (2018)). For the case of unmeasurable states, observers are used. These observers have been proposed in the works (Xiao et al. (2020), Tong et al. (2011)). Fast convergence is one of the key performance indicators in automatic control systems, which characterizes the corresponding convergence speed. Furthermore, to obtain better performance, faster stabilization time has been a focus of research in practice. The works that present a relatively complete relationship between Lyapunov theory and stabilization time and its application to systems are presented in (Bhat, S. P., & Bernstein, D. S. (2000), Yu et al. (2018)). Based on finite-time and fixed-time control techniques, numerous studies have demonstrated their ability to guarantee system states converge to the equilibrium point within a finite time, achieving faster convergence and improved robustness compared to traditional asymptotic stabilization methods. Recent studies (Yu et al. (2018), Polyakov, A. (2012) Tian X. et al. (2023)) have proposed various finite-time control laws that enhance convergence speed and strengthen robustness against model uncertainties. These techniques have been effectively applied to various classes of nonlinear systems such as electromechanical systems, power converters, robotic mechanisms, and systems with state constraints, thereby confirming their superiority in improving control performance and transient response characteristics. However, developing a generalized control law for a broader class of nonlinear models remains challenging due to the inherent complexity in selecting appropriate Lyapunov functions and establishing finite-time stability conditions.

The differential geometry approach to the synthesis of control laws for nonlinear systems allows researchers to consider the fundamental problems of control theory from a much broader perspective, namely the problems of controllability, observability, invariance, decomposability and composability (Jurdjevic, V. (1997). Pupkov, K. A., & Yelutin, N. D. (Eds.). (2004). Miroshnik, I. V. (2006), Agrachev, A. A., & Sachkov, Y. L. (2005), Krasnoshchechenko, V. I., & Krishchenko, A. P. (2005)). This approach is particularly useful for the study of nonlinear control systems. The main idea of differential geometry is the application of mathematical analysis and geometrical images to the explanation of dynamic processes. In the problem of synthesizing control laws for nonlinear systems, the differential geometry approach provides a general idea of the structure and dynamics of states in multidimensional space. As a result, the quality of transient processes can be evaluated explicitly. Once the control laws are parameterized, they can be designed to adjust the process quality according to specific criteria, such as convergence time. The paper by Brockett (2014) discusses the early stages of development in the field of nonlinear control based on differential geometry, including an overview of the broad background of differential geometry-based control theory since the 1970s. The paper (Pan, Z. (2001).) studies the problem of complete linearization by negative feedback for a stochastic nonlinear system with one input – one output. By using the invariant rule under the micromanifold transformation proposed by the author, the coordinate-independent necessary and sufficient conditions for the control signal finding ability are obtained. The study (Kolar et al. (2021)) has demonstrated that any time-discrete nonlinear system with planarity can be decomposed through micromanifold transformations into a subsystem with a lower dimension and an endogenous dynamic response. The study (Marino, R., & Kokotovic, P. V. (1988)) applies two-time-scale singular perturbation methods to nonlinear systems. The main result of this paper is a coordinate-independent description of the time scales through invariant manifolds that represent the conservation and equilibrium properties of the control system. The study (Nguyen et al. (2019)) gives a synthesis of control laws through a quasi-time optimal virtual system based on this approach. The study (Nguyen et al. (2025)) used the differential geometry method to synthesize finite-time control law for flexible-joint manipulator, simulation and experimental results showed the effectiveness of this method.

Based on the above discussions, this paper proposes a finite-time control law synthesis method for a class of objects based on the differential geometry method. First, virtual system is constructed with the property of global stability with finite time. Next, a diffeomorphism between the virtual system and the real control object is constructed and its existence is proven. The control law is synthesized through the transformation from the real object to the virtual system. The finite-time stability of the control law is proven by the Lyapunov method. Unlike previous differential geometry-based methods, our approach integrates finite-time stability with a virtual system framework, offering faster convergence and reduced oscillations compared to traditional backstepping techniques. The rest of this work is organized as follows: Some fundamentals such as the class of control objects, stability and finite-time bounding are presented in Section 2. The virtual system with finite-time stability and the synthesis of the control law are presented in Section 3. In Section 4, some examples of control law synthesis on some higher-order objects and simulation results are presented to demonstrate the effectiveness of the control law. Finally, conclusions and some challenging issues as well as promising directions are presented in the section Conclusion.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Mathematical model of the object

Consider the following class of n-order SISO nonlinear systems:

$$\begin{cases} \dot{x}_1 = f_1(\bar{x}_1) + g_1(\bar{x}_1)x_2 \\ \dot{x}_2 = f_2(\bar{x}_2) + g_2(\bar{x}_2)x_3 \\ \vdots \\ \dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u \end{cases} \tag{1}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ is the state vector with $x(0) = x_0$, $\bar{x}_i = [x_1, x_2, \ldots, x^i]^T$ and u is the control signal. The functions $f_i\,(\cdot)$ and $g_i\,(\cdot)$ (i = 1, 2, .., n) are considered to be known. The symbol x_{1d} denotes the desired reference signal and its derivative is considered to be a smooth, bounded and known function. The control law u needs to be synthesized so that the output x_1 follows the reference signal x_{1d} from any initial condition in finite time and all states of the closed-loop system are bounded in finite time. To synthesize the control law for system (1), the following Assumption 1 must be imposed.

Assumption 1 (Yu et al. (2018)): There exists an open set $\Omega_d \subset R^n$ consisting of the origin and initial condition x_0. The functions $f_i(\cdot)$ and $g_i(\cdot)$ are bounded in the closed set Ω_d; their derivatives up to order n-i are bounded in the closed set Ω_d; there exist known positive constants η and ρ such that $\eta < |gi| < \rho$.

2.2 Differential geometry in equivalence transformation

Nonlinear transformation of state space is a powerful mathematical tool for studying the motion of dynamical systems. According to (Miroshnik, I. V. (2006)), for equivalence transformation of autonomous systems, it is necessary to construct a smooth one-to-one mapping and there exists a corresponding smooth inverse mapping. The model of this transformation can be expressed in the following form. Consider the following autonomous dynamical systems:

$$\dot{x} = f(x), \tag{2}$$

and

$$\dot{z} = \phi(z) \tag{3}$$

where $x \in X \subset R^n$, $z \in Z \subset R^n$, f and ϕ are smooth vector functions defined on some sets X and Z, respectively. The transformation of the state space between (2) and (3) is carried out by the mapping:

$$z = \varphi(x). \tag{4}$$

The equivalence condition of these two systems is determined by the diffeomorphism φ: $X \rightarrow Z$, which has a non-degenerate Jacobian matrix [10], i.e:

$$\det\left[\frac{\partial z}{\partial x}\right] \neq 0. \tag{5}$$

The special feature of this transformation is the preservation of the topological properties of the state space. The theorem on the equivalence of asymptotic stability properties of two systems is presented in the study (Miroshnik, I. V. (2006), Agrachev, A. A., & Sachkov, Y. L. (2005),). In addition, the idea of state space transformation is often used in automatic control systems to study the properties of observability and controllability (Krasnoshchechenko, V. I., & Krishchenko, A. P. (2005)).

2.3 Finite-time control fundamentals

Finite-time stability is defined by the requirement that every trajectory reaches the origin within a finite duration, which represents a stricter condition compared to asymptotic stability. Classical results, such as those in Bhat and Bernstein (2000) and Polyakov (2012), provide sufficient criteria for ensuring that the origin of system (2) is a stable equilibrium in finite time. This property guarantees convergence to the origin within a time that depends on the initial state, whereas fixed-time stability ensures convergence within a uniformly bounded interval that is independent of initial conditions (Polyakov, 2012).

Lemma 1 (Yu et al., 2005): Suppose that there is a Lyapunov function *V(x), c>0, k>*0 and *0< α<1* such that

$$\dot{V}(x) + cV^{\alpha}(x) + kV(x) \le 0, \quad \forall x \in U \tag{6}$$

holds. Then, the equilibrium is practically finite-time stable, and the convergence time of the system (2) is given by:

$$T(x_0) \le \frac{\ln(1 + (k/c)V^{1-\alpha}(x_0))}{k(1-\alpha)} \tag{7}$$

Lemma 2 (Polyakov, A. (2012)): Assume that there is a C^1 function V(x): $R^n \to R^+ \cup \{0\}$ satisfying

$$\dot{V}(x) \le -[\gamma_1 V^p(x) + \gamma_2 V^q(x)]^k \tag{8}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, $p > 0$, $q > 0$, $k > 0$, $p\times k < 1$, and $q\times k > 1$. Then, the equilibrium is practically fixed-time and the and the convergent time of the system (2) is estimated by

$$T(x_0) \le T_{max} = \frac{1}{\gamma_1^k(1-pk)} + \frac{1}{\gamma_2^k(qk-1)} \tag{9}$$

3. SYNTHESIS OF FINITE-TIME CONTROL LAWS FOR SISO NONLINEAR SYSTEMS

3.1 Systems with finite-time characteristics

This section introduces virtual systems designed to exhibit finite-time convergence characteristics to synthesize control laws. The virtual system with fixed finite-time characteristics is built based on the strict-feedback system of the form:

$$\begin{cases} \dot{z}_i = -v_i z_i + z_{i+1}; \quad i = \overline{1, n-1},\ v_i > 0; \\ \dot{z}_n = \tau \end{cases}, \tag{10}$$

where $z_i, i = \overline{1,n}$ are the states of the system, τ is the control signal.

Lemma 3. Consider system (10) with control law (11):

$$\tau = \left(-\frac{c}{2^{\alpha}}\mathrm{sgn}(s)|s|^{2\alpha-1} - \frac{k}{2}s + \sum_{i=1}^{n-1}(v_i c_i z_i - c_i z_{i+1}) \right) / c_n \tag{11}$$

and matrix A ((n-1)×(n-1)) is Hurwitz matrix and satisfies equation:

$$A^T P + PA = -I \tag{12}$$

where: P is positive definite symmetric matrix and $s = \sum_{i=1}^{n} c_i z_i$; $k > 0$; $\alpha \in (0.5,1)$; $c_i > 0, i = \overline{1,n}$.

If matrix A is of the form:

$$A = \begin{bmatrix} -v_1 & 1 & \cdots & 0 \\ 0 & -v_2 & 1 & \cdots \\ 0 & \cdots & \ddots & 1 \\ -\frac{c_1}{c_n} & \cdots & -\frac{c_{n-2}}{c_n} & -v_{n-1} - \frac{c_{n-1}}{c_n} \end{bmatrix}, \tag{13}$$

then system (10) will converge to the neighborhood of origin Δ from any initial value z_0 in a finite time.

Proof:

Consider Lyapunov function:

$$V = \frac{1}{2}s^2 \tag{14}$$

By taking the time derivative, the following expression is obtained:

$$\dot{V} = s\dot{s} = s\left(\sum_{i=1}^{n-1}(-v_i c_i z_i + c_i z_{i+1}) + c_n \tau \right) \tag{15}$$

Substituting *τ* gives us:

$$\begin{aligned} \dot{V} &= s\left(\sum_{i=1}^{n}(-v_i c_i z_i + c_i z_{i+1}) - \frac{c}{2^{\alpha}}\mathrm{sgn}(s)|s|^{2\alpha-1} - \frac{k}{2}s + \sum_{i=1}^{n-1}(v_i c_i z_i - c_i z_{i+1}) \right) = \\ &= s\left(-\frac{c}{2^{\alpha}}\mathrm{sgn}(s)|s|^{2\alpha-1} - \frac{k}{2}s \right) \\ &= -\frac{c}{2^{\alpha}}(s^2)^{\alpha} - \frac{k}{2}s^2 \\ &= -cV^{\alpha} - kV \le 0 \end{aligned} \tag{16}$$

From this, it is evident that the above control law ensures that system (10) evolves on the manifold s = 0. The next step is to prove that the evolution of the system on the manifold s = 0 eads system (10) to converge to the neighborhood Δ of the origin.

For s = 0, it follows that:

$$z_n = -\frac{1}{c_n}\sum_{i=1}^{n-1} c_i z_i \tag{17}$$

Substitute (17) into equation (10) and write it in the following form:

$$\dot{\bar{z}} = A\bar{z} \tag{18}$$

where $z = [z_1, z_2, \ldots, z_{n-1}]^T$ is the vector ($z_{n-1} \in R^{n-1}$), A is a square matrix ((n-1)×(n-1)) of the form (13). The control time for the system of equations (18), given the initial condition $z(0) = z_0$, is analyzed. Let t_p be the minimum time after which $|\bar{z}|$ does not exceed a given value Δ.

First consider the convergence of the solution, consider the Lyapunov function of the form:

$$V(\bar{z}) = \bar{z}^T P \bar{z} \tag{19}$$

By taking the time derivative of V and applying condition (12), the following is obtained:

$$\dot{V}(\bar{z}) = -\bar{z}^T\bar{z} = -|\bar{z}|^2 \tag{20}$$

Using the properties of the quadratic form:

$$\lambda_m|\bar{z}|^2 \le V(\bar{z}) \le \lambda_M|\bar{z}|^2 \tag{21}$$

$$-\frac{V(\bar{z})}{\lambda_m} \le -|\bar{z}|^2 \le -\frac{V(\bar{z})}{\lambda_M}, \tag{22}$$

with λ_m being the smallest eigenvalue and λ_M the largest eigenvalue of the matrix P.

From (20) and (21), the following inequality is obtained:

$$-\frac{dt}{\lambda_m} \le \frac{dV(\bar{z})}{V(\bar{z})} \le -\frac{dt}{\lambda_M} \tag{23}$$

By integrating this inequality, the following is obtained:

$$-\frac{dt}{\lambda_m} \le \frac{dV(\bar{z})}{V(\bar{z})} \le -\frac{dt}{\lambda_M} \tag{24}$$

where:

$$V_0 = \bar{z}_0^T P \bar{z}_0. \tag{25}$$

From inequality (24), combined with inequality (22), the following is obtained:

$$\frac{V_0}{\lambda_M} e^{\frac{-t}{\lambda_m}} \le |\bar{z}|^2 \tag{26}$$

If t_p denotes the response time of the system to move from the initial position to the desired region $(|z|^2 \le \Delta^2)$, the following is obtained:

$$t_p \le \lambda_M \ln \frac{V_0}{\lambda_m \Delta^2} \tag{27}$$

From the above results, it can be seen that system (10) is stable in finite time about the manifold s = 0 and then the system automatically brings the state *z* to a value smaller than the given Δ in a finite time t_p (27). From here, it can be concluded that the system will be globally asymptotically stable in finite time. In addition, the convergence time of system (10) with control laws (11) can be adjusted through the convergence time to the manifold $s = 0$ and the convergence time to the region Δ of the system of equations (18).

Lemma 4. Consider the system (10) with the control law (28)

$$\tau = \left(-\mathrm{sgn}(s) \left(\frac{\gamma_1}{2^p} |s|^{2p-\frac{1}{k}} + \frac{\gamma_2}{2^q} |s|^{2q-\frac{1}{k}} \right)^k + \sum_{i=1}^{n-1} (v_i c_i z_i - c_i z_{i+1}) \right) / c_n \tag{28}$$

and the matrix A ((n-1)×(n-1)) is the Hurwitz matrix, satisfying the equation:

$$A^T P + PA = -I \tag{29}$$

where P is a positive definite symmetric matrix, and $s = \sum_{i=1}^{n} c_i z_i$; $k > 0; p > \frac{1}{2k}; q > \frac{1}{2k}$; $\gamma_1 > 0; \gamma_2 > 0; c_i > 0, i = \overline{1,n}$.

If the matrix A is as in formula (28), the system (10) will converge to the region Δ from any initial value z_0 in a finite time.

Proof: The proof is similar to Lemma 3, the system of equations (10) is stable in a fixed finite time about the manifold s = 0 and the system automatically brings the state z to a value smaller than Δ given in the time interval in formula (27).

3.2 Synthesis of finite-time/fixed-time control laws for nonlinear SISO systems

The main idea of applying differential geometry theory in this work is to find the differential transformation between the original system and some virtual systems. Consequently, this transformation is undertaken by the control law that will be synthesized and have finite-time characteristics. Pointing out the differential transformation will ensure that the synthesized control law is able to stabilize the system in fixed finite-time. The original system is described by the mathematical model (1) and serves as the basis for designing the virtual system (10). This virtual system maintains similar dynamics while offering qualitative improvements, such as finite and fixed convergence times.

Theorem 1. Consider the system (1) with Assumption 1 and the virtual system of equations (10) satisfying Lemma (3). If the diffeomorphism is constructed as in the following system of equations (30):

$$\begin{cases} z_1 = \varphi_1(\bar{x}_1) = x_1 - x_{1d}; \\ z_2 = \varphi_2(\bar{x}_2) = \dot{z}_1 + v_1 z_1 \\ \qquad = f_1(\bar{x}_1) + g_1(\bar{x}_1) x_2 \\ \qquad -\dot{x}_{1d} + v_1(x_1 - x_{1d}); \\ \vdots \\ z_n = \varphi_n(\bar{x}_n) = \dot{z}_{n-1} + v_{n-1} z_{n-1} = \\ = \sum_{i=1}^{n-1} \frac{\partial \varphi_{n-1}}{\partial x_i} (f_i(\bar{x}_i) + g_i(\bar{x}_i) x_{i+1}) - \\ \quad x_{1d}^{(n-1)} + v_{n-1} \varphi_{n-1}(\bar{x}_{n-1}) \end{cases}, \tag{30}$$

and the control signal is constructed as in (31), then the system of equations (1) converges to a region around the origin with radius Δ in a finite time interval (27).

$$u = \left(\frac{\partial \varphi_{n-1}}{\partial x_{n-1}} g_{n-1}(\bar{x}_{n-1}) g_n(\bar{x}_n) \right)^{-1} \times \left(\begin{array}{l} \tau - \sum_{j=1}^{n-1} \frac{d}{dt} \left(\frac{\partial \varphi_{n-1}}{\partial x_j} \right) (f_j(x_j) + g_j(x_j) x_{j+1}) \\ - \sum_{j=1}^{n-2} \frac{\partial \varphi_{n-1}}{\partial x_j} \frac{d}{dt} (f_j(\bar{x}_j) + g_j(\bar{x}_j) x_{j+1}) - \\ - \frac{\partial \varphi_{n-1}}{\partial x_{n-1}} \left(\begin{array}{c} \sum_{i=1}^{n-1} \frac{\partial f_{n-1}}{\partial x_i} \dot{x}_i + \\ x_n \sum_{i=1}^{n-1} \frac{\partial g_{n-1}}{\partial x_i} \dot{x}_i + \\ g_{n-1}(\bar{x}_{n-1}) f_n(\bar{x}_n) \end{array} \right) + \\ x_{1d}^{(n)} - v_{n-1} \sum_{i=1}^{n-1} \frac{\partial \varphi_{n-1}}{\partial x_i} \dot{x}_i \end{array} \right) \tag{31}$$

Proof: Based on condition (5) and the transformation from the original system (1) to the virtual system (10) as in (30), the determinant of the Jacobian matrix is obtained:

$$\det \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & 0 & \cdots & 0 \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \varphi_n}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_2} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{bmatrix} = \prod_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_i} = \prod_{i=2}^{n} g_i(\bar{x}_i) \neq 0 \tag{32}$$

By assumption 1, condition (5) is always true. Therefore, transformation (30) always exists. Then the control law is found recursively from the last equation of system (10) and from the diffeomorphism (30) for system of equations (1) as follows:

$$\dot{z}_n = \tau \tag{33}$$

$$\frac{d\begin{pmatrix} \sum_{i=1}^{n-1}\frac{\partial \varphi_{n-1}}{\partial x_i}(f_i(\bar{x}_i)+g_i(\bar{x}_i)x_{i+1}) \\ -x_{1d}{}^{(n-1)}+v_{n-1}\varphi_{n-1}(\bar{x}_{n-1}) \end{pmatrix}}{dt}=\tau$$

Combined with the system of equations (1), the following is obtained:

$$u=\left(\frac{\partial \varphi_{n-1}}{\partial x_{n-1}}g_{n-1}(\bar{x}_{n-1})g_n(\bar{x}_n)\right)^{-1}\times$$
$$\times\begin{pmatrix} \tau-\sum_{j=1}^{n-1}\frac{d}{dt}\left(\frac{\partial \varphi_{n-1}}{\partial x_j}\right)\left(f_j(x_j)+g_j(x_j)x_{j+1}\right) \\ -\sum_{j=1}^{n-2}\frac{\partial \varphi_{n-1}}{\partial x_j}\frac{d}{dt}\left(f_j(\bar{x}_j)+g_j(\bar{x}_j)x_{j+1}\right)- \\ -\frac{\partial \varphi_{n-1}}{\partial x_{n-1}}\begin{pmatrix} \sum_{i=1}^{n-1}\frac{\partial f_{n-1}}{\partial x_i}\dot{x}_i+ \\ x_n\sum_{i=1}^{n-1}\frac{\partial g_{n-1}}{\partial x_i}\dot{x}_i+ \\ g_{n-1}(\bar{x}_{n-1})f_n(\bar{x}_n) \end{pmatrix}+ \\ x_{1d}{}^{(n)}-v_{n-1}\sum_{i=1}^{n-1}\frac{\partial \varphi_{n-1}}{\partial x_i}\dot{x}_i \end{pmatrix} \qquad (34)$$

It should be noted that if the control law determined from (34), when substituted into (1) ensures the condition (32), then it ensures the stability of the entire system. Furthermore, it can be proved that the system (1) with the control law (34) ensures the system is stable with the finite time of the virtual system (10) according to the similarity of differential geometry.

Theorem 2. Consider the system (1) with Assumption 1 and the virtual system of equations (10) satisfying Lemma (4). If the differential transformation is constructed as in the system of equations (30), the control signal is constructed as in the form (31), then the system of equations (1) converges to a region around the origin with radius Δ in the finite time interval (27).

Proof: Similar to Theorem 1.

Note: In the components of τ in formulas (11) and (28) contains the sign function component *sign(s)*. But this component multiplies expressions that depend on s and will go to 0 as *s* goes to 0. Therefore, the control law does not induce chattering as the system approaches the sliding surface, which is commonly observed in conventional sliding-mode control.

4. NUMERICAL SIMULATIONS AND DISCUSSION

This section presents simulation results for two nonlinear systems to demonstrate the effectiveness of the proposed control synthesis method. The performance is compared with the backstepping method to highlight improvements in tracking accuracy and convergence time.

Example 1:

This example presents a comparison between the technique of this paper and the backstepping method proposed in (Khalil, H. K., 2003). The control object here is the electromechanical system whose diagram is shown in Figure 1 (Tong et al., (2011), Yu et al. (2018)).

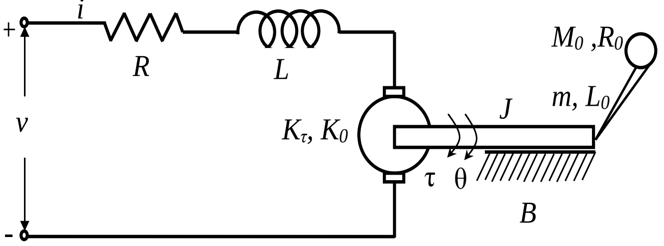

Figure 1. Electromechanical system diagram.

The dynamics of an electromechanical system is described by the following equation (Tong et al., (2011)):

$$\begin{aligned} M\ddot{\theta}+B\dot{\theta}+N\sin(\theta)&=I \\ L\dot{I}+RI&=V-K_B\dot{\theta} \end{aligned} \qquad (35)$$

where $M=\frac{1}{K_\tau}\left(J+\frac{mL_0^2}{3}+M_0L_0^2+\frac{2M_0R_0^2}{5}\right)$, $N=\frac{L_0G}{K_\tau}\left(\frac{m}{3}+M_0\right)$, *J* is the rotor inertia, *M* is the linkage mass, M_0 is the load mass, L_0 is the linkage length, R_0 is the load radius, *G* is the gravity coefficient, B_0 is the viscous friction coefficient at the joint, *θ* is the angular position of the motor (and therefore the load), *I* is the motor armature current, and K_τ is the coefficient characterizing the electromechanical conversion of the armature current into torque. *L* is the armature inductance, *R* is the armature resistance, K_B is the reverse emf coefficient, and *V* is the input control voltage. The values of the selected parameters are $J = 1.625\ 10^{-3}\ Kgm^2$, $m = 0.506\ Kg$, $R_0 = 0.023\ m$, $M_0 = 0.434\ Kg$, $L_0 = 0.305\ m$, $B_0 = 16.25\ 10^{-3}\ Nms/rad$, $L = 15\ H$, $R = 5.0\ \Omega$, $K_\tau = 0.90\ Nm/A$, $K_B = 0.90\ Nm/A$.

Let the variables $x_1=\theta;\ x_2=\dot{\theta},\ x_3=I$ and $u = V$, and the dynamics given by (25) can be rewritten as:

$$\begin{aligned} \dot{x}_1&=x_2 \\ \dot{x}_2&=-\frac{B}{M}x_2-\frac{N}{M}\sin(x_1)+\frac{1}{M}x_3 \\ \dot{x}_3&=-\frac{K_B}{L}x_2-\frac{R}{L}x_3+\frac{1}{L}u \end{aligned} \qquad (36)$$

First, the finite time control law parameters according to Lemma (3) are selected as follows: α=0.99, c=100, v_1=20, v_2=20, v_3=20, c_1=1, c_2=1, c_3=1, k=1. The finite time control law parameters according to Lemma (4) are selected as follows: *p=0.99, q=1.2,* $v_1=20$; $v_2=20$; $v_3=20$, $c_1=1$, $c_2=1$, $c_3=1$, *k=1,* $\gamma_1=100$, $\gamma_2=100$. The simulation is performed with the initial condition chosen to be zero, and the reference signal is $x_d = 0.5\sin(t) + 0.5\sin(0.5t)$. Next, the backstepping method in [2] is also used to control the system (26), with the corresponding controller parameters being $c_1=60$, $c_2=20$, $c_3=40$ and the remaining parameters are chosen the same as the method proposed in this paper. The control signal is limited in the range *[-150,150]*.

The simulation results for the electromechanical, compared with the backstepping control law, are shown in Fig. 2–4. The tracking performance is shown in Fig. 2, from which it can be seen that the tracking performance is still guaranteed and the tracking error is small. For the backstepping method ($x_{1Back.}$), there is still oscillation at the initial time, while the proposed control law ensures that $x_{1Pro.1}$ and $x_{1Pro.2}$ of the motor are stable at the set value without oscillation. It should be noted that this method ensures the system is globally stable, that is, independent of the initial conditions. Fig. 3 presents the tracking performance, from which it can be seen that the tracking error of the backstepping control law is larger and oscillates, while the proposed control law ensures that the error is close to 0 after a finite period of time. Fig. 4 shows the control law values of the three methods, from which it is clear that the proposed control law only changes slightly in the initial time, while the backstepping control law fluctuates strongly in the initial time. This is not good for engineering systems. From this result, it can be seen that the proposed finite-time control method provides better tracking performance than the traditional backstepping method.

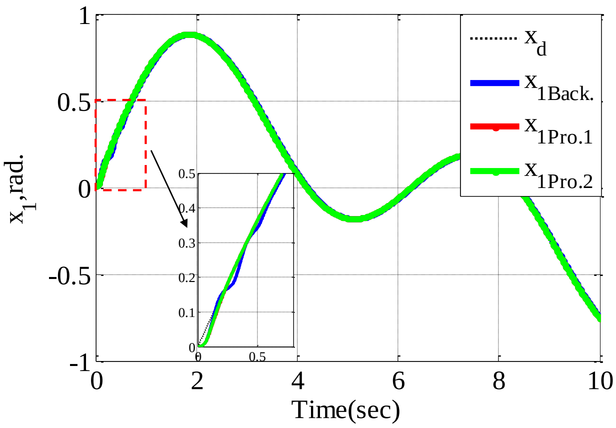

Fig. 2. Tracking performance comparison between the proposed and the backstepping method.

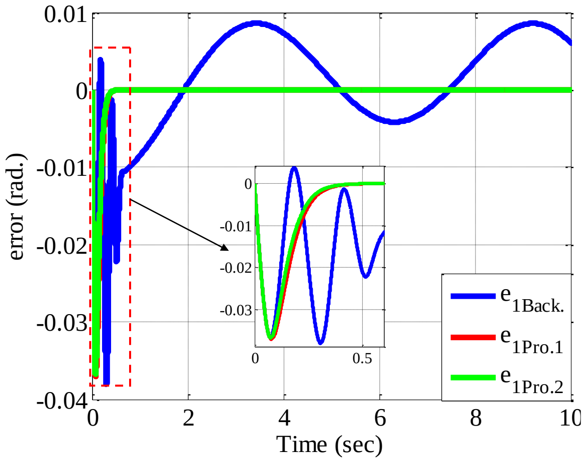

Fig. 3. Control error by the proposed and the backstepping methods.

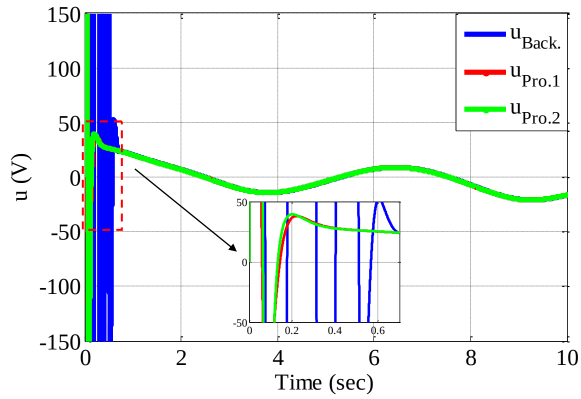

Fig. 4. Control signals by the proposed and the backstepping methods.

Example 2:

In this example, the proposed control laws are synthesized for a flexible joint system. The dynamic model of the flexible joint, presented in the study by Nguyen et al., is given as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{1}{I}\left(Mgsin(x_1) + K(x_1 - x_3)\right) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{1}{J}\left(K(x_1 - x_3) + u\right) \end{aligned} \tag{37}$$

In this system, u denotes the control input applied to the soft joint actuator. The model parameters are defined as follows: $M=1$ *(kg)* is the mass of the link; $K=50$ *(Nm/rad)* represents the stiffness of the joint modeled by a linear torsional spring; $I=1(kg\,m^2)$ and $J=1(kg\,m^2)$ denote the link inertia and motor inertia, respectively; $\ell=1$ *(m)* is the distance to the center of mass of the link; and g =9.8 (m/s^2).

The initial conditions of the system are set at the origin, and the desired reference trajectory for the link angle is given by $x_d=cos(t)$. The parameters of the finite-time control law according to Lemma (3) are selected as: $\alpha=0.99$, $c=100$, $v_1=20$, $v_2=20$, $v_3=20$, $v_4=20$, $c_1=1$, $c_2=1$, $c_3=1$, $c_4=1$, $k=1$. The parameters of the finite-time control law according to Lemma (4) are chosen as: $p=0.99$, $q=1.2$, $v_1=20$; $v_2=20$, $v_3=20$, $v_4=20$, $c_1=1$, $c_2=1$, $c_3=1$, $c_4=1$, $k=1$, $\gamma_1=100$, $\gamma_2=100$. The parameters of the Backstepping controller are selected as: $c_1=5$, $c_2=1$, $c_3=5$, $c_4=20$.

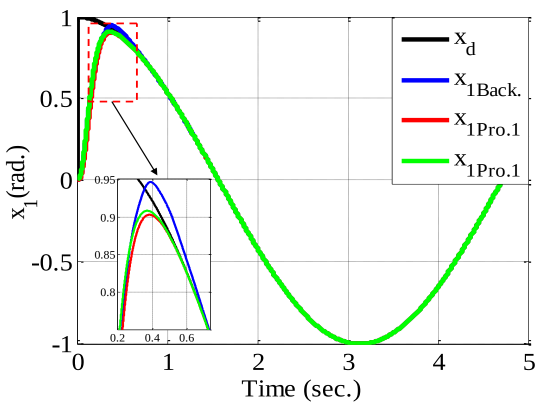

Fig. 5. Response of the link's rotational angle in tracking the desired trajectory x_d.

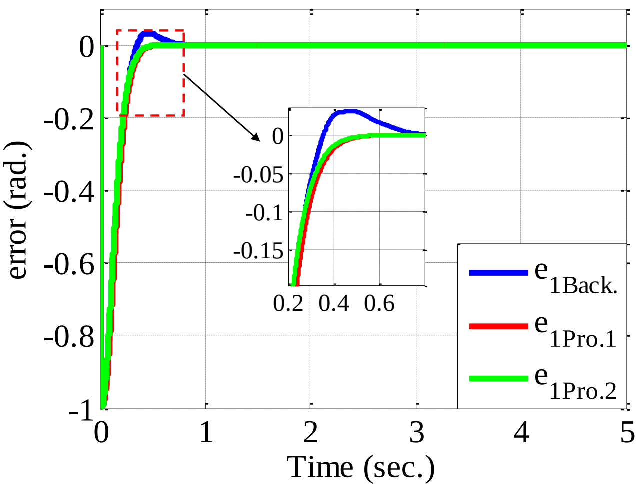

Fig. 6. Tracking error of the link corresponding to the control signals.

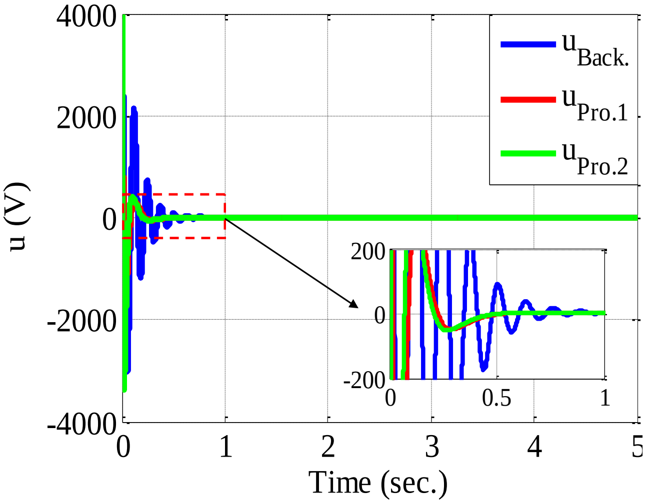

Fig. 7. Control signals generated by the proposed control laws for the flexible-joint manipulator.

The simulation results of Example 2, presented in Figs. 5–7, clearly demonstrate the superior performance of the proposed control laws for the trajectory-tracking problem of the flexible-joint system. Specifically, the link-angle response (Fig. 5) indicates that the two proposed methods ($x_{1Pro.1}$ and $x_{1Pro.2}$) track the desired trajectory rapidly without overshoot and almost coincide with the reference signal. In contrast, the conventional backstepping controller ($x_{1Back.}$) exhibits oscillations and noticeable deviations during the transient phase. The tracking-error analysis (Fig. 6) further shows that the errors corresponding to the proposed methods decrease more rapidly and remain smaller throughout the entire process; in particular, the Pro.2 algorithm achieves near-zero error within a short period of time. Both proposed algorithms suppress the error more effectively, with smaller error magnitudes even from *0.5 s* onward (as highlighted in the zoomed-in region). Moreover, the corresponding control inputs (Fig. 7) also reveal clear advantages, as the proposed methods generate control signals with smaller amplitudes, reduced oscillations, and significantly improved stability compared to the traditional backstepping controller, which produces large and highly oscillatory inputs during the initial phase. These results confirm that the proposed control laws not only enhance trajectory-tracking performance but also offer higher practical feasibility by reducing actuator effort.

Example 3:

Consider the fifth-order nonlinear systems presented in (Yu et al. (2018)) of the following form:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -35.55\sin(x_1) - 0.282x_2 + x_3 \\
\dot{x}_3 &= -0.93x_1x_2 - 5.19x_2 + x_4 \\
\dot{x}_4 &= x_5 \\
\dot{x}_5 &= -\sin(x_3) - x_5 + u
\end{aligned} \tag{37}$$

The initial conditions and reference signals are chosen the same as in [5]. The finite time control law parameters according to Lemma (3) are chosen as follows: $\alpha=0.99$, $c=100$, $v_1=20$, $v_2=20$, $v_3=20$, $v_4=20$, $v_5=20$, $c_1=1$, $c_2=1$, $c_3=1$, $c_4=1$, $c_5=1$, $k=1$. The finite time control law parameters according to Lemma (4) are chosen as follows: $p=0.99$, $q=1.2$, $v_1=20$; $v_2=20$, $v_3=20$, $v_4=20$, $v_5=20$, $c_1=1$, $c_2=1$, $c_3=1$, $c_4=1$, $c_5=1$, $k=1$, $\gamma_1=100$, $\gamma_2=100$. Similarly, the backstepping method is evaluated using different parameter sets: $c_1=90$, $c_2=100$, $c_3=100$, $c_4=100$, $c_5=10$.

The simulation results for the object with equation (37) are presented in Fig. 8 to 10. The simulated controllers all ensure that the system tracks the given signal. For high-order objects, the tracking performance with the control law is still guaranteed as shown in Fig. 8, the backstepping controller gives worse tracking quality. The tracking error shows that the proposed controllers quickly converge to 0 in a finite time but there are still errors, especially at the peak points of the tracking signal. However, the tracking error results are much better than the backstepping method (Fig. 9). In Fig. 10, the control law values of the 3 methods are shown, the control signals do not exhibit excessive peaking or oscillations and do not fluctuate greatly. However, choosing the controller parameters for the backstepping method for high-order systems will be difficult. But with the proposed method it can be clearly seen that the choice of controller parameters can evaluate the stabilization time through the system (10).

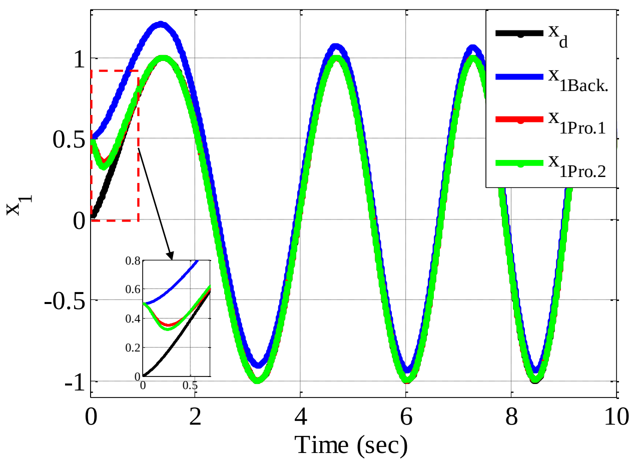

Fig. 8. Control objectives by the proposed and the backstepping methods.

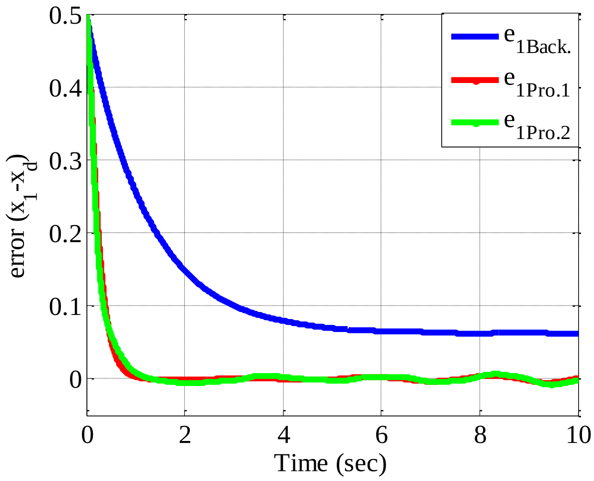

Fig. 9. Control error by the proposed and the backstepping method.

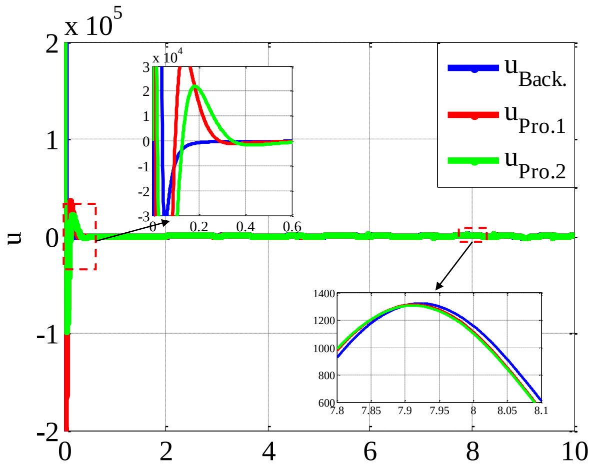

Fig. 10. Control signals by the proposed and the backstepping method.

6. CONCLUSIONS

In this paper, a finite-time control law synthesis method for a class of high-order nonlinear objects is based on the differential geometry method. A finite-time globally stable system was proposed as the virtual system. The finite convergence property of the virtual system can be adjusted through its parameters. The diffeomorphism between the real system and the virtual system is proposed and the existence of this diffeomorphism is confirmed. The proposed control law ensures the similarity of finite stability between the real system and the virtual system. The stability analysis and finite-time stability of the virtual system are proven based on the Lyapunov method. The simulation results through two examples have demonstrated the effectiveness of the control law and the quality compared with the traditional backstepping control law. Future research will aim to extend the proposed approach to high-order nonlinear systems containing model uncertainties and external disturbances.

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